Andreas Eckert

Contributing to develop contributions
– a metaphor for teaching in the reform mathematics classroom
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Abstract

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This thesis aims at contributing to the theoretical research discourse on teaching mathematics. More precise, to explore a teacher's role and actions while negotiating meaning of mathematical objects in discursive transformative practices in mathematics. The focus is to highlight the teacher as an active contributor to the classroom mathematical discourse, having an important role in shaping the mathematics. At the same time, the teacher is acknowledged as an individual who learns and develops as a lesson and semester progress.

Three research papers illustrate the state, at that time, of an inductive analysis of three teachers, teaching a series of lessons based on probability theory at two Swedish primary schools. The teachers worked together with the students to explore an unknown sample space, made up out of an opaque bottle with coloured marbles within that showed one marble at each turn of the bottle. They had to construct mathematical tools together to help them solve the mystery. The analysis focused on teacher–student interactions during this exploration, revealing complex connections in the process of teaching.

The three papers presented the development of a theoretical framework named Contributing to Develop Contributions (CDC). The frameworks' fundamental idea is that teachers learn as they teach, using the teaching metaphor learning to develop learning. That metaphor was developed, in light of the ongoing empirical analysis, into CDC by drawing on a theoretical idea that learning can be viewed as contributing to the collaborative meaning making in the classroom. Teaching and teacher learning are described and understood as reflexive processes in relation to in-the-moment teacher-student interaction.

Contributing to develop contributions consists of three different ways of contributing. The analytical categories illustrate how students' opportunities to contribute to the negotiation of mathematical meaning are closely linked to teachers' different ways of contributing. The different ways are Contributing one's own interpretations of mathematical objects, Contributing with others' interpretations of mathematical objects, and Contributing by eliciting contributions. Each way of contributing was found to have the attributes Transparency, Role-taking and Authority. Together, these six categories show teacher–student interaction as a complex dynamical system where they draw on each other and together negotiate meaning of mathematical objects in the classroom.

This thesis reveals how the teaching process can be viewed in terms of learning on different levels. Learning as thought of in terms of contributing to the negotiation of meaning in the moment-to-moment interaction in the classroom. By contributing you influence the collective's understanding as well as your own. A teacher exercises and develops ways of contributing to the negotiation of meaning of mathematical objects, in order to develop students' contributions. In a wider perspective, the analysis showed development over time in terms of transformation. The teachers were found to have transformed their understanding of classroom situations in light of the present interactions. Contributing to the negotiation of meaning in the classroom was understood as a process in such transformation, in the ever ongoing becoming of a mathematics teacher.

Key words: Teaching mathematics, teaching as learning, professional development, learning to develop learning, contributing to develop contributions.
Abstract


This thesis aims at contributing to the theoretical research discourse on teaching mathematics. More precise, to explore a teacher’s role and actions while negotiating meaning of mathematical objects in discursive transformative practices in mathematics. The focus is to highlight the teacher as an active contributor to the classroom mathematical discourse, having an important role in shaping the mathematics. At the same time, the teacher is acknowledged as an individual who learns and develops as a lesson and semester progress.

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My beloved colleagues, both inside and outside of the university, has also had a part in this project. Thank you for insightful and challenging discussion, for including me although I often worked from home and for all the laughs. Good laughs kept me going many a day.

My biggest gratitude, and love, goes to my devoted family. To Hanna, for your support, and patience during late nights of work, long stays away and my blank stares into the night. And to my daughter Stella: you transformed my view of life by simply being born in the middle of this project. Thank you also for checking in on me from time to time, sharing your energy and helping me to pause.

