Fedorchuk's compacts in topology

Cardinal characteristics of Fedorchuk's compacts
Master Thesis

Abstract

Master’s thesis is devoted to the study of cardinal invariants in the F-compact spaces class. Here and throughout the paper, the concept “compact” would mean a compact Hausdorff space. In my thesis I have tried to present and explain all necessary concepts and statements necessary for the reader to get acquainted with F-compact spaces class. In order to understand the idea of F-compact spaces, it is necessary to understand what the inverse spectrum is from itself, it is necessary to know about the cardinality of sets and to understand that two infinite sets can have different cardinalities, know about closed and open sets, and much else that you will find in this paper.

In the thesis the analysis of the scientific literature sources is presented; the theorems about the relationship between the characteristics of cardinality invariants in the F-compact spaces class are investigated; the relationships between the properties of perfect normality and hereditary normality in the F-compact spaces class of countable spectral height are studied. In the process of the investigation some propositions were found, proved and filled in the missing fragments of evidence.

Conclusion: At present, the method of fully closed mappings (which is used in constructing of F-compact spaces) is the most productive method of constructing counterexamples in general topology. I believe, that this paper will be interesting to all who wants to go beyond the ordinary, habitual way of thinking, because only by studying topology we can speak clearly and precisely about things related to the idea of continuity and infinity!
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1 Introduction

I would like to start with the words of greatest mathematician Mr. David Hilbert, he is recognized as one of the most influential and universal mathematicians of the 19th and early 20th centuries. Hilbert often compared mathematics with a magical, enchanting garden. Many different entrances lead to this garden. One of them is the general topology.

General topology became a part of general mathematical language a long time ago. Topology as a science was emerged in the eighteenth century. August Ferdinand Möbius was one of the main contributors of the topological theory of manifolds. In 1865, Möbius presented an article in which he decomposed several orientations of surfaces in polygonal nets. His most famous example was a non-orientable surface, which is now called the Möbius strip.

But the Möbius strip is not only the exercise for the mind, it is quite practical used. The belt conveyor is made in the form of Möbius strip, which allows it to work longer, because the entire surface of the tape evenly wears out. Möbius strips are also used in recording systems for continuous film (in order to double the recording time), in dot matrix printers, the ribbon also had the form of a Möbius sheet to increase the expiration date.

General topology grew out of a number of areas, one of the most key concepts in topology are the detailed study of subsets of the real line (once known as the topology of point sets; this usage is now obsolete), metrizable spaces, compactness.

It is known that many facts of mathematical analysis are based on a lemma of Heine-Borel-Lebesgue.

Heine-Borel Theorem (modern): If a set \( S \) of real numbers is closed and bounded, then the set \( S \) is compact. That is, if a set \( S \) of real numbers is closed and bounded, then every open cover of the set \( S \) has a finite subcover.

In 1968 V.V. Fedorchuk laid the foundations of the method of constructing counterexamples to the theory of compact spaces, which later received the name of the method of fully closed mappings. In the Western literature Fedorchuk’s method is called the method of resolutions. Fedorchuk gave a survey of the results and ideas of this method.
The method proved to be so productive that there was a need to study the properties of a class of compact sets, based on this method. This class received the name **F-compact spaces**.

The idea of the **F-compact spaces** based on the concept of the **inverse spectrum**, which plays an important role in the modern topology. Using the inverse spectrum complex spaces are may be approximated by a simple spaces, and to design a complex spaces, which greatly facilitates the process of their research.

A compact $X$ is named the **F-compact spaces** if there exists a well-ordered continuous inverse spectrum of compacts $S = X_\alpha, \rho_\beta^\alpha : \alpha, \beta < \tau$, so that

1. $X_0$ is a point;
2. All proections $\rho_\alpha^{\alpha+1}$ are fully closed, $\alpha + 1 < \tau$;
3. Weight of prototypes $(\rho_\alpha^{\alpha+1})^{-1}$ is countable for all points $x \in X_\alpha$,
   \[ \alpha + 1 < \tau; \]
4. $\lim S = X$;

This spectrum begins with a point and it is continuous. The least possible length spectrum is called the **spectral height** ($sh(X)$) of $X$.

Basically, the **F-compact spaces** are the spaces that can be constructed by the method of resolutions.

The spectral height is an important characteristic of the **F-compact spaces** which their topological properties depend on.[2]
1.1 The set theory.

The purpose of this introduction is to familiarize the reader with the basic definitions of the set theory.

It is said that $x$ is an element of the set $A$, if $x$ belongs to the set $A$ and is denoted as $x \in A$. When the $x$ is not an element of the set $A$, i.e. $x$ not belongs to the set $A$ is denoted as $x \notin A$. Let us introduce with such concepts as union, intersection and difference of the sets.

The union of the sets $A$ and $B$, it is written as $A \cup B$, is the set of elements, each of them belongs either to $A$, or to $B$. So, $x \in A \cup B$, if and only if either $x \in A$, or $x \in B$.

\[ A \cup B = \{a, b, c, d, e, f, g\} \]

![Figure 1: Union of sets](image1)

**Example 1.1.** If $A = \{a, b, c, d\}$ and $B = \{b, c, e, f, g\}$, then $A \cup B = \{a, b, c, d, e, f, g\}$

The intersection of the sets $A$ and $B$, it is written as $A \cap B$, is the set of elements, each of them belongs both to $A$ and to $B$. So, $x \in A \cap B$, if and only if $x \in A$ and $x \in B$.

\[ A \cap B = \{b, c\} \]

![Figure 2: Intersection of sets](image2)

**Example 1.2.** If $A = \{a, b, c, d\}$ and $B = \{b, c, f, g\}$, then $A \cap B = \{b, c\}$
The difference of the sets $A$ and $B$, it is written as $A \setminus B$, is the set of elements, which belong to the set $A$, but don’t belong to the set $B$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{difference_sets.png}
\caption{Difference of sets}
\end{figure}

**Example 1.3.** If $A = \{a, b, c, d\}$ and $B = \{b, c, e, f, g\}$, then $A \setminus B = \{a, d\}$

The designation $A \subseteq B$ or $B \supseteq A$ means that each element of the set $A$ is the element of the set $B$. This dependence between sets is called the inclusion.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{inclusion.png}
\caption{Inclusion}
\end{figure}

It is said that $A$ is the subset of the set $B$ when $A \subseteq B$.

The set consisting of a finite number of elements $x_1, x_2, x_3 \ldots x_k$ is denoted as $\{x_1, \ldots x_k\}$.

If $f$ is the function which actions from the set $X$ to the set $Y$ and $x \in X$, then there exists a unique $y$ which satisfies to the conditions $(x, y) \in f$ and $y$ is denoted as $f(x)$ and is called the value of function $f$ in the point $x$.

*The image* of the set $A \subseteq X$ with the mapping $f$ is a set

$$f(A) = \{y \in Y : y = f(x) : \text{for some } x \in X\},$$

*The prototype* of the set $B \subseteq Y$ with the mapping $f$ is a set

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

Sinyakova Evgenia
There exists injective, surjective and bijective mappings.

**Definition 1.** Let $X$ and $Y$ be sets. Suppose that $f : X \to Y$ is a function. Then

- It is said that $f$ is the injective mapping (the injection) is called the one-to-one mapping as well, if for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$
that is, $f(x_1) = f(x_2)$ implies $x_1 = x_2$. 7

Figure 7: Injective mapping

- It is said that $f$ is the surjective mapping (surjection) is called the onto mapping as well, if and only for every $y \in Y$ there exists at least one $x \in X$ (depending on the choice of $y$), so that $f(x) = y$. 8

Figure 8: Surjective mapping

- It is said that $f$ is the bijective mapping (bijection), if for every $y \in Y$ there exists exactly one $x \in X$ so that $f(x) = y$. In other words, $f$ is the bijective mapping if $f$ is both surjective and injective mappings. 9
**Example 1.4.** Consider the function $f: \{1, 2, 3, 4\} \to \{3, 5, 9, 11\}$ defined by $f(1) = 11, f(2) = 5, f(3) = 9, f(4) = 11$. Then $f$ is not injection, since $f(1) = f(4)$ is performed, but $1 \neq 4$. Therefore, the function $f$ is not bijection.

**Example 1.5.** Consider the function $f: \mathbb{N} \to \mathbb{N} \cup \{0\}$ defined by $f(n) = n - 1$.
We claim that $f$ is the bijection. To see that $f$ is the injection we must show that $n, m \in \mathbb{N}$ are such that $f(n) = f(m)$ is performed then, $n = m$ is fulfilled.

Suppose that $n, m \in \mathbb{N}$ are such that $f(n) = f(m)$. Then $n - 1 = f(n) = f(m) = m - 1$. Therefore, $n - 1 = m - 1$ hence, $n = m$. That is $f$ is the injection.

To see that $f$ is the surjection we must show that for every $k \in \mathbb{N} \cup \{0\}$ exist $n \in \mathbb{N}$ (depending on $k$) such that $f(n) = k$. Suppose that $k \in \mathbb{N} \cup \{0\}$. Then $k + 1 \in \mathbb{N}$. Moreover $f(k + 1) = (k + 1) - 1 = k$. Hence $f$ is the surjection.
Hence $f$ is the bijection.

The bijective mappings have some very nice properties.

**Lemma 1.** Let $X$ and $Y$ be sets and suppose $f: X \to Y$ is the bijection. Then for each $y \in Y$ there exists a unique element $x_y \in X$ such that $f(x_y) = y$ (that is, if $f$ is the bijection and $y \in Y$ there exists exactly one element of $X$ that $f$ maps to $y$).

**Proof.** Let $X$ and $Y$ be sets, and suppose $f: X \to Y$ is the bijection.
Fix $y \in Y$. There are two parts to the theorem: there exists an $x \in X$ such that $f(x) = y$, and there exists only one such element.
Since $y \in Y$ and $f$ is the surjection, there exists the element $x_y \in X$ such that $y = f(x_y)$.

Suppose that $x \in X$ such that $f(x) = y$. Then $f(x) = f(x_y)$, since $f$ is the injection, $f(x) = f(x_y)$ implies that $x = x_y$. Hence $x_y$ is the unique element of $X$ that $f$ maps to $y$.

