Model calibration methods for mechanical systems with local nonlinearities

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Yousheng Chen
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Appended papers

This doctoral thesis is based on the following papers:


The appended papers were prepared in collaboration with the co-authors. The author of this thesis was responsible for the major progress of the work including taking part in planning the papers, developing theories, carrying out the modelling, simulations and measurements and writing of the papers.
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1 Introduction

1.1 Background

For a long time, product development progress was mainly based on testing on product prototypes. Nowadays, increasing demands on product performance and a strong desire to shorten the product development lead-time in combination with the accessibility of powerful computers have resulted in a considerable shift toward computer based simulations (virtual prototyping). In the field of structural dynamics, mathematical models are widely used. Such models are made under assumptions of material properties, product geometry, properties of structural joints, boundary condition, loading conditions etc. To make the results of virtual prototyping more reliable, the models which they rely on need to be verified and validated with experimental data, when available, throughout the development process. Practical testing can sometimes reveal significant deviations between the numerical predictions and the structure’s behavior. In many situations, the deviations are caused by structural nonlinearities not taken into account in the models.

Linear finite element (FE) models, in which nonlinear effects are neglected by linearization, are often employed in the analysis of vibration due to their simplicity and efficiency. However, linearity is an idealization and for a real structure, some degree of nonlinearity is always present. Recent demands on lighter products lead to more flexible structures with potentially more significant nonlinear behavior.

Nonlinear phenomena are often observed during vibration tests made on prototype structures. For instance, nonlinear behavior was observed from various ground vibration tests (GVTs) on the Cassini–Huygens spacecraft [1] and the Airbus A400M aircraft [2]. In [1], it was found that the frequency of the longitudinal mode of the Huygens Probe decreased as the excitation level increased. In [2], the nonlinearities were attributed to the pylon mounted turboprop engines of the aircraft. The nonlinear behavior was also observed in between the wing-to-payload mounting interface of an F-16 aircraft [3, 4]. An example of a GVT setup of an F-16 aircraft with a mounting connection is shown in Figure 1.
In the automobile industry, viscoelastic engine mounts are often used; they show significant nonlinear behavior which, among other variables, depend on the amplitude, frequency and preload [5]. In civil engineering, nonlinear features can for instance be found in grandstands demountable structures with loose joints. Nonlinearity may also display in a damaged structure.

Nonlinearity can generate complex dynamics, such as shifted resonance frequencies, rapid transitions between responses with totally different amplitudes, sub- and super-harmonic responses, modal interactions, quasi-periodicity and chaos. When structures show significant nonlinear behavior, linear models may fail to predict the structural response with the necessary accuracy. Hence, to improve the predictive fidelity of structural models, it is important to take nonlinear effects into account. Based on the localized nature of nonlinear features, most mechanical systems can be accurately represented by linear FE models combined with nonlinear elements [6], which makes it possible to combine high prediction accuracy with high computational efficiency. The linear FE models may have many degrees of freedom (DOFs) while the number of DOFs connected to nonlinear elements is relatively small. The local nonlinearities cause the overall model nonlinear effects.

To properly define candidate parameters that can describe the nonlinearities, it is necessary to understand the physics behind the nonlinear phenomena. A wide variety of methods have been developed to support in detection [7], localization [8] and identification [9, 10] of structural nonlinearities. Other methods for nonlinear system identification in the field of structural dynamics were reviewed in [6, 11].

The parameters chosen to control nonlinear effects of a structural model introduce more sources of uncertainties. In this work, these parameters are calibrated with experimental data. Improving the structural model through comparison with data from vibration measurements made on the structure is referred to as structural model updating/calibration. Model calibration is thought to be very crucial for model based design and maintenance of engineering structures. Model calibration methods for linear structural models were well documented in [12]. In [13], model calibration methods, for linear

Figure 1: GVT setup of an F-16 aircraft with a mounting connection [3, 4].
FE models, concerning test design, parameter identifiability and test data informativeness, were developed. Recently, a damping equalization method which avoids mode pairing for model calibration of linear systems was proposed [14]. The damping equalization method was applied to calibrate and validate an FE model representing a car front subframe [15]. It is also combined with probabilistic tools and methodologies for stochastic model calibration, which deals with the uncertainties arising from missing information [16].

Model calibration methods for nonlinear structural models are relatively sparse and there is no universal method that can be applied for all types of nonlinear systems.

Transient time-domain data was used to update a nonlinear model representing a structure containing a hyperelastic polymer in [17]. An FE model with local nonlinear elements was updated using a function of the measured time history in [18]. The proper orthogonal decomposition (POD) of the covariance matrix constructed from response time histories was applied for model calibrations of nonlinear structures [19]. Nonlinear normal modes (NNMs) can be regarded as the extension of the concept of linear normal modes for nonlinear systems. NNMs offer a rigorous mathematical tool for interpreting a wide class of nonlinear dynamical phenomena and have been utilized for model calibration [20, 21].