Lastly, there is a saying in Swedish when it comes to acknowledgements, “no one mentioned, no one forgotten”. I’m going to challenge that notion by mentioning several who come to mind at this moment. If not all, then many of you who have challenged and encouraged me to develop my ideas into what is now a thesis in mathematics education: Thank you Simon, Astrid, Alexandra, Odour, Koeno, Barbara, Jan, Anna, Jorryt, Johan, Malin, Marcus, Jeppe, Magnus, Maike, Jonas, Karin, Hanna, Elisabet, Kirsti, Linda, Yukiko, Helen, Magnus, Uffe, Lena, Abdel, Pauline, Ewa, Elin, Niklas, Tuula, Andreas, Yvonne, Helena, Miguell, Kerstin, Andreas, Trude, John, Aaron, Anna, Jörgen, Linda, Helena and Håkan.
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INTRODUCTION

"To teach is to learn twice over" is a quote from the French moralist and essayist Joseph Joubert, and it indicates that teaching and learning are a two-way street. Perhaps you yourself have engaged in explaining something to another person and have watched your own understanding develop through this process. This thesis is an attempt to conceptualize and explain that process, as well as to capture the social artistry of teaching mathematics. This paper seeks to develop ideas about teachers' crucial role in the mathematics classroom discourse through which all participants develop in the process of making sense of mathematics.

Mathematics is a social activity, a science of patterns allowing us to organize and explain principles either via logical process or via abstraction from the real world (Schoenfeld, 1992). A pedagogy entailing interactive engagement with students, such as the creation of experiment-based learning environments, requires teachers and students to recreate mathematics together. Schoenfeld (1992) has suggested, among others, that mathematics instruction should provide students with exploratory situations and should engage them in the practice of reasoning and communicating mathematically to solve problems. Freudenthal (1991) used the term "horizontal mathematizing" to describe a process in which students are asked to move from the world of concrete objects to the world of symbols. This transformation marks a shift from the tangible to the abstract.

A learning environment marked by high levels of student engagement and student autonomy, along with a focus on problem-solving, is often called student-centred instruction. This approach is the opposite of teacher-directed instruction, where the emphasis is on the teacher presenting mathematical rules for the students to mimic (e.g., Gersten et al., 2008). Boaler (2008) has argued that it is not fruitful to pay too much attention to this proposed dichotomy and has instead claimed that we should seek to understand what 1 In this context, the term discourse refers to the specialized and situated communication of mathematics that includes some actors and excludes others (Sfard, 2008a).
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1 In this context, the term discourse refers to the specialized and situated communication of mathematics that includes some actors and excludes others (Sfard, 2008a)
comprises effective mathematics instruction, regardless of approach. I propose that mathematics is something we do together and that it includes joint investigation, exploration, experimentation, and problem-solving. This idea of mathematics is in line with what has long been known as the reform of mathematics education. “The teacher works to orchestrate the content, representations of the content, and the people in the classroom in relation to one another” (Franke, Kazemi, & Battey, 2007, p. 227), and this conceptualization highlights the relationship between teachers, students, and content. The key to understanding mathematics instruction is to view both students and teachers as interactively engaged in the task and as joint contributors to the development of the mathematical discourse. Both teachers and students are learners, but the teacher has a central role in teaching and is responsible for shaping meaning-making in the classroom.

The word teach generally means to impart knowledge to, to instruct, or to give information about (OxfordDictionaries.com, 2016). Furthermore, the word has its roots in Old English and Germanic languages, where the term originally meant to show, to present, to point out, or to represent. There is no consensus regarding what it means to teach that transcends the colloquial meaning of the word. Mathematics education research has overlooked the development of theories around the practice of teaching (Jaworski, 2006). Lester (2005) has problematized teaching mathematics by raising what has become one of the largest questions within mathematics education research: What is the teacher’s role in instruction? If rephrased to focus on the teacher’s action, this question reads: What is the teacher’s contribution to the interactions within students’ learning process? Lave (1996) wrote that, “[t]hose most concerned with relations between learning and teaching must untangle the confusions that mistakenly desubjectify learners’ and teachers' positions, stakes, reasons, and ways of participating, and then inquire anew about those relations” (p. 162). This study thus starts with the epistemological assumption that we should understand teaching—in terms of teachers’ contributions to students’ learning through teacher-student interactions—as an outcome of those interactions rather than as a product of a stable construct of teacher knowledge (cf. Eckert & Nilsson, 2015). The problem lies in unveiling the complex relations involved in meaning-making, as well as in disentangling the teacher’s role in it.