**Definition 2.** Let $X$ and $Y$ be sets and suppose $f: X \to Y$ is the bijection. Then there exists the unique function $g: Y \to X$ such that

1. $g(f(x)) = x$ for all $x \in X$.
2. \( f(g(y)) = y \) for all \( y \in Y \).

The function \( g \) is called the inverse of \( f \) and is denoted as \( f^{-1} \).

**Lemma 2.** Let \( X \) and \( Y \) be sets and suppose that \( f: X \to Y \) and \( g: Y \to X \) are functions such that

1. \( g(f(x)) = x \) for all \( x \in X \).
2. \( f(g(y)) = y \) for all \( y \in Y \).

Then \( f \) and \( g \) are bijections.

**Theorem 3.** Let \( X \) and \( Y \) be sets and suppose \( f: X \to Y \) is the bijection. Then \( f^{-1}: Y \to X \) is the bijection as well. Moreover \( (f^{-1})^{-1} = f \) (that is, the inverse of \( f^{-1} \) is \( f \)).

**Definition 3.** The sets, elements of which are the sets are called the families or classes of sets.

**Definition 4.** The indexed family \( \{A_s\}_{s \in S} \), strictly speaking, a function that bijectively assigns to each element \( s \in S \) the set \( A_s \) and if not indexed family is just named the set of sets.

The union and intersection of the family of sets \( \{A_s\}_{s \in S} \) is denoted as \( \bigcup_{s \in S} A_s \) and \( \bigcap_{s \in S} A_s \) respectively.\[4\]
1.2 The cardinality of the sets

At first this looks like a very simple concept. To find the cardinality of the set, just count its elements. If \( X = \{a, b, c, d, e, f\} \), then the cardinality of \( X \) is equal 6, is denoted as \( |X| = 6 \).

Actually, the idea of cardinality becomes quite subtle when the sets are infinite. The main idea of this chapter is to explain how there are numerous different kinds of infinity, and some infinities are bigger than others.

What does it mean for two sets to have the same cardinality? It is said that \( |X| = |Y| \) if \( X \) and \( Y \) have the same number of elements. This is applies for a finite sets. We need a new approach that applies to both finite and infinite sets.

**Definition 5.** Let \( X \) and \( Y \) be the sets and have the same cardinality and are called equicardinal, if there exists the bijection \( f \) of the set \( X \) to the set \( Y \). If there is no such bijective function, then the sets have unequal cardinalities.

Let us consider the set of natural numbers \( \mathbb{N} \) and the set of integer numbers \( \mathbb{Z} \). These sets are equicardinal since there exists the bijection between them. In a certain sense we can count the elements of \( \mathbb{N} \); we can count its elements as 1, 2, 3, 4, ..., but we have to continue this process forever to count the whole set. Thus \( \mathbb{N} \) is called a *countably infinite set*, and the same term is used for any set whose cardinality equals to the cardinality of natural numbers \( \mathbb{N} \).

**Definition 6.** Let \( X \) be the set. Then \( X \) is countably infinite if \( |\mathbb{N}| = |X| \) that is, if there exists the bijection \( \mathbb{N} \to X \). The set \( X \) is uncountably if \( X \) is infinite and \( |\mathbb{N}| \neq |X| \), that is, if \( X \) is infinite and there is no bijection \( \mathbb{N} \to X \).

The cardinality of the natural numbers is denoted as \( \omega_0 \). Thus any countably infinite set has cardinality \( \omega_0 \).

The elements of any countably infinite set \( X \) can be written in the infinitely long list \( a_1, a_2, a_3, \ldots \), that begins with some element \( a_1 \in X \) and includes every element of \( X \).

The set \( X \) is countably infinite if and only if its elements can be arranged in the infinite list \( a_1, a_2, a_3, \ldots \).

Note that the set \( \mathbb{R} \) is not countably infinite as meaning that it is impossible to write out all the elements of \( \mathbb{R} \) in the infinite list.

Let us consider the definition of comparing cardinalities.

**Definition 7.** Let \( X \) and \( Y \) be the sets.

- \( |X| = |Y| \) means there exists the bijection \( X \to Y \);

- \( |X| < |Y| \) means there exists the injection \( X \to Y \), but no surjection \( X \to Y \);

- \( |X| \leq |Y| \) means \( |X| < |Y| \) or \( |X| = |Y| \).

For example, let us consider \( \mathbb{N} \) and \( \mathbb{R} \). The function \( f: \mathbb{N} \to \mathbb{R} \) defined as \( f(n) = n \) is clearly injective, but it is not surjective because given the element \( \frac{1}{2} \in \mathbb{R} \), we have \( f(n) \neq \frac{1}{2} \) for every \( n \in \mathbb{N} \). Hence, \( |\mathbb{N}| < |\mathbb{R}| \).

The cardinality of the family of all subsets of the set \( X \) is named the power
set) satisfying |X| = m, for every cardinal number m, is equal to 2^m; this cardinality is denoted as \(\mathcal{P}(X)\). Thus the \(\mathcal{P}(\mathbb{N}) = 2^{\omega}\).

Since the power set of the set, finite or infinite, always has more members than set itself, it is always possible to find the set with larger cardinality by simply taking the power set of a given set. Thus, one can obtain a sequence of sets having larger and larger infinities.

\[\omega_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \ldots\]

**Theorem 4.** The cardinality of the power set of natural numbers is equal to the cardinality of the real numbers. That is, \(|\mathcal{P}(\mathbb{N})| = 2^{\omega_0} = c\).

This cardinality \(c\) have received the name **continuum**.

If now call \(\omega_1\) the next larger infinite number after \(\omega_0\), the interesting question arises. Is there the cardinality larger than \(\omega_0\) but less than \(c\)? Stated another way, is \(c = \omega_1\)? Cantor believed the assertion was true but was never able to prove or disprove it. Cantor called the hypothesis \(c = \omega_1\) the **continuum hypothesis**.

### 1.3 The order relation on the set.

**Definition 8.** The set \(X\) consisting of any desired items is called a partially ordered if a relation of partial order is defined on it, i.e., for some pairs \(x, x'\) of its various elements it is known that one of them precedes the other, for example, the element \(x\) precedes the element \(x'\) which is written as follows:

\[x < x' \text{ or } x' > x\]

(1)

It is assumed that the order relation satisfies the condition of transitivity:

If \(x < x'\) and \(x' < x''\), then \(x < x''\).

**Definition 9.** Let \(X\) be the set and \(<\) - some order relation on \(X\). The relation "<" is called the linear order relation on \(X\) if it satisfies the following properties:

- If \(x < y\) and \(y < z\), then \(x < z\);
- If \(x < y\), then the order \(y < x\) not fulfilled;
- If \(x \neq y\), then \(x < y\) or \(y < x\).

The set \(X\) with the linear order relation on \(X\) is called a linearly ordered set.

It is said that the element \(x_0\) of the linearly ordered set \(X\) is called the **smallest element** of \(X\) if \(x_0 < x\) for any \(x \in X \setminus \{x_0\}\).

Similarly we define the **largest element**.

Since every subset of the linearly ordered set is the linearly ordered set, then the smallest and largest elements of the subset of the linearly ordered set are defined correctly.
Definition 10. A linear order $<$ on the set $X$ is called a well-ordered, and the set $X$ endowed with the order $<$ is called the well-ordered set if the order $<$ has the property that

- every non-empty subset of $X$ has the smallest element. For example, the set of natural numbers $N$ is the well-ordered set.[1]

Every set of the cardinal numbers is the well-ordered set.[1]

Let the set $X$ be linear ordered set and endowed with the linear order $<$ and let the set $Y$ be linearly ordered set and endowed with the linear order $'$. It is said that the mapping $f$ from the set $X$ to the set $Y$ save the order if $f(x) < f(y)$ for every couple $x, y \in X$ such that $x < y$.

Definition 11. If there exists the order-preserving mapping of the linearly ordered set $X$ to the linearly ordered set $Y$, then the sets $X$ and $Y$ are called similar.

Definition 12. The partially ordered set $A$, i.e. the set endowed with the partial order $(A, \leq)$ is called a directed set if for any two elements $a, b$ from the set $A$ there exists the element $c$ from $A$ such that $c \geq a, c \geq b$.

Every well-ordered set $X$ is assigned an order number or ordinal; this one is called an order type of the well-ordered set. For example: the order type of the null set is $0$, the order type of the set which consists of 1 element is 1, the order type of natural numbers is $\omega$.

The order types of the well-ordered sets $X$ and $Y$ are the same in that case, when $X$ and $Y$ are similar.

Every order-preserving mapping is injective therefore, if $X$ and $Y$ are similar, then $| X | = | Y |$. The cardinal is called the cardinality of ordinal $\alpha$ and it is denoted as $| \alpha |$. If $| \alpha | \leq \omega_0$, then the ordinal $\alpha$ is called countable.

Let $\alpha$ and $\beta$ be the ordinals which are the order types of $X$ and $Y$ respectively. We will say that $\alpha$ is less than $\beta$ or $\beta$ is larger than $\alpha$ and write $\alpha < \beta$ or $\beta > \alpha$ if there exists a such point $y_0 \in Y$, that the sets $X$ and $\{ y \in Y : y < y_0 \}$ are similar. Every set of ordinals is the well-ordered set endowed with the relation $<$.

Definition 13. The ordinal $\lambda$ is called a limit ordinal, if there is no the order number, which immediately precedes the $\lambda$, so if for any $\xi < \lambda$ there exist the ordinal $\alpha$, such that $\xi < \alpha < \lambda$.

Definition 14. The infinite ordinal $\alpha$ (i.e. the order type of some infinite well-ordered set) is called an initial ordinal if for any ordinal $\beta < \alpha$ the inequality $| \beta | < | \alpha |$ holds.