Many statistical methods have also been employed to calibrate nonlinear models [22-27]. One of most widely used statistical methods is based on Bayes’ theorem. Yue and Beck [24] used a Bayesian probability approach to calibrate nonlinear structural models. The Bayesian theorem can be combined with efficient Markov Chain Monte Carlo simulation techniques [28, 29], unscented Kalman [30] or particle filters [31-34] for nonlinear system identification.

A classic frequency domain approach based on the first order harmonic balance (HB) method was employed in the calibration of nonlinear parameters [35, 36]. Only the fundamental harmonic is considered in that method. When the structure is strongly nonlinear, it is important to also include higher order harmonics and multiharmonic frequency response functions (FRFs) have also been investigated for nonlinear model calibration. The multiharmonic balance method was combined with the extended constitutive relation error (ECRE) for nonlinear model calibration in [37]. The higher order FRFs can also be calculated by using Volterra kernels, which were used to calibrate a three-dimensional portal frame with a local nonlinearity [38]. Another nonlinear model calibration method was also developed by Canbaloglu and Özguven [39]. They calculated the underlying linear system and the structural nonlinearity by using FRFs measured at various forcing levels.
1.2 Aim and scope

The aim of this work is to develop a framework to calibrate models of complex mechanical systems with local nonlinearities. The research work is centred on the following topics:

1. Development of a model calibration method suitable for FE-models with structural nonlinearities.
2. Development of efficient simulation methods for nonlinear systems.
3. Development of methods for pre-test planning aiming at achieving informative data for nonlinear model calibration.
4. Development of practical vibrational testing techniques to generate test data for calibration of nonlinear models.
2 Linear and nonlinear systems

This chapter describes the mathematical definitions of linear and nonlinear systems. In addition, it provides an introduction to the fundamental concepts used throughout the research work.

2.1 Mathematical definition

The creation of mathematical models accurately representing a structure may be the most demanding step in any dynamic analysis. Analytical models fall into two groups: continuous models and discrete models. Partial differential equations are frequently employed to build the continuous models, which give natural mathematical descriptions of physical phenomena. However, such models are often too complicated to be solved. Therefore, discrete models are more widely utilized.

Structural models are often built with the assumption that the structural behavior is linear. A linear system satisfies the principle of superposition which states that the system is additive and homogenous [7, 40]. Linearity means that the system’s response to any combination of dynamic loads, simultaneously applied, equals the sum of responses due to each of the loads acting separately, which can be stated as

\[ H\{a_1f_1(t) + a_2f_2(t)\} = a_1H\{f_1(t)\} + a_2H\{f_2(t)\} \]  

where \(a_1\) and \(a_2\) are constants and \(H\{f_1(t)\}\) denotes the response due to the excitation \(f_1(t)\). If the system is linear and the parameters that describe the function \(H\) are independent of time, then \(H\) represents a linear time invariant (LTI) system.

A lot of LTI mechanical systems can be described by the second order differential equation known as the governing equation of motion:

\[ M\ddot{q}(t) + V\dot{q}(t) + Kq(t) = f(t) \]  

(2)
where $M$, $V$, and $K$ are the $n$-by-$n$ mass, viscous damping, and stiffness matrices respectively. $x(t)$ is the $n$-by-$1$ displacement vector and $f(t)$ is the $n$-by-$1$ force vector. Using the Laplace transform, the outputs can be related to the inputs via a transfer function matrix, $[H(s)]$

$$q(s) = H(s)f(s), \quad H(s) = [Ms^2 + Vs + K]^{-1}$$  

An evaluation of the transfer function along the $j\omega$ (frequency) axis gives an FRF representation. The FRF is the most generally used method to visualize the input-output properties of a linear dynamic system. Information such as resonance frequencies, anti-resonances and mode shapes can be extracted from FRFs and the FRFs are invaluable for model calibration studies.

In contrast to a linear system, a system that does not fulfil the principle of superposition is denoted a nonlinear system. The equation of motion for a general nonlinear mechanical system can be written as:

$$M\ddot{q}(t) + V\dot{q}(t) + Kq(t) + f_{NL}(q(t), \dot{q}(t)) = f(t)$$  

where the vector $f_{NL}$ is a non-linear function of the displacement and velocity vectors. Many of the properties which hold for linear structures break down for nonlinear ones. The properties introduced with linear theory, for example FRFs and resonance frequencies, are often dependent on the characteristics of the excitation when applied to a nonlinear system.

### 2.2 Common types of nonlinearities

Typical nonlinearities are introduced by large displacements, contact surfaces which vary with time, friction in joints and nonlinear material properties. These causes are usually amplitude, velocity and frequency dependent.

For a cubic nonlinear restoring force

$$f_{NL}(x(t)) = k_3q^3(t)$$  

where $k_3$ is the coefficient for the nonlinear spring; it can be positive or negative. If $k_3<0$, the effective stiffness will decrease with an increasing forcing level and such a system will therefore show a softening characteristic. Whereas, if $k_3>0$, the effective stiffness will increase with an increasing forcing level and such a system will show a hardening characteristic.