Previous conceptualizations of teaching have followed different paths. Some have focused on distinguishing particular teacher attributes, such as teacher knowledge, beliefs, and intentions, and these approaches have also considered how combinations of these traits can explain teachers’ actions in the classroom (e.g., Schoenfeld, 1999; Stahnke, Schueler, & Roesken-Winter, 2016). Others have focused on teachers’ actions and strategies for creating fertile conditions for classroom learning (e.g., Conner, Singletary, Smith, Wagner, & Francisco, 2014; Jaworski, 1994; Staples, 2007). Thus, the
literature has emphasized teachers’ role as organizers of the learning environment, centring on their application of knowledge regarding both students and mathematics. Mason (2016) has argued that scholars have perhaps overemphasised the significance of assumptions about the teacher’s mind, claiming that the focus should be on the teacher’s actions and roles in mathematical discourse. He argued that this approach would reveal the artistry of teaching mathematics. Goodchild (2014), on the other hand, has argued that perspectives that do not consider cognitive aspects might not be capable of explaining the art of teaching. Voigt (1994), and others with him, has suggested a shift in focus towards interactions in the classroom and the balancing of social and individual aspects. He demonstrated the benefits of viewing teaching and meaning as something that arise in interactions. If the meaning-making process is viewed as a reciprocal responsibility amongst teachers and students, teaching becomes more than imparting knowledge or giving instruction.

Meaning-making, as a reciprocal responsibility of teachers and students, harmonizes with mathematics as a social practice and suggests that teaching is a dynamic process. Teaching is a transformative practice for both teachers and students, both of whom develop their own approaches to mathematics via participation in classwork. Instead of imparting knowledge, teachers contribute to meaning-making in the classroom, leading the way towards an alignment with mathematical practice. As teachers are continuously developing through their participation—here referred to as a transformative practice—they learn to develop learning (Jaworski, 2006).

**Aims of the thesis**

The overarching aim of this research project, which is comprised of three research articles and this compilation, is to add to the theoretical discourse on teaching mathematics. More precisely, this study seeks to explain teachers’ roles and actions in negotiating the meaning of mathematical objects via discursive and transformative practices. The aim is two-fold; the study seeks to provide insights from practice into how teaching affects the mathematical discourse in the classroom, and it also explores practice-induced teacher change.

The first part, which examines the effect of teaching on the mathematical discourse in the classroom, is significant, because, as Jaworski (2006) argued, the academic discourse on teaching mathematics lacks theoretical perspectives. Introducing an interactive perspective on teaching has the potential to complement the existing tradition of relying on professional knowledge to explain actions and decision-making in the classroom. Hence, the goal is to facilitate a theoretical discussion on teaching, similar to the debate over learning, which has grown considerably over the years.
The second part seeks to provide insights into practice-induced teacher change, as this perspective is missing in the academic literature on teaching mathematics. This section treats teaching as a transformative practice, on the basis of the idea that teachers’ interpretations of the world transform as they teach. This approach has the potential to complement existing ideas and explanatory models, such as that of missed opportunities as a consequence of lacking professional knowledge, or teachers re-engaging in a multitude of prior practices.