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1 The cardinal numbers, or cardinals for short, are generalization of the numbers used to measure the cardinality of sets.
2 Ordinals - measure the "length" or "shape" of a set.
Cardinals - measure the "size" of a set.
All natural numbers and ω are the initial ordinals. At the same time the ordinals
\[ \omega + 1, \omega + 2, \ldots, \omega + n, \ldots, \omega + \omega, \ldots, \omega + \omega + \omega \ldots \]
have the countable cardinality and aren’t the initials ordinals. The order relation can be established on any set in such a way that the set will be the well ordered set. Thereby, the cardinality of any set is equal to the cardinality of the some initial ordinal. For this reason the cardinalities (in other words, the cardinal numbers) are identified with the initials ordinals and for example, \(| n | = n, | \omega | = \omega\), note that \(| \omega + 1 | = \omega\). The order type of the set \(\{ \alpha : | \alpha | \leq \omega\}\) which consists of ordinals, the ordinal cardinality is at most countable is denoted as \(\omega_1\).

**Definition 15.** Let \(X\) be the set, and \(\leq\) be the order on this set.
It is said that \(\leq\) is the order on \(X\) if the following properties are fulfilled:

- If \(x \leq y\) and \(y \leq z\), then \(x \leq z\);
- For any \(x \in X\), we have \(x \leq x\);
- If \(x \leq y\) and \(y \leq x\), then \(x = y\).

The set \(X\) with the order \(\leq\) is called an ordered set.
Two elements \(x, y\) of the ordered set can be incomparable; it can happen when both inequalities have no place neither \(x \leq y\) nor \(y \leq x\). Any family of sets are ordered by the way inclusion \(\subset\).
If \(X\) is the linearly ordered set endowed with the relation \(<\), then it is assumed that for any \(x, y \in X, x \leq y\), if and only if,

when \(x < y\) or \(x = y\), we get some order on \(X\).

Hence, we can consider every linearly ordered set as some ordered set.

If for every couple \(x, y\) which are the elements of subset \(A\) of the ordered set \(X\), one of the relations \(x \leq y\) or \(y \leq x\) take place, then assume, that

\[ x < y\] if and only if \(x \leq y\) or \(x \neq y\)

we get some linearly order on \(A\).
It is said that \(A\) is the linearly ordered subset of set \(X\), which ordered by the relation \(\leq\).

**Definition 16.** Let \(X\) be the arbitrary set and \(\leq\) be the order on \(X\). We will say that \(\leq\) directs the set \(X\) or \(X\) is being directed by an order \(\leq\) if the order \(\leq\) has the following properties:

- If \(x \leq y\) and \(y \leq z\), then \(x \leq z\);
- For any \(x \in X\), we have \(x \leq x\);
- For any \(x, y \in X\), there exists the element \(z \in X\), such that \(x \leq z\), \(y \leq z\)

The subset \(A\) of the set \(X\) is directed by the relation \(\leq\) is cofinal in \(X\) if for every \(x \in X, a \in A\) the inequality \(x \leq a\) is performed.
The cofinal subsets of linearly ordered sets and ordered sets are defined similarly.
THE AXIOM OF CHOICE

Suppose we are given the set $X$ and the property $P$ is pertaining to subsets of $X$; it is said that $P$ is the property of finite character if the empty set has this property $P$ and the set $A \subset X$ has the property $P$ if and only if all finite subsets of $A$ have this property.

**Lemma 5.** Suppose we are given the set $X$ and the property $P$ of subsets of $X$. If $P$ is the property of finite character, then every set $A \subset X$ which has the property $P$ is contained in the set $B \subset X$ which has the property $P$ and the set $A$ is maximal subset in the family of all subsets of $X$ that have property $P$ ordered by the way $\subset$. 
2 Topological spaces.

2.1 Metrizable spaces.

**Definition 17.** Let $X$ be the non-empty set.
It is said that the metric is defined on the set $X$ if the real number $\rho(x, y)$ is corresponded to every couple $x, y \in X$ such that the following axioms are fulfilled:

1. $\rho(x, y) \geq 0$ and $\rho(x, y) = 0 \Leftrightarrow x = y$;
2. $\rho(x, y) = \rho(y, x)$;
3. $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$;

$\rho$ is called the distance.
The couple $(X, \rho)$ is called the metric space.

**CONSIDER THE EXAMPLES OF THE METRIC SPACES**

**Example 2.1.** Let us consider the Affine Euclidean space $(M, V)$.
Where $M$ is the set of point, $V$ is the set of vectors.
Let $A, B$ be the points and belong to $M$.
The distance from the point $A$ to the point $B$ is defined as the length of the vector $\overrightarrow{AB}$, so $\rho(A, B) = |\overrightarrow{AB}|$.

**Example 2.2.** Let us consider the Affine Euclidean space $(\mathbb{R}^n, \mathbb{R}^n)$ and $\rho(x, y)$.
In this space $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$.
Let us introduce the distance $\rho(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$.
That is $(\mathbb{R}^n, \rho)$ is the metric space.

**Example 2.3.** Let us consider the set $X$ and the metric $\rho_d$ is defined on $X$, $\rho_d$ is the discrete metric.

\[ \rho_d(x, y) = \begin{cases} 1, x \neq y \\ 0, x = y \end{cases} \]
It is true for any non-empty $X$.

**Definition 18.** Let $\varepsilon$ be larger then 0.
The (open) $\varepsilon$ - neighborhood of the point $x \in X$ is called the following set:

\[ O(x, \varepsilon) = \{ y : \rho(x, y) < \varepsilon \} \]

The point $x \in O(x, \varepsilon), \rho(x, x) = 0 < \varepsilon$.

**Definition 19.** The set $X$ with the metric $\rho$, $(X, \rho)$ is the metric space.
Any set $U \subset X$ is called the open set in the topology induced by the metric, if for any $x \in U \exists \varepsilon > 0 : O(x, \varepsilon) \subset U$.

The family of all open sets in the topology induced by the metric is denoted as $\tau_\rho$ and $\tau_\rho$ is called the topology induced by metric.

**The properties of open sets in the topology induced by the metric.**

1. $\varnothing, X \in \tau_\rho$.
   (The empty set and $X$ are open in the topology induced by the metric.)
2. If \( U_\alpha \in \tau_\rho, \alpha \in A \), \( A \) is the set of indexes, then \( \bigcup_{\alpha \in A} U_\alpha \in \tau_\rho \).
(The union of open sets is open in any quantity.)

3. \( U_1, \ldots U_n \in \tau_\rho \) then \( \bigcap_{i=1}^n U_i \in \tau_\rho \).
(The intersection of finite number of open sets is open.)

**Proposition 6.** \( \varepsilon \) - neighborhoods \( O(x, \varepsilon) \) are open in any points.

*Proof.* Let \( y \) be the point and belongs to \( O(x, \varepsilon) \) and let us put in \( \varepsilon' = \varepsilon - \rho(x, y) \).
\((\rho(x, y) < \varepsilon, \text{ by definition \[18\] and hence } \varepsilon' > 0)\).

It is necessary to prove, that \( O(y, \varepsilon') \subseteq O(x, \varepsilon) = U \).

Let \( z \) be the point and belong to \( O(y, \varepsilon') \), hence \( \rho(z, y) < \varepsilon' \), hence it must be \( z \in O(x, \varepsilon) \), that is \( \rho(x, z) < \varepsilon \).

And by the third property of \( \rho \): \( \rho(x, z) \leq \rho(x, y) + \rho(y, z) < \rho(x, y) + \varepsilon' = \varepsilon \). \( \square \)

**Proposition 7.** The set is open in the topology induced by the metric if and only if the set can be presented as the union of \( \varepsilon \) - neighborhoods.
\((U \in \tau_\rho \iff U = \bigcup_{\alpha \in A} O(x_\alpha, \varepsilon_\alpha))\).
2.2 Topological spaces. Open and closed sets and their properties.

Definition 20. Let $X$ be the set. The family $\tau$ consisting of subsets of the set $X$ is called the topology on $X$, if the following axioms are fulfilled:

1. $\emptyset, X \in \tau$;
2. If $U_\alpha \in \tau, \alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \tau$;
3. $U_1, \ldots, U_n \in \tau$ then $\bigcap_{i=1}^n U_i \in \tau$.

The set $X$ with the topology $\tau$ on it $(X, \tau)$ is called the topological space.

Definition 21. The elements of family $\tau$ $(U \in \tau)$ are called the open sets of the topological space $(X, \tau)$.

Definition 22. The subset $F$ of $X$ ($F \subseteq X$) is called a closed set, if the complement $(X \setminus F)$ is open. So, we have the topological space $(X, \tau)$ and the closed subset $F$ belongs to $X$, then $(X \setminus F)$ is open.

**THE PROPERTIES OF CLOSED SETS**

1. There are open and closed sets at the same time.
   For example, the set $X$ and the empty set $\emptyset$ are both open and closed sets.
2. The intersection of the closed sets $\bigcap_{\alpha \in A} F_\alpha$ is the closed set.
3. The union of the closed sets $\bigcup_{i=1}^n F_i$ is the closed set.

Definition 23. Let $(X, \tau)$ be the topological space and the set $A$ belongs to $X$. The topology of subspaces or induced topology on $A$ is called the family $\tau_A = \{U \cap A : U \in \tau\}$.

Definition 24. Let $(X, \tau)$ be the topological space.

The neighborhood of the point $x \in X$ in the topological space is any open set containing the point $x$.

Let us note the neighborhood of the point $x$ in the topological space is denoted as $O_x \in \tau$.

The concept "\( \varepsilon \)-neighborhood " arises in the metric space; there is no distance in the topology.

Definition 25. Let $A$ be the subset of $X$. (hereinafter in this chapter)

The point $x$ in $X$ is a "contact point" of $A$ if every neighborhood of $x$ contains at least one point of $A$.

Definition 26. The set of all contact points of $A$ is called the "closure" of $A$.

**THE PROPERTIES OF CLOSURE**

1. The set $A$ always "lying" in it's closure. $A \subseteq [A]$.
2. If $A \subseteq B$, then $[A] \subseteq [B]$.
3. The closure of $A$ is always closed.
4. The closure of $A$ is equally the set $A$ if and only if $A$ is closed.