For a clearance type of nonlinearity, the nonlinear force can be represented as

$$f_{NL}(x(t)) = \begin{cases} k_1(q(t) - e_1)^r, & q(t) \geq e_1 \\ k_2(q(t) + e_2)^s, & q(t) \leq -e_2 \\ 0, & -e_2 < q(t) < e_1 \end{cases}$$  

where $r$ and $s$ are exponents which can be chosen to be equal to the integer value 1 or 2.
where \( e_1 \) and \( e_2 \) denote the gaps whereas \( r \) and \( s \) are constants. This type of nonlinearity can arise in assemblies such as pylon-wing assemblies. It is also common to have nonlinear damping effects in structures. The most common form of polynomial damping is the quadratic one which can be written as

\[
f_{NL}(\ddot{x}(t)) = c_2 \dot{q}(t) |\dot{q}(t)|
\]  

(7)

where \( c_2 \) is the constant for the quadratic damping force. Idealized forms of a cubic stiffness force, symmetric clearance and quadratic damping force are illustrated in Figure 2.

![Figure 2: Idealized forms of some common structural nonlinearities.](image)

**2.3 Nonlinear frequency response functions**

Nonlinear structures respond differently to different types of excitations. Stepped-sine, impact, rapid sine sweep (chirp) and random excitation are four types of excitation commonly used in vibrational testing.

The usage of sinusoidal excitations of nonlinear systems usually produces the most vivid effects such as bifurcation (the jump phenomenon seen in the FRFs). With a stepped sinusoidal excitation, the signal-to-noise ratio is high compared to the situation for a random or transient excitation and the resulting FRFs clearly show the distortions arising from nonlinearities, particularly when a constant magnitude of the excitation is used.
A possible drawback of using stepped sinusoidal excitations is that such a test can be time consuming compared to using transient or random excitations. The reason for this is that some time is required for the response to attain a steady-state condition for each frequency increment, before the response at that frequency can be measured. However, this is usually a secondary factor compared to the importance of obtaining high-quality FRFs. Another drawback is that a mono-sinusoidal excitation is difficult to achieve when nonlinear structures are concerned [41]; the intended pure sine often becomes a multi-sinusoidal excitation. Therefore, a multi-sinusoidal excitation is designed and applied for vibrational testing which results in multi-harmonic FRFs. The multi-harmonic FRFs are discussed in detail in section 4.2.

A classical nonlinear system is the Duffing oscillator, which is a single-input-single-output (SISO) system, with a cubic hardening spring force:

\[ m\ddot{q}(t) + c\dot{q}(t) + kq(t) + pq^3(t) = f(t) \]  

(8)

The excitation \( f \) is here chosen as a pure sinusoidal excitation and an example of a nonlinear FRF for this type of system is shown in Figure 3. The Eq. (8) is solved under the assumption that the response to a sinusoidal excitation is a sinusoid at the same frequency. As seen in this example, there is a frequency region with three possible steady-state response solutions; two stable and one unstable.

![Figure 3: The FRF of a single degree-of-freedom nonlinear oscillator with a cubic hardening spring force subjected to a sinusoidal input. Parameters: \( m=450 \text{ kg, } c=250 \text{ Ns/m, } k=1.5 \cdot 10^7 \text{ N/m, } p=1.7 \cdot 10^{15} \text{ N/m}^3 \) and the amplitude of the excitation force is 2 N.](image)


3 Simulation techniques

Simulation refers to the computation of the dynamic responses of a model representing a mechanical system with given system parameters, stimuli and initial conditions. Simulation has an important role in the understanding of the dynamic behavior of nonlinear structures. In this chapter, the time domain and the frequency domain numerical simulation techniques used in this work are presented.

3.1 Time domain simulations

The time domain simulation methods used in this thesis are computations of the motion (displacement, velocity and acceleration) of a system subjected to dynamic stimuli using direct time integration. Time domain simulations of nonlinear mechanical systems can be applied in model calibration and validation purposes.

Traditional time integration schemes such as the Runge-Kutta [42] and the Newmark methods [43] are commonly used to calculate the responses of mechanical systems. These methods may be computationally expensive when a nonlinear system is concerned. This problem will only become worse when large-scale systems are considered. To address this issue, many efforts have been put into developing different strategies such as model reduction [44, 45], parallel simulation methods [46, 47] and efficient time-integration procedures [48-51].

In Paper II, a fast time domain scheme was developed. The method was formulated in a state-space form which was developed for large-scale linear systems with local nonlinearities. The system was divided into a linear part and a nonlinear part. The nonlinear part was treated as external forces acting on the linear system rendering in a force feedback system. Then, equation (4) can be written as

\[ M\ddot{q}(t) + V\dot{q}(t) + Kq(t) = f(t) - f_{NL}(q(t), \dot{q}(t)) \]  

Equation (9) can be put into a state-space formulation as

\[
\dot{x}(t) = Ax(t) + B^Lf(t) + B^{NL}f_{NL}(t) \\
y(t) = Cx(t) + D^Lf(t) + D^{NL}f_{NL}(t)
\]

in which \( y(t) \) is the system’s output and the state vector \( x(t) = [q(t) \ \dot{q}(t)]^T \). \( A \) is the state matrix of the underlying linear system. The applied forces \( f(t) \).
and $f_{NL}(t)$ are projected onto the state-space through the input matrices $B_L$ and $B_{NL}$. $C$ is the state to output matrix and $D_L$ and $D_{NL}$ are the feedthrough matrices. The state-space matrices are constant when the system is time invariant.