**Thesis overview**

This thesis relies on an inductive approach, in which the close relationship with practice is of great importance. For that reason, the thesis begins by summarizing the results, with the goal of immersing readers in the data and the findings. This overview is intended to bring the reader closer to the practice that this study examined. The next section comprises the literature review, which marks the beginning and the end of the analytical work. Previous findings were the starting point for the methodology, and it also added meaning and context to the emerging categories. Next is the methodology section, which provides a sense of context for the analytical process used in this study, as well as a deeper description of the practice from which it emerged. The structure of the compilation mirrors the process of the project, with the theoretical work following the description of the methodology. In other words, the research process was empirically motivated. The goal was to identify explanations grounded in practice, rather than to impose ready-made analytical categories that might increase the distance between the researcher and the data. Instead, theory and theorization were inseparable parts of the inductive approach, and they developed in symbiosis, taking inspiration from the analysis and then contributing ideas capable of enriching the overall process. The final section discusses the results of the three research papers in relation to the wider literature review, and it also reflects on the entire research project.
SUMMARY OF THE RESULTS

This summary of the results covers each publication included in this thesis, and the three papers are presented in chronological order.

- Paper 1 “Introducing a symbolic interactionist approach on teaching mathematics: The case of revoicing as an interactional strategy in the teaching of probability” (Eckert & Nilsson, 2015),
- Paper 2 “Theorizing the interactive nature of teaching mathematics: Contributing to develop contributions as a metaphor for teaching” (Eckert, in press)
- Paper 3 “An emerging framework on Contributing to develop contribution in whole-class mathematics discussions”.

The three papers are connected in that they all analyse a case of interactive experiment-based approach to teaching probability. Specifically, Paper 1 discusses traditional means of analysing the practice of teaching mathematics, and it presents an alternative centred on symbolic interactionism (Blumer, 1986). In this manner, it highlights the interactive nature of teaching mathematics. Paper 2 builds on the perspective developed in Paper 1, and it theorizes teaching mathematics by coordinating symbolic interactionism (Blumer, 1986), learning as contributions (Stetsenko, 2008) and teaching as learning to develop learning (Jaworski, 2006). Paper 3 draws on this foundation and develops a framework for making sense of teaching mathematics. It addresses how teachers’ mathematical contributions in whole-class discussions play a role in the development of students’ contributions and in the development of their own future contributions.

Paper 1

Title: “Introducing a symbolic interactionist approach on teaching mathematics: The case of revoicing as an interactional strategy in the teaching of probability”
The aim of this study was to investigate teachers’ interaction patterns as they negotiated the meaning of experimentally based concepts of probability within their teaching practice. This study and its approach was motivated by an analysis of a teacher who struggled to adapt during a discussion with students about chance and sampling (Eckert & Nilsson, 2013). The initial goal of the project was to examine teachers’ mathematical knowledge for teaching probability, but the results from Eckert and Nilsson (2013) raised the question of whether teacher knowledge was the appropriate tool for understanding these complex situations. The subsequent methodology thus adopted a broader approach to exploring the topic of teaching probability. Paper 1 focused on revoicing as a theoretical case of a teacher action to understand its role in negotiating meaning of probability concepts in interaction with students. Interaction patterns emerged from data, which was generated from a series of classroom lessons on probability. This data was analysed via an inductive approach, using ideas from symbolic interactionism as sensitizing concepts rather than as preconceived analytical categories. We found that the teachers revoiced differently depending on the situation. Their actions seemed to rely on their interpretations of the mathematical object in the specific situation, rather than on any predefined teacher knowledge of the topic.

The analysed lessons were on the topic of probability. The teacher and the students interacted with an unknown sample space and were tasked with negotiating the meaning of fundamental probability concepts (e.g., chance, sample, sample space, relative frequency, and the law of large numbers). As the lesson sequence progressed and the classes discussed both small and large samples of observations from this unknown sample space, an increasing number of these concepts emerged. Paper 1 focused on how they negotiated the meaning of chance. Previous research has demonstrated that chance is an ambiguous concept, without a clear and usable definition suiting students’ level of mathematical knowledge. Colloquial interpretations—such as that chance is an unlikely outcome of an unpredictable process—sometimes clash with formal mathematical understandings—such as that chance refers to the randomness of a sequence. Both teachers’ and students’ interpretations represent important contributions to the process of negotiating the meaning of chance in the classroom.