5. The closure of the closure of the set $A$ is just the closure of $A$. $\mathcal{A} = [A]$

6. The closure of the set $A$ is the smallest closed set containing $A$.

7. The closure of the set $A$ is the intersection of all closed sets containing $A$. $A \subset \bigcap_{\beta \in B} F_{\beta}$, where $F_{\beta}$ are closed sets and $[A] = \bigcap_{\beta \in B} F_{\beta}$.

**Definition 27.** The point $x$ is called the 'limit point' of $A$ if every neighborhood of $x$ contained infinitely many points of $A$.

**Definition 28.** The point $x$ is called the "interior point" of the set $A$ if there exist the neighborhood Ox of the point $x$, which contained in $A$.

**Definition 29.** The interior of the set $A$ is the set of all interior points of $A$. The interior of $A$ is denoted as $\text{int}(A)$ or $\text{Int}(A)$.

The interior of the set has the following properties:

1. $\text{Int}(A)$ is the open subset of $A$.

2. $\text{Int}(A)$ is the union of all open sets contained in $A$.

3. $\text{Int}(A)$ is the largest open set contained in $A$.

4. $\text{Int}(A) = A$ if and only if the set $A$ is open.

5. $\text{Int}(\text{Int}(A)) = \text{Int}(A)$.

6. If $A$ is the subset of $T$, then $\text{Int}(A)$ is the subset of $\text{Int}(T)$.

7. If $S$ is the open set, then $S$ is the subset of $A$ if and only if $S$ is the subset of $\text{Int}(A)$.

**Definition 30.** The subset $A \subset X$ is called the everywhere dense if the closure of $A$ equal to $X$, so $[A] = X$.

The nowhere dense set $A$ is the set whose closure has empty interior, $\text{Int}([A]) = \emptyset$.

**Definition 31.** The density $d(X)$ of the topological space $X$ is the least cardinality of the everywhere dense subset of $X$.

A vital role in the mathematics plays such the concept as the continuity of function.

A basic concept have presented in mathematical analysis:

Let $f$ be the real-valued function defined on the subset $E$ of the real numbers $\mathbb{R}$ that is $f: E \rightarrow \mathbb{R}$. It is said that $f$ is the continuous at the point $x_0 \in E$ if for any $\epsilon > 0$ there exist $\delta > 0$, so that for all $x \in E$ with $|x - x_0| < \delta$ the inequality $|f(x) - f(x_0)| < \epsilon$ is valid. If one denotes by

$$U(x_0, \delta) = (x_0 - \delta, x_0 + \delta)$$

and

$$V(f(x_0), \epsilon) = (f(x_0 - \epsilon, f(x_0 + \epsilon))$$

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the $\epsilon$- and $\delta$-neighborhoods of $x_0$ and $f(x_0)$ respectively, then the definition above can be rephrased as follows: $f$ is called continuous at the point $x_0 \in E$ if for each $\epsilon$-neighborhood $V = V(f(x_0), \epsilon)$ of $f(x_0)$ there exists the $\delta$-neighborhood $U = U(x_0, \delta)$ of $x_0$, so that $f(U \cap E) \subset V$.

We are interested in the definition of the continuity of the function in the topology.

**Definition 32.** Let $X, Y$ be topological spaces. The function $f : X \to Y$ is continuous if and only if for every point $x_0 \in X$ and for every neighborhood $V = V(f(x_0))$ of its image $f(x_0)$ there exists the neighborhood $U = U(x_0)$ of $x_0$ such that $f(U) \subset V$.

There are many equivalent definitions of continuity. Thus for the continuity of the function $f : X \to Y$ it is necessary and sufficient that any one of the following conditions hold:

- The inverse image $f^{-1}(C)$ of every open set $C$ in $Y$ is open in $X$;
- The inverse image $f^{-1}(F)$ of every closed set $F$ in $Y$ is closed in $X$;
- The image of the closure is contained in the closure of the image.
2.3 Base, character and weight of topological spaces.

Let \((X, \tau)\) be the topological space.

A family \(\sigma \subset \tau\) is called a base of topology \(\tau\) if every open non-empty subset of \(\tau\) can be represented as the union of elements of \(\sigma\).

\[
U = \bigcup_{\alpha \in A} V_{\alpha}, V_{\alpha} \in \sigma, U \in \tau
\]

The family \(\sigma\) is the base of topology \(\tau\) and elements of \(\tau\) is the union of elements of \(\sigma\), if the following properties are fulfilled:

- For every \(x \in X\) there exists \(U \in \tau\), so that \(x \in U\).
- For any \(U_1, U_2 \in \tau\) and every point \(x \in U_1 \cap U_2\) there exists \(U \in \tau\), so that \(x \in U \subset U_1 \cap U_2\).

If for some \(x \in X\) and the open set \(U \subset X\) we have \(x \in U\), we say that \(U\) is the neighborhood of \(x\).

The family \(\sigma(x)\) of neighborhoods of \(x\) is called the base for topological space \((X, \tau)\) at the point of \(x\) if for any neighborhood \(V\) of \(x\) there exists \(U \in \tau(x)\), so that \(x \in U \subset V\).

**Definition 33.** The weight of the topological space \((X, \tau)\) is called the smallest cardinality of the base of topology \(\tau\) and is denoted as \(w(X, \tau)\).

**Definition 34.** The family of neighborhood of the point \(x\) is called the neighborhood base or local base for the point \(x\) if for any neighborhood \(U\) we can find the neighborhood \(V\) in the neighborhood basis that is contained in \(U\).

\[
\forall U \in \tau(x) \exists V \in \sigma(x) : V \subset U.
\]

**Definition 35.** The character of the topological space \(X\) at a point \(x\) is called the smallest cardinality of local base of \(x\) and is denoted as \(X(x, (X, \tau))\).

**Definition 36.** The character of the topological space \(X\) is called a supremum of all characters at the points of \(X\) and is denoted as \(\mathcal{X}(X, \tau)\).

\[
\mathcal{X}(X, \tau) = \sup \mathcal{X}(x, (X, \tau), x \in X).
\]

When \(\mathcal{X}(X, \tau) = \omega_0\) the space \(X\) is said to be first countable. When \(w(X, \tau) = \omega_0\) the space \(X\) is said to be second countable.

Let \(\tau_1\) and \(\tau_2\) be the topologies on the space \(X\), then we say that topology \(\tau_1\) is finer than the topology \(\tau_2\), or that topology \(\tau_2\) is coarser than \(\tau_1\).

---

3We remember that the neighborhood of the point \(x\) in topological space \((X, \tau)\) is any open set containing the point \(x\).

4Here, we write \(\sigma(x), \tau(x)\) in order to show the dependence of \(x\).
3 Compactness.

3.1 Compact spaces and Subspaces

Let $X$ be the topological space. The system $M = \{M_\alpha, \alpha \in A\}$, $M_\alpha \subset X$ is called a **cover** of $X$, if $X$ can be represented as the union of $M_\alpha$.

$$X = \cup M_\alpha.$$ 

The cover is called an **open cover** (closed cover), if all subsets $M_\alpha$ are open (closed) sets. The subsystem of the system $M$ is called a **subcover** of cover $M$, if $X$ can be represented as the union of subcover’s elements or another words when the subcover is the cover of $X$.

**Definition 37.** The topological space $X$ is called the compact space if and only if every open cover of $X$ has the finite subcover.

**Definition 38.** The subspace $A$ of $X$ is called the compact space if and only if every open cover of $A$ has the finite subcover.[6]

**Example 3.1.** Examples of the compact spaces and non compact spaces.

1. Any space consisting of the finite number of points is the compact space.
2. The real line $\mathbb{R}$ with the finite complement topology is the compact spaces.
3. The infinite set $X$ with the discrete topology is non compact space.
4. The open interval $(0; 1)$ is non compact space. $A = \{(1/n, 1) | n = 2, \ldots \infty\}$ is the open cover of $(0, 1)$. However, there is no the finite subcover of these sets which will be cover $(0; 1)$.

**Theorem 8.** The interval $[0, 1]$ is the compact space.

**Proof.** Let $O$ be the open cover. Assume that $[0; 1]$ is non compact space. Then either $[0, 1/2]$ or $[1/2, 1]$ is not covered by finite number of elements of $O$. Let $[a_1, b_1]$ be the half that is not covered by the finite number of elements of $O$.

Let us apply the same reasoning to the interval $[a_1, b_1]$.

One of the halves, which we will call $[a_2, b_2]$ is not finitely coverable by $O$ and has length $1/4$.

We can continue this reasoning inductively to create a nested sequence of closed intervals $\{a_n, b_n\}_{n=1}^{\infty}$, none of which is finitely coverable by $O$. Also, by construction, we have that,

$$b_n - a_n = \frac{1}{2^n}$$

so the diameters of these intervals goes to zero.

By the Cantor Nested Intervals Theorem[5] we know that there exists precisely one point in the intersection of all of these intervals; let $p \in [a_n, b_n]$ for

---

5Cantor Nested Intervals Theorem. If $\{a_{n}, b_{n}\}_{n=1}^{\infty}$ is the nested sequence of closed and bounded intervals, then $\cap_{n=1}^{\infty} [a_{n}, b_{n}] \neq \emptyset$. If , in addition, the diameters of intervals converge to zero, then the intersection consist of precisely one point.
Since \( p \in [0, 1] \) there exists the open interval \( C \subset O \) with \( p \in C \). Thus, there exists the positive number \( \varepsilon > 0 \), so that \( (p - \varepsilon, p + \varepsilon) \subset C \). Let \( N \) be the positive integer, so that \( 1/2^N < \varepsilon \). Then, since \( p \in [a_N, b_n] \) it follows that

\[
[a_n, b_n] \subset (p - \varepsilon, p + \varepsilon) \subset C.
\]

This contradicts the fact that \([a_N, b_N]\) is not finitely coverable by \( O \) since we just covered it with one set from \( O \). This contradiction shows that \([0, 1]\) is finitely coverable by \( O \) and is the compact space.