Equation (10) is a system of continuous-time, ordinary differential equations which generally need to be transformed into a discrete time recursive algorithm with a time step, $T$, to enable a numerical solution. When working with sampled data, approximations between two successive sample points have to be made by a discretization method, such as the impulse invariant, the zero-order-hold (ZOH), the first-order-hold (FOH) or the Lagrange second-order-hold (LSOH) method. Different approximations of a discrete load are shown in Figure 4.

![Figure 4: Different assumptions of a discrete load, in the interval $[nT, nT+T]$ (from Paper III).](image)

The discretization methods introduce bias errors, which are critical for the predictive fidelity of time simulation methods. Therefore, the effects of different discretization methods were studied for linear and nonlinear systems in Paper III. By studying the convolution kernels and the Fourier transforms, theoretical expressions of the bias error were derived; they are plotted in Figure 5.
Figure 5: Bias errors for the ZOH, the FOH and the LSOH methods. Left: amplitude error, Right: phase error (from Paper III).

The theoretical expressions of bias errors were evaluated using the fourth order Runge-Kutta, the digital filter [48] and the proposed state-space based simulation method. It was shown that the ZOH method gives large errors and it is therefore not suitable for time integration. The LSOH method gives the smallest bias error, but it is computationally more expensive than the FOH method. The FOH method constitutes a good compromise between accuracy and efficiency for time simulation methods. Therefore, an exponential FOH integrator was used in the state-space based simulation method proposed in Paper II. An example of a time response of a nonlinear system subjected to a mono-sinusoidal excitation, calculated using the proposed method, is shown in Figure 6. The general process for solving Eq. (4) using the proposed method is shown in Figure 7.

Figure 6: An example of a time response of a nonlinear system subjected to a mono-sinusoidal excitation.
3.2 Frequency domain simulations

Time domain methods may be inefficient if the main concern is to obtain the steady-state solution, since it can take a long time until the transients vanish. To overcome this, frequency domain methods can be applied to obtain the nonlinear FRFs without first computing the transient responses.

Among the frequency domain methods, the harmonic balance (HB) method is probably the most widely used and it was developed to solve forced periodic responses of nonlinear systems. The HB method is based on the assumption that a system’s responses will be periodic when the excitation is periodic. Then, the responses can be represented in the form of truncated Fourier series.

The first mechanical applications of this method were on an SDOF dry friction damped system [52] and on a beam incorporating dry friction [53]. The computation expense of the HB method has been further improved using the alternating frequency time domain method [54], receptance-based perturbation [55] and arc-length continuation [56]. More recently, the adaptive HB method was developed to adaptively select the number of harmonics to include in the responses using the observation of the relative variation of an approximate strain energy for two consecutive numbers of harmonics [57]. The HB method has been applied to many nonlinear systems [58-61] and it has also been implemented for bifurcation tracking [62].
The performance of the HB method for calculation of nonlinear FRFs was compared with the time-domain methods Runge-Kutta, Newmark and the state-space based method presented in this research work [63]. It was shown that the HB method is a fast simulation scheme for calculation of FRFs of a nonlinear system under harmonic excitation. The HB is normally applied for a system excited with a pure harmonic excitation, i.e. a mono frequency excitation. As mentioned before, a mono-sinusoidal excitation may be difficult to realize in real measurements and a multi-sinusoidal excitation was therefore proposed in Paper I. The receptance based HB method was extended for multi-sinusoidal excitations, which was applied in Papers I, IV and V to calculate multi-harmonic FRFs of a nonlinear system subjected to a multi-sinusoidal excitation.

With an, infinite in time, multi-sinusoidal excitation; if periodic steady-state responses exist, the responses and the nonlinear force can be expressed in the frequency domain using Fourier series. Thus, Eq.(4), can be written in the frequency domain as:

$$ZQ + F_{NL} = F$$

(11)

where $Z$ denotes the complex valued dynamic stiffness of the underlying linear system and $Q$, $F_{NL}$ and $F$ are the Fourier coefficient of the displacement, the nonlinear force and the externally applied force respectively. When $N$ harmonics are included in the calculation, Eq. (11) consists of a system containing $n$ times $N$ equations, where $n$ is the number of degrees of freedom of the mechanical system. For each frequency of interest, the system’s response $Q$ is estimated by balancing the Fourier coefficients in Eq. (11). The accuracy of the HB method will be improved by increasing the number of harmonics in the truncated Fourier series, at the expense of an increased computational time. By solving the system’s frequency responses over a specified interval, it is possible to calculate FRFs. The FRFs obtained by the HB method is often regarded as the analytical equivalence to the FRFs obtained by stepped sine testing [7].