We observed that the teachers used a discursive action called revoicing to influence the negotiation of meaning. This technique involves re-uttering and paraphrasing students’ mathematical reasoning. Forman, Larreamendy-Joerns, Stein, and Brown (1998) have defined it as a strategy for “shar[ing] the responsibility and authority for explaining and evaluating mathematical problems” (p. 313). They proposed that revoicing might serve to either align or contrast students’ arguments, by highlighting certain aspects of their reasoning. We found that the two teachers conducting the lessons used revoicing in two distinct ways, which we named active and inactive revoicing.
The distinction lay in the degree to which the teacher’s own interpretation was explicit in the interaction and the categorisation depended on how the teachers interpreted each situation.

Inactive revoicings are mere repetitions of a student’s own words. The act gives little or no indication of the teacher’s interpretation or of the reason that he or she has opted to revoice that specific contribution. Even though a teacher interprets the task and the students’ contributions, his or her communication does not reflect that process or its contents. The data demonstrated how the teachers’ use of inactive revoicing effectively terminated that particular strand of negotiation. Although it was not possible to uncover exactly why that was the case, two possible explanations are as follows: (1) the student’s reasoning did not fit the teachers’ intentions in that particular situation, or (2) insufficient subject knowledge impeded the teachers’ ability to assess and actively react. An overview of similar situations and lesson outcomes indicated that it was more likely that the student contributions did not fit into the negotiation that the teachers were trying to create in these instances.

Active revoicings are re-utterances of student contributions with minor alterations on the part of the teachers. These modifications indicate to students the teachers’ interpretations and intentions. Moreover, they have the potential to influence the negotiation of meaning. Examples from the data revealed a range of situations in which active revoicings were used. In some cases, the goal was to influence the negotiation, but in other cases, this was not the objective. In the latter instances, the teachers seemed to interpret the students’ contributions as “harmless” or as obviously out-of-context. They thus actively revoiced to encourage further contributions, guiding the discussion in such a way that the previous contribution did not risk moving the negotiation in a different direction. In contrast, in the former cases, the teachers seemed to view the students’ contributions as productive and as in alignment with their own intentions and interpretations.

When applied in interactions with students, this theoretical concept of revoicing—a teacher action centred on interpretations of probability concepts—provided us with a justification for challenging the dominant perspective that teachers’ successes and shortcomings are products of their mathematical knowledge. Adopting a symbolic interactionist approach enabled us to provide alternative explanations for the teachers’ actions in classroom interactions, and so we were not compelled to merely distinguish between sufficient and insufficient teacher knowledge. We observed how the teachers occasionally switched between alternative interpretations of the mathematical concepts. We also observed how they developed their own use of these concepts during the course of the lesson sequence. Symbolic interactionism also allowed us to view the teaching process as a system of interactions, rather than as a linear model in which expert knowledge was transformed and then transferred. Shulmans’ view of teacher knowledge
characterizes this more traditional approach (Steinbring, 1998). Our theoretical lens permitted us to uncover the interactional and social character of teaching strategies as dynamic and situation-dependent. When teachers use active or inactive teacher strategies, coupled with an interpretation appropriate to the particular interaction, with the goal of negotiating the meaning of a concept, learning opportunities should be viewed as skipped rather than as missed. Skipping of an opportunity then becomes a conscious choice, a result of teacher professionalism. We offered an alternative interpretation: Rather than deeming the teachers in the study as less knowledgeable when they did not follow up on every student utterance, active and inactive teacher actions can be understood as strategies intended to direct the negotiation of meaning towards the teacher’s goal.

**Paper 2**

**Title:** “Theorizing the interactive nature of teaching mathematics: Contributing to develop contributions as a metaphor for teaching”

The second paper was inspired by the perspective on teaching mathematics presented in the first paper and by the observation that the teachers developed their practices during the course of the lesson sequence. This insight sparked an interest in investigating the potential of Joubert’s quote that “to teach is to learn twice over”, or, as Jaworski (2006) phrased it, that teaching is learning to develop learning. Jaworski (2006) also argued that the academic discourse on teaching mathematics was insufficiently theorized as compared to that on learning mathematics. The aim of Paper 2 was thus to further develop the notion of teaching as “learning to develop learning” by theorizing interactional processes in teacher-student interactions in the classroom mathematics discourse. Paper 2 started in an analysis of the use of symbolic interactionism in the existing mathematics education research.