Compactness is defined in terms of open sets. The duality between open and closed sets and if \( K_\alpha = X \setminus C_\alpha \),

\[
X \setminus (\bigcap_{\alpha \in I} K_\alpha) = \bigcup_{\alpha \in I} C_\alpha
\]

leads us to believe that there exists a characterization of compactness by closed sets.

\[\square\]

**Theorem 9.** Each closed subset of the compact space is the compact space.

**Theorem 10.** Each compact subset of the Hausdorff space is closed.

**Proof.** Let \( A \) be the compact subset of the Hausdorff space \( X \).

In order to show that \( A \) is closed subset, we will show that its complement \( X \setminus A \) is open. Let \( x \) be the point and belongs to \( X \setminus A \).

Then for each \( y \in A \) there exist disjoint sets \( U_y \) and \( V_y \) with \( x \in V_y \) and \( y \in U_y \).

The collection of open sets \( \{U_y | y \in A\} \) forms the open cover of \( A \).

Since \( A \) is the compact space, this open cover has the finite subcover \( \{U_y_i | i = 1, \ldots, n\} \).

Let

\[
U = \bigcup_{i=1}^{n} U_{y_i}, \quad V = \bigcap_{i=1}^{n} V_{y_i}
\]

Since each \( U_{y_i} \) and \( V_{y_i} \) are disjoint, we have \( U \) and \( V \) are disjoint. Also \( A \subset U \) and \( x \in V \). Thus, for each point \( x \in X \setminus U \) we have found the open set \( V \) containing \( x \) which is disjoint with \( A \). Thus, \( X \setminus A \) is the open set and \( A \) is the closed set. \(\square\)

---

6The topological space is Hausdorff is termed if given any two distinct points \( x \neq y \) in the topological space, there exist disjoint open sets containing the two points respectively.

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3.2 Separation axioms

The separation axioms $T_i$ stipulate the degree to which distinct points or closed sets may be separated by open sets. These axioms are statements about the richness of topology.

Let $(X, \tau)$ be the topological space.

$T_0$ axiom: If $a, b$ are two distinct elements in $X$, there exists the open set $U \in \tau$ so that either $a \in U$ and $b \notin U$, or $b \in U$ and $a \notin U$ (i.e. $U$ containing exactly one of these points).

$T_1$ axiom: If $a, b \in X$ and $a \neq b$, there exist the open sets $U_a, U_b \in \tau$ containing $a, b$ respectively, so that $b \notin U_a$ and $a \notin U_b$.

$T_2$ axiom: If $a, b \in X$ and $a \neq b$, there exist disjoint open sets $U_a, U_b \in \tau$ containing $a, b$ respectively.

$T_3$ axiom: If $A$ is the closed set of $X$ and $b$ is the point in $X$, so that $b \notin A$, there exist disjoint open sets $U_A, U_b \in \tau$, so that $b \in U_b$ and $A \subset U_A$, $U_A \cap U_b = \emptyset$.

$T_4$ axiom: If $A$ and $B$ are disjoint closed sets in $X$, there exist disjoint open sets $U_A, U_B \in \tau$ containing $A$ and $B$ respectively.

It is said that $U$ and $V$ separate $A$ and $B$ if $A \subset U, B \subset V$ and $U \cap V = \emptyset$.

If $(X, \tau)$ satisfies a $T_i$ axiom that $X$ is called a $T_i$ space.

A $T_0$ space is called the Kolmogorov space.

A $T_1$ space is called the Frechet space.

A $T_2$ space is called the Hausdorff space.

A $T_3$ space is called the regular space.

**Corollary 11.** Let $f, g$ be two continuous functions of the topological space $X$ to the Hausdorff space $Y$, then the set of all $x \in X$ so that $f(x) = g(x)$ is closed in $X$.

**Corollary 12.** Let $f, g$ be two continuous functions of the topological space $X$ to the Hausdorff space $Y$. If $f(x) = g(x)$ at all points of the dense subset of $X$, then $f = g$.

In other words, the continuous function of $X$ to $Y$ (the Hausdorff space) is uniquely determined by its values at all points of the dense subset of $X$.

**Proposition 13.** If every point of topological space $X$ has the closed neighborhood which is the Hausdorff subspace of $X$, then $X$ is the Hausdorff space.

Each of separation axioms is independent of the axioms for the topological space; but they are not independent of each other, for instance, axiom $T_2$ implies axiom $T_1$, and axiom $T_1$ implies $T_0$.

More importantly than the separation axioms themselves is the fact that they can be employed to define successively stronger properties. To this end, note that if the space is both $T_3$ and $T_0$ it is $T_2$ space, while the space that is both $T_4$ and $T_1$ must be $T_3$ space.

Specifically the space $X$ is said to be regular space if and only if it is both a $T_0$ and a $T_3$ space; to be normal space if and only if it is both a $T_1$ and $T_4$ space.

If $X$ is the normal space, this implies that $X$ is regular space.
Proposition 14. Every subspace of the regular space is the regular space.

Proof. Let $A$ be the subspace of the regular space $X$. Since $X$ is the Hausdorff space, so $A$ is the Hausdorff space; on the other hand, every neighborhood of a point $x \in A$ with respect to $A$ is of the form $V \cap A$ where $V$ is a neighborhood of $x$ in $X$. Since $X$ is regular there is a neighborhood $W$ of $x$ in $X$ which is closed in $X$ and contained in $V$; $W \cap A$ is then a neighborhood of $x$ in $A$, closed in $A$ and contained in $V \cap A$. Hence the result.

Corollary 15. If every point $x$ of the topological space $X$ has the closed neighborhood which is the regular subspace of $X$, then $X$ is the regular space.
3.3 The normal and hereditarily normal spaces

The normal spaces is widely used in the theory of functions and in the function analysis. A lot of functions is defined on the normal spaces, thus it is possible to construct and examine the analysis on these spaces.

Normality do not inherited by subspaces, due to this, let us introduce with the concept of hereditary normality.

Hereditarily normal spaces are such spaces in which any subset is the normal space. Accordingly to the concept of hereditary normality if \( X \) is not the hereditarily normal spaces that there exist the open subset of \( X \) which is not the normal subspace.

Let \( (X, \tau) \) be the normal space and \( F \) is the closed subset of \( X \), then \( (F, \tau_F) \)- is the normal space.

The normality is inherited by the closed subspaces.

**Definition 39.** Let \( X \) be the topological space and the set \( A \) is the subspace of \( X \). Is said to be the set \( A \) have the type \( G_\delta \) if

\[
A = \bigcap_{i=1}^{\infty} U_i
\]

where \( U_i \) are open in \( X \).

**Definition 40.** Let \( X \) be the topological space and the set \( A \) is the subspace of \( X \). Is said to be the set \( A \) have the type \( F_\delta \)

\[
A = \bigcup_{i=1}^{\infty} F_i
\]

where \( F_i \) are closed in \( X \).

There exists a narrow class of hereditarily normal spaces is the perfectly normal space.

**Definition 41.** The normal space \( X \) is said to be the perfectly normal space if any closed set of \( X \) have the type \( G_\delta \), that is can be represented as the countable intersection of open subsets or equivalently, any open set of \( X \) can be represented as the countable union of closed subsets.

Let us show the equivalence:

\[
X \backslash F = V
\]

\( V \) is open in \( X \).

\[
F = \bigcap_{i=1}^{\infty} U_i
\]

\[
V = X \backslash F = X \backslash \bigcap_{i=1}^{\infty} U_i = \bigcup_{i=1}^{\infty}(X \backslash U_i)
\]

We represented the open subset \( V \) as the union of closed subsets. Perfectly normal spaces are hereditarily normal spaces.
3.4 Compactness and Continuity

Compactness is the generalization to topological spaces of the property of closed and bounded subsets of the real line. Compactness was introduced into topology with the intention of generalizing the properties of the closed and bounded subsets of $\mathbb{R}^n$.

**Theorem 16.** Let $X$ be the compact space and the function $f: X \to Y$ is the continuous function from $X$ to $Y$. Then $Y$ is the compact space.

**Corollary 17.** Let $X$ be the compact space and the function $f: X \to Y$ is the continuous function. The image $f(X)$ of $X$ in $Y$ is the compact subspace of $Y$.

**Theorem 18.** Let $X$ be the compact space, $Y$ be the Hausdorff space and the function $f: X \to Y$ is the continuous one-to-one function. Then $f$ is the homeomorphism.

---

7The function $f: X \to Y$, where $X$ and $Y$ are topological spaces, is called the homeomorphism if $f$ satisfies the three conditions:
1. The function $f$ is the bijection.
2. The function $f$ is the continuous.
3. The inverse function $f^{-1}$ is the continuous function.
4 F - compacts

4.1 Inverse spectrum

Will be assume the set \( A \) is the directed set \([12]\).

**Definition 42.** Let \( X_a \) be the family of topological spaces, where \( a \in A \) and let for each pair of indices \( a, b \in A, \ a \geq b \) assigned the continuous mapping:

\[
\rho^a_b : X_a \to X_b
\]

and if \( a \geq b \geq c \) the following transitivity condition is performed:

\[
\rho^a_c = \rho^b_c \cdot \rho^a_b, \text{ and if } a = b, \text{ then } \rho^a_b = \text{id}_{X_a}.
\]

Then the system of spaces and mappings:

\[
S = \{ X_a, \rho^a_b : a, b \in A \}
\]

is called the inverse spectrum of topological spaces.

Let \( S = \{ X_a, \rho^a_b : a, b \in A \} \) be the inverse spectrum of topological spaces. The point

\[
x = \{ x_a : a \in A \}
\]

lies in the product \( \prod_{a \in A} X_a \) is called the thread of \( S \) if for any \( a, b \in A \) satisfying the condition \( a \geq b \), we have the following equality:

\[
\rho^a_b x_a = x_b.
\]

**Limit of the spectrum** \( \lim S \) is the set of its threads with the induced topology \( \prod_{a \in A} X_a \) of Tikhonov product.\[8\]

**Definition 43.** Let us consider the case where the index set \( A \) - is the well-ordered set, in other words \( A = W(\tau) \), where \( W(\tau) \) - the set of ordinal numbers which less than some ordinal number \( \tau \). Then the system of topological spaces and mappings:

\[
S = \{ X_\alpha, \rho^\alpha_\beta : \alpha, \beta \in A \}
\]

is called the well-ordered inverse spectrum, or simply the well-ordered spectrum.