4 Model calibration framework

The proposed model calibration process for nonlinear systems is described in this chapter. The underlying theory is presented together with the analysis methods used for the pre-test planning and validation.
4.1 Overview

A mechanical system with local nonlinearities is often modelled as a linear system together with polynomial expressions with the aim of taking the nonlinear effects into account. It is desirable to, if possible, first calibrate the linear part of the model, before calibrating nonlinear parameters. The underlying linear system of the nonlinear structure can often be measured using low level forcing; systems with nonlinearities from dry friction may be exceptions. However, when there are only friction types of nonlinearities, FRFs measured at high response levels can accurately represent FRFs of the underlying linear system.

The proposed FE model calibration procedure consists of two main phases: linear and nonlinear model calibrations. In the linear model calibration made in this work, the underlying linear structure was calibrated using low level forcing data resulting in calibrated global mass, linear damping and stiffness matrices.

Here, the nonlinear model calibration refers to the calibration of the parameters representing the causes of the nonlinear behavior. The nonlinear model calibration consists of four parts; a pre-test planning, a multi-sinusoidal excitation, an efficient optimization routine and a validation. The pre-test planning renders in a test design for measurements and forms a calibration metric for the optimization. The optimization is conducted by minimizing the difference between the calculated and measured multi-harmonic nonlinear FRFs using a selected parametric starting point. After the model is calibrated, the optimum is used as the starting point for a $k$-fold cross validation to obtain parameter uncertainty. An overview of the proposed model calibration framework can be seen in Figure 8.

4.2 Multi-harmonic FRFs

In vibrational testing of a nonlinear structure, a mono-frequency excitation is difficult to achieve and harmonic distortions due to the exciter-structure interaction are always present [41]. Instead of trying to achieve a pure sinusoidal excitation, one may have a better chance to enforce a multi-harmonic stimulus with a distortion level that overshadows the intrinsic test setup distortion. It was also shown in Paper I that the frequency responses at the side harmonics contain valuable information for calibration of models of locally nonlinear structures. Therefore, a multi-sinusoidal excitation was designed.
Let a multi-sinusoidal excitation at DOF $j$ with the fundamental frequency $\Omega$ be defined as

$$f_j(t) = \Re \left\{ \sum_{v=1}^{\overline{v}} F_v e^{iv\Omega t} + \sum_{v=2}^{\overline{v}} F_{1/v} e^{i\Omega t/v} \right\}$$  \hspace{1cm} (12)$$

where $\Re\{\cdot\}$ denotes the real part, $F_v$ and $F_{1/v}$ are complex valued force amplitudes, $(\overline{v} - 1)$ and $(\overline{v} - 1)$ are the number of super-harmonic and sub-harmonic components included in the excitation force. The force magnitudes at side harmonics are designed to be small compared to the magnitude of the fundamental harmonic. The periodic steady-state time response, if such exists, at DOF $i$ can be expressed as a series.

*Figure 8: An overview of the proposed nonlinear model calibration framework, NL denotes nonlinear.*
\[ q_i(t) = \Re \left\{ \sum_{v=1}^{\bar{N}} Q_v e^{i\omega_v t} + \sum_{v=2}^{N} Q_{1/v} e^{i\Omega t/v} \right\} \] (13)

\( Q_v \) and \( Q_{1/v} \) are the complex valued harmonic coefficients of the displacement whereas \((\bar{N} - 1)\) and \((N - 1)\) are the number of super-harmonic and sub-harmonic displacement components taken into account. A harmonic nonlinear FRF between the response at DOF \( i \) and the excitation at DOF \( j \) is defined as

\[ H_{ij}^v(\Omega) = \frac{Q_v}{F_v} \quad \text{and} \quad H_{ij}^{1/v}(\Omega) = \frac{Q_{1/v}}{F_{1/v}} \] (14)

in which \( H_{ij}^v(\Omega) \) denotes the \( v \):th-order nonlinear FRF with the fundamental frequency \( \Omega \) and \( H_{ij}^{1/v}(\Omega) \) represents the \( 1/v \):th order nonlinear FRF. Examples of multi-harmonic FRFs, from a mono-sinusoidal excitation, compared with the underlying linear FRFs are shown in Figure 9, in which the superharmonic FRFs are plotted with respect to the fundamental frequency with a scaling.

![Figure 9: An example of multi-harmonic FRFs. The solid lines show FRFs from the underlying linear model whereas the dotted lines show nonlinear FRFs (from Paper 1).](image)

### 4.3 Pre-test planning

A pre-test planning is crucial for successful vibration tests, especially when testing a complex structure using limited resources. The objective is often to find optimal sensor and actuator placements to be used in experimental modal analysis [13, 64].
A natural aim of a pre-test planning is to maximize the data informativeness and parameter identifiability using a nominal analytical model. The requirement for data informativeness is that a change of the value of any parameter should change the test data in a noticeable way. Parameter identifiability is referred to that test data should differentiate changes of different parameters. When these two requirements are not fulfilled, the estimated parameter values obtained using that test data are unreliable [13]. Data informativeness and parameter identifiability are coupled. They can jointly be stated as regardless of which parameters that are estimated, precise estimates require a small parameter covariance matrix. The Fisher information matrix (FIM) and the Cramer-Rao lower bound are useful quantities to assess test data informativeness and parameter identifiability [65]. The reason for this is that the Cramer-Rao theoretical lower bound establishes a limit on the expectation of the covariance matrix of the estimates of the parameter values. This limit is coupled to the inverse of the FIM according to