Bauersfeld (1988) and Voigt (1994) have elaborated on the relevance of the interactionism perspective in mathematics education research. Since then, other researchers have highlighted the interactive nature of meaning-making, mathematics as a social practice, and the balance between social and individual aspects of interaction. I identified different traditions of using symbolic interactionism in the mathematics education literature. These approaches differ in terms of how they understand learning, with one school viewing it as the transformation of human doing and another understanding it as the transformation of the person. Sfard (1998) summarized this division via two metaphors for learning: learning as acquisition and learning as participation. In the symbolic interactionism tradition combined with the “learning as acquisition” metaphor understands knowledge as arising in interaction and through the process of negotiating meaning. Participants
acquire knowledge via a process of interpretation. Meanings are thought of as acquired and individual, and as the result of previous interactions. However, even though meanings are viewed as individual, interaction is still possible since we act on the assumption that others interpret situations in a manner similar to how we do, also known as taken-to-be-shared. This tradition relies on an analytical separation between the process of learning mathematics and the social interactions that learning occurs in. On the other hand, in the symbolic interaction tradition combined with the “learning as participation metaphor” takes care not to separate the learning process from the participation in classroom interaction. The focus is on knowing, or becoming able, to participate in mathematical practices. An example of the contrast between the two traditions is the conceptualization of norms. In the acquisitionist tradition, norms create the conditions for learning, shaping a learning environment into which students must fit themselves. In the participationist tradition, alignment with norms is part of the learning process, as students learns to participate in accordance with the norms of mathematical practices.

The literature review on symbolic interactionism in mathematics education research concluded that the learning metaphor was the key to understanding each tradition. That was also true when formulating what it means to teach in each approach. The acquisitionist tradition paints a picture of a complex caretaker who maintains a well-functioning micro-culture in which students have the opportunity to learn. The participationist model envisions an expert participant, a representative of the mathematical community, whose contributions to the social interaction are connected to his or her prior engagements in other (mathematical) practices. Both traditions conceptualize the teacher as a contributor to the classroom learning process. The teacher actively contributes by basing his or her actions concerning the mathematical objects on his or her interpretations. What a contribution is, and its role, was not developed beyond its colloquial meaning in the analysed literature. I argued that this needs to be amended to better understand the process of teaching mathematics within this theoretical frame. Hence, I drew the outlines of a third tradition, for theorizing regarding the interactive nature of teaching mathematics. This approach relies on contributions as the primary metaphor for learning.

Stetsenko (2008) has argued that learning can be understood as contribution to the continuous flow of actions as part of a collaborative purposeful transformation. That is, by contributing to the negotiation of meaning, one transforms both the collective understanding and his or her own view of the object in question. Individuals play an active role since, their actions transform their world, just as the world also transforms them (Stetsenko, 2008). Thus, by contributing to the negotiation of meaning, a person actively transform his or her understanding of prior events.
Subsequently, learning can be viewed as the act of contributing. However, if contributing is learning, this raises the question of what constitutes teaching.

Jaworski (2006) has argued that it is not only meaning-making that is distributed amongst teachers and students but also learning. She has suggested that teaching can be viewed as a process of learning, and more specifically, as the process of learning to develop learning. In terms of the suggested symbolic interactionism perspective, the teaching metaphor was developed as a main result in Paper 2 into contributing to develop contributions (CDC). This framework describes the interactive nature of teaching mathematics via a layered metaphor. On the one hand, it considers the ongoing development of the teacher as she is a part of the learning practice of the mathematics classroom. On the other hand, it signals the active role of the teacher in shaping the classroom mathematical discourse. Two classroom examples were used to exemplify the layers of the metaphor and to indicate the future direction of the framework’s development.