**Definition 44.** Let \( S = \{ X_\alpha, \rho^\alpha_\beta : \alpha, \beta < \tau \} \) be the well-ordered spectrum. For each limit ordinal\[9\] \( \gamma < \tau \) consider the spectrum

\[
S |_{\gamma} = \{ X_\alpha, \rho^\alpha_\beta : \alpha, \beta < \gamma \}
\]

and let \( \overline{X}_\gamma = \lim X |_{\gamma} \). There exists the natural mapping \( \overline{\rho}_\gamma : X_\gamma \to \overline{X}_\gamma \) that each point \( x \in \overline{X}_\gamma \) assigns the thread \( \{ \rho^\alpha_\beta : \alpha < \gamma \} \) of spectrum \( S |_{\gamma} \).

The well-ordered spectrum \( S = \{ X_\alpha, \rho^\alpha_\beta : \alpha, \beta < \tau \} \) is called continuous if for any limit ordinal \( \gamma < \tau \) the mapping \( \overline{\rho}_\gamma \) is the homeomorphism.

\[8\] The Tikhonov product is the topological space which received as the Cartesian product of initial topological spaces. The Cartesian product - is the set which consist of the pairs of elements of \( A \) and \( B \).

\[ A \times B = \{(a, b)|a \in A, b \in B\} \]

\[9\] Ordinal \( \gamma \) is called the limit ordinal if there exists no a serial number immediately preceding the \( \gamma \), i. e there exists no a such serial number \( \xi \) which satisfies the condition: \( \gamma = \xi + 1 \).
4.2 Cardinal invariants of topological spaces

Definition 45. The density $d(X)$ is the smallest cardinality of the dense subset of $X$.

Definition 46. The Suslin number $c(X)$ is the least infinite cardinal number $\tau$, so that the cardinality of every family of pairwise-disjoint non-empty open sets does not exceed $\tau$.

Definition 47. The Lindelöf number $l(X)$ is the least infinite cardinal number $\tau$, so that every open covering of $X$ has the subcovering with the cardinality $\tau$.

It is possible to do arithmetic with cardinals: to multiply and to add them, and to raise them to the power. Correspondingly, it is possible to do arithmetic with cardinal invariants: to multiply and to add them as the functions, etc. This opens up the new possibilities for comparing cardinal invariants, using arithmetic. This inequality always fulfilled:

$$c(X) \leq d(X) \leq w(X),$$

where $w(X)$ is the weight of topological space $X$. And

$$l(X) \leq w(X).$$

Definition 48. The hereditary density $hd(X)$ is the upper bound of the densities of all of its subspaces.

Definition 49. The spread or hereditary Suslin number $hc(X)$ is the upper bound of cardinalities of uncountable discrete subspaces of $X$.

Definition 50. The hereditary Lindelöf number $hl(X)$ is the upper bound of cardinals $l(A)$ for all subspaces $A \subset X$.

Proposition 19. If $f: X \rightarrow Y$ is the continuous mapping, then

$$hd(X) \geq hd(Y), \; hc(X) \geq hc(Y), \; hl(X) \geq hl(Y).$$

Inequalities, connecting these cardinals between them:

$$hc(X) \leq hd(X), \; hc(X) \leq hl(X)$$

Definition 51. The continuous mapping $f: X \rightarrow Y$ is called the closed mapping if for every closed set $A \in X$ the image $f(A)$ is closed in the space $Y$.

Definition 52. Let $f: X \rightarrow Y$ be the continuous mapping, the small image of the set $A \in X$ is called the set of points from $Y$, so that their prototype is contained in $A$, i.e.,

$$f^\#(A) = \{y \in Y : f^{-1}(y) \subset A\}$$

Therefore, for the closed mapping $f: X \rightarrow Y$ and arbitrary set $Z \in Y$ and for every neighborhood $Of^{-1}(Z)$, the set $f^\#(Of^{-1}(Z))$ is the neighborhood for the set $Z$. 

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Proposition 20. The mapping $f: X \to Y$ is closed if and only if for every point $y \in Y$ and any neighborhood $O f^{-1}(y)$, the set $f^\#(O f^{-1}(y))$ is the neighborhood of the point $y$.

Definition 53. Surjective mapping $f: X \to Y$ is called fully closed at the point $y \in Y$ if for every finite cover of its prototype $f^{-1}(y)$ by the open sets $U_1, \ldots, U_s$ in $X$, the set
\[ \{y\} \cup \bigcup_{i=1}^{s} f^\#(U_i) \]

is the neighborhood of the point $y$. (open in $Y$)

It is said that mapping $f$ is fully closed at $x \in X$ if $f$ is fully closed at $f(x)$. If $f: X \to Y$ is fully closed at every point $y \in Y$, then the mapping $f$ is called fully closed.

If in this definition is limited to the covers consisting of the single element, we get one of the equivalent definitions of the closed mapping.

Proposition 21. If $f: X \to Y$ is fully closed mapping, then the open subsets $U \subset X$, for which the equality holds
\[ f(U) = f^\#(U) \cup \{f(x)\}, \]
forms the basis in $X$.

Proof. Let $x \in X$ and $O$ be the neighborhood of the point $x$. Take the such neighborhood $O_1$ of the point $x$, which $[O_1] \subset O$. The sets $O$ and $X \setminus [O_1]$ from the cover of $X$, because sets $O$ and $X \setminus [O_1]$ are open in $X$.

Therefore, by virtue of closure mapping $f$, closed set in $Y$
\[ A = Y \setminus (f^\#(O) \cup f^\#(X \setminus [O_1])) \]
discretely. Consider the set
\[ U = (O_1 \cup f^{-1}(f^\#(O))) \setminus f^{-1}(A \setminus \{f(x)\}). \]
The set $U$ is open, it contains the point $x$ and itself contained in $O$. As $f(O_1) \setminus A \subset f^\#(O)$, we have
\[ f(U) = ((f(O_1) \cup f^\#(O)) \setminus A) \setminus f(x) \subset f^\#(U) \cup \{f(x)\}. \]
We have $f(U) \subset f^\#(U) \cup \{f(x)\}$, on the other hand, $f(U) \supset f^\#(U) \cup \{f(x)\}$.

So, $f(U) = f^\#(U) \cup \{f(x)\}$. □
Definition 54. The compact space $X$ is called the F- compact space if there exists the such well-ordered continuous inverse spectrum of compact spaces $S = \{X_\alpha, \rho_\beta^\alpha : \alpha, \beta < \tau\}$, that

1. $X_0$ - is the point;
2. All projections $\rho_\alpha^{\alpha+1}$ are fully closed, $\alpha + 1 < \tau$;
3. Weight of prototypes is countable for all points $x \in X_\alpha$, $\alpha + 1 < \tau$;
4. $\lim S = X$;

Definition 55. Spectral height of F-compact space $X$ (sh($X$)) is called the minimum length $\tau$ of spectrum $S$, satisfying the conditions 1 - 4.

Recall that, here and throughout the paper, the concept "compact" means the compact Hausdorff space.

In fact, F - compacts are such spaces which can be constructed by the method of fully closed mappings. This spectrum starts at the point and is continuous.

For example, F - compact with the spectral height 1 is the space, consisting of the single point. F - compact with the spectral height 2 - is not the single-point space.

Actually, if $X$ is the compact space, and the cardinality of $X$ larger than 1 ($|X| > 1$), let $X_1 = X$, $X_0$ - is the point and $\rho_1^0 : X_1 \to X_0$ is defined as the constant mapping. Then the spectrum $S = \{X_1, X_0, \rho_1^0\}$ satisfies all the conditions in the definition of F-compact spaces and $\lim S = X_1 = X$.

Accordingly to this, $X$ - F-compact and $sh(X) = 2$.

Rearward if $X$ - F- compact and $sh(X) = 2$, then there exists the spectrum $S = \{X_1, X_0, \rho_1^0\}$ satisfies the conditions in the definition of F-compact spaces. Then, $X = X_1$, $w(X) \leq \omega_0$ and $|X| > 1$. Due to this, $X$ is non the single-point compact space.

We shall always assume that $X$ - is the topological space.

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\[10\] If in the spectrum $S = \{X_\alpha, \rho_\beta^\alpha : \alpha, \beta \in A\}$ the index set A has the largest element $\alpha$, then $\lim S = X_\alpha$. 

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Theorem 22. The following equation is performed for $F$-compacts with the counting spectral high.

\[ \text{hd}(X) = \text{hc}(X) = \text{hl}(X) \] (11)

Proof. In order to prove this theorem we use the following lemma:

Lemma 23. Let $f : X \to Y$ be the fully closed mappings of compacts, and all prototypes of points $f^{-1}(y)$ are metrizable. If $\text{hd}(Y) = \text{hc}(Y) = \text{hl}(Y)$ then $\text{hd}(X) = \text{hc}(X) = \text{hl}(X)$ (12)

Proof. Let $\text{hc}(X) = \tau$, where $\tau$ - some infinite cardinal.
Let us show that $\text{hd}(X) \leq \tau$, i.e. the upper bound of densities of all subspaces of $X \leq \tau$, or in other words, any subspaces $A \subset X$ contains the dense set in $A$, which has a cardinality $\leq \tau$. Since Proposition 19

\[ \tau = \text{hc}(X) \geq \text{hc}(Y) = \text{hd}(Y) \]

Therefore in the set $f(A)$ we can choose the dense subspace $B$, which has the density $\leq \tau$.
Let $C$ be the subspace of $A$, which is one-to-one mapped to $B$.
Let us denote by $D = A \setminus C$.

Let us show that the cardinality of $f(D) \leq \tau$. We will show by contradiction, so let $|f(D)| > \tau$. Then in the set $D$ we can choose subset $E$, which has cardinality $> \tau$ and is one-to-one mapped to $f(D)$.

Any point $x \in E$ has the such neighborhood $O(x)$ that its intersection with $C$ is empty $O(x) \cap C = \emptyset$. Since $E$ is contained in $D$, and $D = A \setminus C$. This implies that the small image of the neighborhood of $x$ does not intersect with $f(C)$, i.e

\[ f^\#(Ox) \cap f(C) = f^\#(Ox) \cap B = \emptyset. \]

Thus in $f(A)$ we have pointed the open subset $f^\#(Ox)$, which does not intersect with $B$, since the set $B$ is the dense in $f(A)$, that $f^\#(Ox) \cap f(A) = \emptyset$.