\[ \Xi[p - p^*][p - p^*]^T \geq \text{FIM}^{-1} \] (15)

in which \( \Xi \) denotes the expectation and \( p^* \) denotes the true parameter setting. The Cramer-Rao bound states that, irrespective of the unbiased estimator used to quantify the parameters from the data, the co-variance of the estimator is at least as large as the inverse of the FIM. To rank the goodness of the FIM or its inverse, a number of criteria such as the T-optimality [66] and D-optimality [67] have been proposed. The T-optimality aims at maximizing the trace of the FIM whereas the D-optimality maximizes the determinant of the FIM. Data informativeness can also be optimized by minimizing the largest magnitude of the individual diagonal elements of the inverse of the FIM, which is motivated by its capability to make the test data as equally informative for all parameters as possible [68]. When the FIM is truly or nearly rank deficient, one way to increase the test data informativeness or the parameter identifiability is to re-parameterize; this is studied in Paper I. An alternative solution may be to change the test design. In [69], an orthogonality/collinear index was developed to facilitate such a re-parameterization.

The traditional pre-test planning methods may not be applicable for testing of nonlinear systems in which the measured responses are not only dependent on the placement of sensors and actuators, but also on the excitation levels and the excitation frequency ranges. Therefore, a FIM based pre-test planning method concerning the sensor and actuator positions, the excitation levels and the excitation frequency ranges is proposed in this work. The optimal configuration is found by a three step procedure:

1. Select candidates for actuators, sensors, frequency ranges and excitation levels.
2. Combine the candidates from all the variables above to obtain all the possible candidate configurations.
3. Calculate the resulting FIM for each candidate configuration and rank the resulting FIMs using the D-optimality criterion. The configuration resulting in the FIM with the largest determinant is considered as the best among the candidates.

4.4 Vibrational testing

Vibrational testing was carried out to verify and validate the developed model calibration framework. The experimental techniques used in this work are illustrated in this chapter.

The test-rigs were selected to enable studies of two types of sources of nonlinearities. Linear models which provide good starting points for test-analysis correlations were developed. Such linear models perform well when the dynamics remain linear but fail when nonlinear effects are significant. The test structures used in this work are shown in Figure 10.

In the initial modal testing, see Figure 8, the linear behavior of the test structures were measured and analyzed using an LMS (Siemens) measurement system. Both hammer and shaker excitations were used to stimulate the linear parts of the test-rigs and the responses were measured using sensors such as

![Experimental setups of the test-rigs used (a) a cantilever beam connected with a membrane (b) a closer perspective of the cantilever and membrane intersection (c) a cantilever beam with a bump stop (d) a closer perspective of the gap mechanism.](image)

*Figure 10: Experimental setups of the test-rigs used (a) a cantilever beam connected with a membrane (b) a closer perspective of the cantilever and membrane intersection (c) a cantilever beam with a bump stop (d) a closer perspective of the gap mechanism.*
accelerometers, force transducers and impedance heads.

When nonlinearities are present, a pure mono-sinusoidal excitation is difficult to achieve due to the interaction between the shaker and the structure and a multi-sinusoidal excitation is commonly obtained. This effect is usually most significant around resonances where the systems display large deflections [41]. Therefore, a multi-sinusoidal excitation, see Eq.(12) was designed using MATLAB in combination with a National Instrument (PXIe-8135) data acquisition system interface. The system was excited with one fundamental frequency together with a few side harmonics at a time and the steady state responses were measured.

It is also necessary to have control of the applied forces, due to the force dropout in frequency ranges around resonance [70, 71]. An example of magnitudes of a measured multi-sinusoidal excitation is shown in Figure 11; the force drop-outs close to the resonance frequency can be clearly seen. This effect is important and may lead to considerable errors in the model calibration process. A way to solve this problem is to compensate the force drop-outs through iterative off-line tuning of the harmonic excitation signal used in the measurement. However, it can be very time consuming to control the amplitude of the harmonics of the excitation force to be perfectly constant for each frequency step. An alternative solution may be to control the amplitudes by a few iterations using an off-line feedback force controller, which can dramatically shorten the measurement time. The amplitudes are then kept constant for each frequency step within some tolerances. To keep the accuracy of the measurement data, the applied force amplitudes were also measured and used to calculate the analytical multi-harmonic nonlinear FRFs in this work. An overview of the measurement setup for nonlinear testing is shown in Figure 12. This measurement technique was applied in both Paper IV and Paper V.

Figure 11: Measured multi-harmonic force magnitudes as functions of frequency. The force magnitude of the fundamental frequency is plotted in blue, the force magnitude of the second harmonic is plotted in green and the force magnitude for the third harmonic is plotted in red.
4.5 Optimization routine

Optimization is an important task within the model calibration process; many strategies have been proposed during the last decades. It is common to use several starting points for optimization studies to increase the likelihood of finding the global optimum. With a limitation in computational time it may be wise to trade complete optimization runs in favor of examinations of more starting point candidates.