To better understand the active role of the teacher in shaping the classroom mathematical discourse, a teacher’s use of symbols and the manner in which she made her own interpretations available via interactions was analysed. The evaluation yielded preliminary results, and Paper 3 assessed the findings in greater detail. Similar to the analysis presented in Paper 1, the examples demonstrated how a teacher’s actions could potentially influence the negotiation of meaning in the classroom. It also illustrated the interactive nature of meaning-making in the classroom, highlighting that teaching means constantly interpreting the situation and contributing to the negotiation. This analysis, to a greater extent than that performed in Paper 1, indicated how the teacher’s contributions not only influenced the negotiation but also transformed her interpretations of prior events. This highlighted the theorization’s unique contribution to the field of mathematics education research. Namely, to understand the teacher’s role in classroom mathematical discourse, one must take into account how contributing to that discourse transforms the teacher’s own interpretations of the world.

**Paper 3**

Title: “An emerging framework on Contributing to develop contributions in whole-class mathematics discussions”

The third paper is related to the first two papers, as it relied on their insights to create an initial categorization of the CDC framework. It further examined the preliminary classroom data, which primarily served as an example in the first two papers, into a more complete, practice-based inductive analysis. The aim of the paper was to further develop the conceptualization of transformative teaching as contributing to develop contributions to describe and understand
teaching and teacher learning as reflexive processes in relation to in-the-moment teacher-student interaction.

The complexity of teachers’ in-the-moment classroom decision-making has received considerable attention in the literature (Stahnke et al., 2016). Mathematics teachers’ in-the-moment interactions are believed to be influenced by teacher knowledge, beliefs, and goals (Schoenfeld, 1999; Stahnke et al., 2016). Moreover, Stahnke et al. (2016) have argued that teacher knowledge and beliefs (or clusters of beliefs) are predictors of teachers’ situation-specific skills and instructional practices. These factors should not be considered separately, but as a dynamic system with a connection to other resources, such as social and material resources (Schoenfeld, 2011). Mason (2016) has argued that scholars have overemphasized assumptions about teachers’ minds, overlooking their actions and roles within mathematical discourse. Thus, the art of teaching mathematics has not been fully explored. In line with Mason’s (2016) critique, Paper 3 was linked to Paper 1 in that it suggested an alternative approach focusing on the interactional aspects of teaching. Instead of regarding teachers’ decision-making from the point of view of stable teacher knowledge, Eckert and Nilsson (2015) emphasized teachers’ interpretation of the past and present, and thus adopted a more dynamic view. Eckert and Nilsson (2015) argued that focusing on interactional aspects could enrich the literature on teaching mathematics by offering alternative means of conceptualizing the role of the teacher. Eckert (in press) have amended this idea, drawing on Jaworski (2006) to argue that the CDC conceptualization of teaching mathematics is centred on interactional aspects of teaching through which teachers and students develops their contributions continuously. Teaching mathematics is viewed as a dynamic and transformative practice, as is mathematics itself. It is viewed as transformative in the sense that it is subject to continuous development (Stetsenko, 2008).

Paper 3 continues the theorization of CDC that was initiated in Paper 2 with the results of the grounded theory inspired analysis. It also extended the analysis, including interactions with a third teacher that the earlier papers had not considered. The motivation for extending the case was to gain additional insight on the preliminary results from Paper 1 and 2. It provided an opportunity to code more instances, yielding greater insight into the previously coded instances from the first two teachers. A total of three lesson series with three different teachers generated the data analysed in Paper 3.