By virtue the proposition 21 as the open set $U$ we can choose the neighborhood $Ox$, such that

\[ f(Ox) = f^\#(Ox) \cup \{f(x)\} \]

and then

\[ f(Ox) \cap f(E) \subset (f^\#(Ox) \cup \{f(x)\}) \cap f(A) = \{f(x)\}. \]

\[ \text{Let the sets } A, B \subset X \text{ and } A \subset B, \text{ then the set } A \text{ is called the dense set in } B, \text{ if the any neighborhood of any point of } B \text{ contains at least one point from } A. \text{ The set } A \text{ is called the everywhere dense set, if the set } A \text{ is dense in } X. \]
It means that $Ox \cap E = x$. Thereby the set $E$ is discretely. We have supposed that $|f(D)| > \tau$, and have pointed that the set $X$ contains the discrete subset $E$ and $|E| > \tau$, this contradicts the fact that $hc(X) = \tau$. Therefore $|f(D)| \leq \tau$.

The prototype of every point $f^{-1}(y)$ is metrizable, i.e. we can point the countable everywhere dense set in $X$. If the prototype $f^{-1}(y)$ to intersect with $D$, then we can specify the some countable everywhere dense set $B_0$ by all $y \in f(D)$ in this intersection. Let us denote the union of all $B_y$ as $P$.

The cardinality of $P$ is the maximum of cardinality $f(D)$ and $\omega_0$, hence the cardinality of $P \leq \tau$.

It is obvious that $P \cup C$ is densely in $A$ and $|P \cup C| \leq \tau$. The equality $hc(X) = hd(X)$ is proved.

Let us suppose that $hl(X) > \tau$, then there exists the increasing sequence of open sets $\{U_\alpha : \alpha < \tau^+\}$ in $X$ which has the lengths $\tau^+$, in which for any $\alpha, \beta, \alpha < \beta$, $U_\alpha$ is strictly contained in the $U_\beta$.

Since $hl(Y) \leq \tau$, the sequence of open sets $\{f^\#(U_\alpha) : \alpha < \tau\}$ of the set $Y$ is stabilized, i.e. beginning from some $\gamma_0$, where $\alpha > \gamma_0$ we have:

$$f^\#(U_\alpha) = f^\#(U_{\gamma_0})$$

Let us denote $\bigcup f^\#(U_\alpha) = W$ which is contained in $Y$.

Thereby, $\{f^\#(U_\alpha) : \alpha < \tau\}$ is the open cover of the set $Y$. Then there exists the subcover $\{f^\#(U_\alpha) : \beta < \tau\}$.

Let us set $\sup\{\alpha, \beta, \beta < \tau\} = \gamma_0 < \tau^+$. It is clear that for two different indexes $\alpha, \alpha' (\alpha' > \alpha)$ we have

$$f^\#(U_\alpha) \subset f^\#(U_{\alpha'})$$

Because the $\{f^\#(U_{\alpha'})\}$ is a subcover, so we have:

$$f^\#(U_{\gamma_0}) \supset \{f^\#(U_{\alpha})\} \text{ and } f^\#(U_{\gamma_0}) \supset f^\#(U_\alpha) \text{ for any } \alpha$$

Thus

$$f^\#(U_{\gamma_0}) = f^\#(U_\alpha), \alpha \geq \gamma_0.$$

Let us denote $f^\#(U_\alpha) = V$ for all $\alpha > \beta$. Let us consider the set

$$M = \bigcup f(U_\alpha) \setminus V : \beta < \alpha < \tau^+$$

and we will show that cardinality of $M > \tau$.

We will show by contradiction. Let the cardinality of $M \leq \tau$.

Let us consider the sequence of open sets for every point $y \in M$ in the set $f^{-1}(y)$.

$$\{U_\alpha \cap f^{-1}(y) : \alpha < \tau^+\}$$

Since $f^{-1}(y)$ is metrizable this sequence is stabilized beginning from some number $\beta_y < \tau^+$. We have supposed that the cardinality of $M \leq \tau$, hence

$$\sup\{\beta_y : y \in M\} = \gamma < \tau^+.$$
We have the contradiction, since we have chosen the different indexes. Therefore, the cardinality of $M > \tau$. Now in the prototype $f^{-1}(y)$ we choose the point $z_y$ for every point $y$ from $M$, which lies in some $U_\alpha$, and let us set $Z = \{z_y : y \in M\}$. Now we will show that the set $Z$ is discretely.

Let the point $z$ belongs to the set $Z$, $z \in U_\alpha$ and $Oz$ is the neighborhood of the point $z$ for which the following equality is fulfilled:

$$f(Oz) = f^\#(Oz) \cup f(z)$$

(Such neighborhood exists according to Proposition 21.) Then we have

$$f(Oz) \cap f(Z) = (f^\#(Oz) \cup f(z)) \cap M \subset (V \cap M) \cup f(z)$$

Since the mapping $f$ is one-to-one on $Z$ ($f$ is the bijection), hence we have $Oz \cap Z$. Thereby, we have shown that $Z$ is discretely and it’s cardinality $> \tau$, but it is contradict with $hc(X) = \tau$.

It means that $hl(X) \leq \tau$ and therefore $hl(X) = hc(X)$.

For ending the proof of theorem it is necessary to consider the case a limit $\alpha$.

**Lemma 24.** If compact $X$ is the limit of countable spectrum of compacts

$$\{X_\beta, \rho^\beta_{\beta'} : \beta, \beta' < \alpha\}$$

then

$$hd(X) = \sup hd(X_\beta), hl(X) = \sup hl(X_\beta), hc(X) = \sup hc(X_\beta)$$

(13)

Let us consider the first inequality $hd(X) = \sup hd(X_\beta)$.

**Proof.**$hd(X) = \sup hd(X_\beta)$.

By the Arhangelsk’s theorem.

**Theorem 25.** Arhangelsk

$$hd(X) = \sup \{d(F) : F \subset X, F \text{ is closed}\}$$

(14)

Let $F$ be the closed subset of $X$.

Compact $X$ is the limit of countable spectrum of compacts $\{X_\beta, \rho^\beta_{\beta'} : \beta, \beta' < \alpha\}$. Let us construct the spectrum:

$$S_F = \{\rho_\beta(F) = F_\beta \subset X_\beta\},$$

where $F_\beta$ is the closed subset of $X_\beta$.

From the definition of hereditary density follow that:

$$d(F_\beta) \leq hd(X_\beta).$$

We have $d(\lim S_F) \leq \sup d(F_\beta)$. So, $d(F_\beta) \leq hd(X_\beta)$, then

$$\sup d(F_\beta) \leq \sup hd(X_\beta),$$
and we can go to another inequality:
\[ d(\lim S_F) \leq \sup \text{hd}(X_\beta). \]

Since \( \lim S_F = F \), we got \( d(F) \leq \sup \text{hd}(X_\beta) \), this is carry out for every closed set \( F \) from \( X \), then since the Arhangelsk's theorem \( \text{hd}(X) \leq \sup \text{hd}(X_\beta) \).

In the other hand \( \text{hd}(X) \geq \sup \text{hd}(X_\beta) \) always carry out, therefore
\[ \text{hd}(X) = \sup \text{hd}(X_\beta) \]

Let us prove the second inequality \( \text{hl}(X) = \sup \text{hl}(X_\beta) \).

Let \( \omega \) be the open cover of \( X \), \( \omega = \{ U_\alpha : \alpha \in A \} \), where \( U_\alpha \) - the open sets, \( A \) - the set of indices.

Then there exists the open subcover:
\[ \omega' = \{ U_\alpha : \alpha \in A' \} \text{and} \bigcup_{\alpha \in A} U_\alpha = \bigcup_{\alpha \in A'} U_\alpha \]

Denote by
\[ \rho^\#(\omega) = \{ \rho^\#_\beta(U_\alpha) : \alpha \in A \} \]

The small images \( U_\alpha \) and \( \rho^\#(\omega) \) are open. Then there exists \( A_\beta \subset A \), so that
\[ \bigcup_{\alpha \in A_\beta} \rho^\#(U_\alpha) = \bigcup_{\alpha \in A} \rho^\#_\beta(U_\alpha) = \bigcup_{\alpha \in A} \rho^\#_\beta(U_\alpha) \]

Obviously that
\[ | \bigcup_{\beta < \alpha} \rho^\#_\beta(U_\alpha) | \leq \sup_{\beta < \alpha} \text{hl}(X_\beta) \]

To complete the proof, we need to verify the following equation:
\[ \bigcup_{\beta < \alpha} \bigcup_{\alpha \in A_\beta} U_\alpha = \bigcup_{\alpha \in A} U_\alpha \]

Inclusion \( \subset \) is obvious, let us check the reverse inclusion \( \supset \).

To do this, we take the point of \( \bigcup_{\alpha \in A} U_\alpha \) and will show that it is contained in \( \bigcup_{\alpha \in A_\beta} U_\alpha \).

Let the point \( x \in U_\alpha \). It’s prototype \( \rho_\beta(x) \subset \rho^\#_\beta(U_\alpha) \).

Therefore,
\[ \rho_\beta(x) \subset \rho^\#_\beta(U_\alpha) \quad \text{where} \quad \alpha_0 \in A_\beta. \]

We get that \( x \in \bigcup_{\alpha \in A_\beta} U_\alpha \).

Accordingly to this, the equality \( \text{hl}(X) = \sup \text{hl}(X_\beta) \) is proved.

And finally let us prove the third equality \( \text{hc}(X) = \sup \text{hc}(X_\beta) \).

Let \( A \) be the discrete subset of \( X \). If we can show that \( |A| \leq \sup \text{hc}(X_\beta) \), this will means that \( \text{hc}(X) = \sup \text{hc}(X_\beta) \).