In this thesis, the model calibration was based on the Levenberg-Marquardt nonlinear programming [72] utilizing a selected parametric starting point. Starting seed candidates are found by the Latin hypercube sampling (LHS) method [73], intended to span the complete parametric space. The candidate resulting in the smallest calibration metric is then selected as the starting point for the iterative calibration.

The LHS method is employed to generate a reduced number of realizations which can still be used to study a given stochastic problem. The basic idea is to make the sampling point distribution in close resemblance to the probability density function. An example of a two dimensional LHS is shown in Figure 13. The example was generated, under the assumption that $p_1$ (parameter 1) and $p_2$ (parameter 2) are independent of each other with the following three steps:

1. Generate stratified, distributed one-dimensional samples of $p_1$
2. Generate stratified, distributed one-dimensional samples of $p_2$
3. Using the LHS strategy to combine the samples into two-dimensional pairs
Figure 13: An example of a two dimensional LHS, the red crosses represent the selected realizations.

4.6 Validation

When the numerical model is calibrated, the model was first validated using the part of the experimental data that were not included in the calibration metric. To obtain an estimate of the calibrated parameter statistics, a $k$-fold cross-validation was used. In the $k$-fold cross-validation, the available data set is partitioned into $k$ equally sized subsamples. Here, one subsample was used for validation while the remaining $k-1$ subsamples were used for calibration, which resulted in $k$ calibration runs. From these, $k$ sets of calibrated parameter settings that minimize the calibration deviation metric for the $k$ different partitions of the available data were obtained. These were then used for statistical evaluation of the mean and covariance of the parameter estimates. Examples of the $k$-fold cross validation results for three different noise levels using two objective functions are show in Figure 14.
Figure 14: An example of k-fold cross validation results for two different calibration metrics and three different noise levels. The parameter is normalized to its known parameter. The blue bars represent the normalized estimated parameter value for each calibration. The red lines indicate the standard deviation for each parameter (from Paper I).

5 Summary of the appended papers

Paper I: Informative data for model calibration of locally nonlinear structures based on multi-harmonic frequency responses

A model calibration framework was proposed to increase the reliability of FE-models representing mechanical systems with local nonlinearities. This was achieved by adding a set of candidate parameters to control the nonlinear effects to the baseline FE model. The model calibration method concerns pre-test planning, test design, optimization and cross validation.

The pre-test planning sets out from a parameter identifiability and data informativeness perspective. Parameter identifiability and data informativeness are coupled and they can be accessed using the Fisher information matrix. The pre-test planning gives a strategy for the test design and forms a calibration metric for the optimization. To obtain informative data, the excitation force is designed to be multi-sinusoidal and the resulting multi-harmonic nonlinear FRFs form the calibration metric. The optimization
is done using the gradient based Levenberg–Marquardt. To increase the possibility of finding the global minimum, starting seed candidates for calibration, which are intended to span the parametric space, are found by the Latin hypercube sampling method. The candidate that gives the smallest deviation to test data is selected as a starting point for the iterative search for a calibration solution. When the model is calibrated, a $k$-fold cross validation is applied to quantify the parameter uncertainties.

In this paper, synthetic test data from a model of a nonlinear benchmark structure are used for illustration. The steady-state responses at the side frequencies are shown to contain valuable information for the calibration process and can improve the accuracy of the parameters’ estimates. The calibration result shows good agreement with the true parameter setting and the $k$-fold cross validation result shows that the variances of the estimated parameters shrink when multi-harmonics nonlinear FRFs are included in the data used for calibration.

**Paper II:** *An efficient simulation method for large-scale systems with local nonlinearities*

In this paper, an efficient simulation method for large-scale systems with local nonlinearities is proposed. The method is formulated in a state-space form and the simulations are done in an MATLAB environment. The nonlinear system is divided into a linearized system and a nonlinear part which is considered as external nonlinear forces acting on the linear system; thus taking advantage in the computationally superiority in the locally nonlinear system description compared to a generally nonlinear counterpart. The exponential triangular-order hold integrator is used to obtain a discrete state-space form. To shorten the simulation time, auxiliary matrices, similarity transformation and compiled C-codes (mex) used for the time integration are studied. Comparisons of the efficiency and accuracy of the proposed method in relation to simulations using the ODE45 solver in MATLAB and MSC Nastran are demonstrated on numerical examples of different model sizes.

**Paper III:** *Bias errors of different simulation methods for linear and nonlinear systems*

Responses of mechanical systems are often studied using numerical time-domain methods. Discrete excitation forces require a transformation of the dynamic system from continuous time into discrete time. Such a transformation introduces errors. Algebraic expressions of the bias error caused by the zero-order, first-order and Lagrange second-order discretization methods were derived. Different simulation methods are also studied for numerical evaluation of the derived theoretical bias errors. The discretization
techniques are implemented for the Runge–Kutta, the Digital Filter and the State Space based methods. The study is carried out for both a linear and a nonlinear system; two numerical examples assist in validating the theory. Perfect matches between the numerically estimated bias errors and the theoretical ones are shown. The results also show that, for the nonlinear example, the fourth order Runge–Kutta method produces data that give less accuracy in the following system identification than the Digital Filter and the proposed State Space based method do.