The set \( A \) is the discrete set, this means that for every \( x \) from \( A \) there exists the neighborhood \( Ox \), so that \( Ox \cap A = \{ x \} \). Let us take the image of the point \( x \) and is denoted this image as \( \rho_\beta(x) \), which is contained in \( X_\beta \). The image \( \rho_\beta(x) \) is contained in the some neighborhood \( U : \rho_\beta^{-1}(U) = Ox \). Thus we see
that the points of $A$ are one to one displayed on $X_\beta$. For any $x$ from $A$ there exists $\beta$, so that the point $x$ is isolated at the level of $\beta$, so we get the set

$$\{\rho_\beta(A) : U \cap \rho_\beta(A) = \rho_\beta(x)\}.$$ 

Hence, $\rho_\beta(A)$ is the discrete set.

Let us denote that $A = \bigcup_{\beta < \alpha} A_\beta$, and $\rho_\beta(A_\beta)$ is the discrete set in $X_\beta$.

Thereat, $|A_\beta| = |\rho_\beta(A_\beta)| \leq hc(X_\beta)$ and let us take the union of all $A_\beta$, ie $\bigcup_{\beta < \alpha} A_\beta$ and $\bigcup_{\beta < \alpha} A_\beta \leq \sup hc(X_\beta)$, we can see that

$$|A| \leq \sup hc(X_\beta),$$

and this means that $hc(X) = \sup hc(X_\beta)$.

So, if

$$hc(X_\beta) = hl(X_\beta) = hd(X_\beta)$$

for all $\beta < \alpha$ then

$$hc(X) = hl(X) = hd(X).$$
The theorem of the relation of properties between the perfectly normality and the hereditary normality in the class of F-compact spectral counting height.

**Theorem 26.** Let $X$ be the separable hereditarily normal F-compact and it’s height is countable $shX < \omega_1$, then $X$ is a perfectly normal space $(2^{\omega_1} > 2^{\omega_0})$.

**Proof.** By assumption, $X$ is separable\(^{12}\) and hereditarily normal space. We will show that uncountable discrete\(^{13}\) subspaces in itself does not exist in $X$. (The separability is not inherited by subspaces).

We will show by contradiction. The discrete subset $A$ with the uncountable cardinality there exists in $X$.

Let $\xi = \{F_\alpha : \alpha \in H\} = P(A)$ be the family of the subsets of $A$ which are indexed by elements of some set of indices $H$. The cardinality of $\xi = 2^{\omega_1}$. (Let $2^k$ be the cardinality of the set of all subsets of the power $k$, by Cantor’s theorem the inequality $2^k > k$ is always fulfilled.)

Let us denote $G_\alpha$ as $A \setminus F_\alpha$ for every $F_\alpha \in \xi$. By our assumption $A$ is discrete subset of $X$, then

$$[G_\alpha] \cap F_\alpha = [A \setminus F_\alpha] \cap F_\alpha = \emptyset$$

Since $X$ is the hereditary normal space then the sets $[G_\alpha] \cap X'$ and $[F_\alpha] \cap X'$ are closed and do not intersect in the normal space $X' = X \setminus ([G_\alpha] \cap [F_\alpha])$. It is fulfilled being that $[G_\alpha] \cap X' = [G_\alpha] \cap (X \setminus ([G_\alpha] \cap [F_\alpha]))$, where $[G_\alpha] \cap [F_\alpha] = A$ and $A \subset G_\alpha$. Similarly for $[F_\alpha]$. Moreover, $G_\alpha \subset [G_\alpha] \cap X$, $F_\alpha \subset [F_\alpha] \cap X'$, hence there exists the disjoint open neighborhoods $U_\alpha \supset F_\alpha$ and $V_\alpha \supset G_\alpha$ which are open in $X'$ and hence these neighborhoods are open in $X$.

Let $B$ be the countable dense subset of $X$. Let us consider the family of the subsets of the set $B$: $\eta = \{U_\alpha \cap B : \alpha \in H\}$

All elements $\eta$ are different. For any two different indices $\alpha, \alpha' \in H, F_\alpha \neq F_{\alpha'}$. Then we will assume that $F_\alpha \setminus F_{\alpha'} \neq \emptyset$ and thereat $F_\alpha \cap G_{\alpha'} \neq \emptyset$.

Hence the intersection $U_\alpha \cap V_{\alpha'}$ is not empty as well. Then we have the non-empty set $U_\alpha \cap V_{\alpha'} \cap B$, which lies in the $U_\alpha \cap B$ and do not intersect with $U_{\alpha'} \cap B$, so

$$U_\alpha \cap B \neq U_{\alpha'} \cap B.$$ 

The sets $U_\alpha$ are the subsets of the set $A$ which has the uncountable cardinality, hence the cardinality $U_\alpha = 2^{\omega_1}$, and the cardinality of the set $B$ is countable. Therefore the family of distinct subsets of $B$, i.e. the set $\eta$ has the cardinality $2^{\omega_1}$ ($|\eta| = 2^{\omega_1}$).

This is a contradiction, since the set $B$ is the countable dense subset of $X$ and by the Continuum Hypothesis (abbreviated CH): $2^{\omega_1} > 2^{\omega_0}$.

\(^{12}\) Separable space $X$ is the topological space containing the finite or countable dense set.

\(^{13}\) Discrete space is the space with the discrete topology in which all points are isolated.
6.1 Comment on the theorem.

Let us note that the theorem \[26\] is true under the assumption that \(2^{\omega_1} > 2^{\omega_0}\). Where

\(\omega_0\) - the counting cardinality;
\(\omega_1\) - the uncountable cardinality;
\(2^{\omega_1}\) - the cardinality of all subsets of the set with the cardinality \(\omega_1\).

This inequality \(2^{\omega_1} > 2^{\omega_0}\) is the simple consequence of the continuum-hypothesis CH, videlicet, the continuum-hypothesis CH claims that \(2^{\omega_0} = \omega_1\), hence \(2^{\omega_1} > \omega_1\) (this follows from the theorem about the cardinality of the set of subsets, which states that \(2^k > k\), by \(2^k\) is denoted the cardinality of the set of all subsets of the set with the cardinality \(k\)) and then, \(2^{\omega_1} > 2^{\omega_0}\).

The requirement of separability in the theorem is essential, since there exists not separable hereditary normal, not perfectly normal F-compact with spectral height of 3.
6.2 Fedorchuk structure

The example of not separable hereditary normal, not perfectly normal F-compact with spectral height of 3 is named "Two circumferences".

This example is based on the known structure B, which was proposed by V.V. Fedorchuk. Let us define this structure.

Let $X$ be the compact and for every point $x$ from $X$ ($x \in X$) is mapped to the compact $Y_x$. For every point $x$ from $X$ the continuous mapping $h_x : X \setminus \{x\} \to Y_x$ is defined.

In this situation the topology can be defined on the set $\bigcup \{Y_x : x \in X\}$ which makes this set to the desired space $B(X, Y_x, h_x)$ (or just $B$).[3]

The base of this topology consists of the sets of the form:

$$O(x, U, V) = V \cap \pi^{-1}(U \cap h_x^{-1}V).$$

Where $V$ is the open subset of $Y_x$, $U$ is the neighborhood of the point $x$ from $X$ and $\pi : \bigcup Y_x \to X$ is the natural projection acting according to the formula $\pi(y) = x$, if $y \in Y_x$. 
7 Examples of application of construction B

7.1 ”Two circumferences”

Let $X$ be the circle, let us denote it as $S^1$ and for every point $x$ from $X$ the $Y_x$ is the compact, according to the Fedorchuk structure is the colon $\{a, b\}$ with the discrete topology.

The mapping $h_x : X \setminus \{x\} \rightarrow Y_x$ is defined as the constant mapping to the point $a$, that is $h_x(y) = a$ for any $y$. The space $B(S^1, \{a, b\}, h_x : x \in S^1)$ is the classical space with the name ”Two circumferences”. The topology on $B$ is defined on the set $\bigcup_{x \in X} Y_x$, which can be represented as the union of two circles $S^1_a$ and $S^1_b$. Thereby $B = S^1_a \cup S^1_b$ and the point from $B$ is usually denoted as $(x, a)$ or $(x, b)$, where $x \in S^1$.

If we will take the set $V$ (the open subset of $Y_x$) which figures among the parameters of the basic of open set $O(x, U, V)$, as the point $\{b\}$ from a layer $Y_x$, and the set $U$ is the neighborhood of the point $x$ from $X$, then we will get that

$$O(x, U, V) = \{(x, b)\}$$

since $h_x^{-1}\{b\} = \emptyset$.

If we will take the set $V$ as the point $\{a\}$, we will get that

$$O(x, U, V) = (U \setminus \{x\}) \times \{a, b\} \cup \{(x, a)\}.$$

The basic of open set of topology $B$, $O(x, U, V)$ - is the union of two exemplars of a punctured neighborhood $U$ of the point $x$ on $S^1_a$ and $S^1_b$, to which is added the point $(x, a)$ and coincides with the ”Two circumferences” topology.

The space $B(S^1, \{a, b\}, h_x : x \in S^1)$ is the F-compact with the spectral height 3.

Indeed, $X_0$ is the point, $X_1$ is the circle $S^1$, $X_2$ are the circles $S^1_a$ and $S^1_b$.

In truth, the set of indices are numbers $\{0, 1, 2\}$, hence the space $B(S^1, \{a, b\}, h_x : x \in S^1)$ is the F-compact with the spectral height $\leq 3$.

In this case it is impossible to build the spectr with height less than 3.

For example, for the spectr with length 2, we have: $X_1 = X$ (the compact), $X_0$ is the point and $\rho_0^1 : X_1 \rightarrow X_0$ is the constant mapping.

The spectr $S = \{X_0, X_1, \rho_0^1\}$ has $\text{lim} S = X_1 = B(S^1, \{a, b\}, h_x : x \in S^1)$. The compact $X$ is the metrizable space, but the space $B(S^1, \{a, b\}, h_x : x \in S^1)$ is non metrizable space.
7.2 "Two arrows"

Let $X$ be the two segments $[0; 1]$. Let us denote (1) - is the 1st segment, (2) - is the 2nd segment. The base of topology is defined as

$$\sigma = \{ (a, b)(1) \cup [a, b)(2) : a < b \}$$
References


[3] V.V.Fedorchuk."The metod of spectrs"