**Paper IV:** Experimental validation of a nonlinear model calibration method based on multi-harmonic frequency responses

The model calibration method proposed in Paper I is improved and experimentally validated. The method is further developed in two aspects. Firstly, the calibration is separated into two parts; the calibration of linear parts and the calibration of nonlinear parts. Secondly, an experimental testing technique is developed to obtain accurate test data.

The experimental validation is conducted using a replica of the École Centrale de Lyon (ECL) nonlinear benchmark test setup. The ECL structure consists of a cantilever beam with a membrane at its free end. It shows complex nonlinear behavior due to the large displacement of the membrane. The linear parts of the ECL structure were calibrated first. Then, the calibration of nonlinear elements was made by minimizing the deviations between the measured multi-harmonic nonlinear FRFs and the analytical counterparts that are calculated using the HB method. After the nonlinear model is calibrated, a $k$-fold cross validation was applied to access the parameter uncertainty. The resulting calibrated model’s output corresponds well with the measured responses.

**Paper V:** Validation of a model calibration method through vibrational testing of a mechanical system with local clearance

The improved version of the model calibration method, which is shown in Paper IV, is further developed with an extended version of the pre-test planning method. The final version of the model calibration method was experimentally validated using a test structure with a clearance type nonlinearity. The test structure consists of a cantilever beam and a bump stop at its free end. The structure shows a rich nonlinear behavior due to the varying contact between the cantilever beam and the stop. From the pre-test planning, an optimal configuration for the data acquisition was determined. The multi-harmonic nonlinear FRFs of the structure under test were then generated by a multi-sinusoidal excitation. The model calibration was conducted by minimizing the differences between the experimental multi-
harmonic nonlinear FRFs and their analytical counterparts. The calibrated nonlinear model was then validated by a comparison between calculated and measured multi-harmonic nonlinear FRFs, which were not included in the calibration metric. To show the advantage of using the optimal configuration for data acquisition, the statistics of the parameter values estimated from the configuration were examined using a k-fold cross validation. The results confirm that the uncertainties of the estimated parameters are small when the optimal configuration is applied.

6 Conclusion and future work

In this thesis, the focus is on the development and validation of simulation and calibration techniques for mechanical systems with local nonlinearities. In **Paper I**, an FE-model calibration framework, which concerns pre-test planning, parametrization, simulation methods, experimental testing and optimization, was proposed for mechanical systems with structural nonlinearities.

A good model of the underlying linear system will increase the possibility of a successful prediction of the dynamic behavior of the nonlinear structure. A parameterized FE-model representing the underlying linear system was perfectly calibrated first to facilitate the calibration of the complete system. In **Paper IV** and **Paper V**, the linear parts of the models were calibrated using the responses of the structure measured with a low level excitation.

A set of parameters that controls the effect of nonlinearities was added to the calibrated linear model. The Cramer-Rao lower bound and the FIM were used to assess the data informativeness and parameter identifiability. The responses at the side harmonics were verified to contain valuable information for the calibration of nonlinear models and they can be used to improve the accuracy of the estimated parameters. Therefore, a multi-sinusoidal excitation was applied to obtain the multi-harmonic FRFs to be used in the calibration metric. To obtain accurate test data efficiently, the amplitudes of the harmonics in the multi-sinusoidal excitation were controlled using an off-line force feedback algorithm. Such an experimental technique was developed and utilized in **Paper IV** and **Paper V**.

In this work, the proposed model calibration method for nonlinear systems includes an optimization based on the Levenberg-Marquardt algorithm. To increase the possibility of finding the global minimum, starting seed candidates which span the complete parametric space were selected using the Latin hypercube sampling method. The candidate that gave the smallest calibration metric was used as the starting point for the iterative search in the optimization. A large number of repeated simulations of the nonlinear FRFs
were required within the optimization. An extended version of the HB method was developed to generate the multiharmonic nonlinear FRFs from simulation models in Paper I.

Fast and accurate time domain methods are necessary to fully understand and predict the dynamic behavior of nonlinear systems. Therefore, an efficient simulation method, for large-scale mechanical systems with local nonlinearities, based on a state-space form was developed in Paper II. To understand the accuracy of different simulation methods, the bias error for the proposed simulation method as well as other simulation methods were studied and compared in Paper III. It was shown that the implicit methods, such as the developed state-space based simulation method, produce more accurate simulation data than the explicit methods for nonlinear systems.

The proposed nonlinear model calibration framework was validated using a simulation model in Paper I, which was then experimentally validated through a test-rig which displays geometrical nonlinearity in Paper IV. The proposed nonlinear model calibration framework was further improved in Paper V through an extension of the pre-test planning approach. The improved version of the model calibration framework was validated using experimental data from a mechanical system with a clearance type of nonlinearity in Paper V.

A suggestion for future work is to validate the proposed model calibration method with other types of nonlinearities, such as dry friction and a combination of different types of nonlinearities. The localization and characterization of the nonlinearities were made by physical insight combined with trial and error in this work. Hence, another suggestion for the future work is to develop methods for localization and characterization of structural nonlinearities.
References


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