Paper 3 demonstrated that students’ opportunities to contribute to the negotiation of mathematical meaning were closely linked to teachers’ different ways of contributing. Three analytical categories of teachers’ ways of contributing and their attributes form the CDC framework. That model captures the dynamic nature of teaching and teacher learning in in-the-moment interactions, and it indicates how these interactions support teachers in transforming their understandings of teaching mathematics. The analytical
categories are contributing one’s own interpretation of mathematical objects, contributing others’ interpretations of mathematical objects, and contributing by eliciting contributions. Each action was found to exhibit different degrees of the following attributes: transparency, role-taking, and authority. These actions and attributes were considered in terms of negotiation and transformation processes.

Contributing one’s own interpretation of mathematical objects occurs when teachers describe or argue in favour of their own position during classroom discussions. The data illustrated that the teachers’ own interpretations influenced both the negotiation of meaning and students’ opportunities to contribute. Contributing with others’ interpretations of mathematical objects constituted a more indirect way of influencing the negotiation. By adjusting other students’ utterances and then repeating them in different ways, the teachers highlighted some ideas and downplayed others. Finally, contributing by eliciting contributions included actions intended to encourage active participation on the part of students (i.e., by contributing to the discussion). Examples included the teachers asking questions and probing for ideas from the students. These contributions ensured that multiple perspectives emerged in the discussion and that a dynamic interchange took place between the teachers and the students.

Attributes in the framework formed a link between the teacher’s contributions, his or her mathematical interpretation, students’ contributions, and the negotiation of meaning. They illustrated the reflexivity of the teaching and teacher-learning processes in relation to in-the-moment teacher-student interactions. Transparency meant that an action made the teacher’s interpretation more or less explicit for the students. It, in turn, influenced how the students responded, creating reflexivity between the teacher’s contributions and those of the students. Role-taking indicated that a teacher had framed his or her contribution in a manner familiar to students. The data demonstrated how the teachers used wordings and ideas put forward by the students to increase their probability of correctly interpreting the teacher’s contributions. Moreover, the findings illustrated how the teachers utilized a version of the imagined other—a student—when framing their contributions. The data also indicated that teachers’ contributions had a quality that led to important shifts in the negotiation. That quality, or attribute, was called authority. This authority gave the contributions a higher social status, which was, in turn, linked to their impact on the negotiation of meaning. The teacher could also “lend” that quality to students’ contributions by contributing others’ interpretations. In this way, the teacher indicated that a student’s ideas deserved the same sense of authority.

Paper 2 discussed the CDC metaphor in terms of layers of learning. On the one hand, teachers play a role in developing students’ contributions. On the other hand, teachers themselves learn as they engage with students. Paper 3
developed this notion of layers of learning to include development over time, or transformation. The distinction differentiates between learning and development. Learning is a local process of in-the-moment interaction, through the notion of contributing to the negotiation. By contributing, one is re-interpreting previous interactions in light of present and intended future interactions. Development, in contrast, refers to a global process of transforming one’s understanding of teaching mathematics. This process entails contributing to multiple negotiations, such as the local school’s pedagogical discourse or the national debate on mathematics education. However, in the present paper, this concept is only considered in relation to the classroom mathematical discourse.

Paper 3 adds to the theoretical literature on teaching mathematics by conceptualizing teaching in terms of CDC. In particular, the framework takes into account the dynamic nature of teaching mathematics by viewing teaching and teacher learning as reflexive processes linked to in-the-moment teacher-student interactions. This model represents an alternative to perspectives that rely on pre-determined mental constructs, such as knowledge and beliefs, to explain teachers’ actions or lack thereof. By acknowledging teachers as individuals who learn and develop as the lesson and the semester progress, one can understand their actions in relation to such processes.
LITERATURE REVIEW

This literature review provides a background for the methodology and the results, and it also enriches the analysis. New perspectives, examples and counterexamples help to further define the emerging categories and to contextualize them within the field of mathematics education research. The works presented in this section were identified via the following search criteria: teaching mathematics; teacher decision-making; teaching probability; and teacher’s role in classroom mathematical discourse. The literature has been organized around the assumption that a teacher’s actions have a beginning, a middle, and an end. Teacher knowledge and decision-making mark the beginning. Teaching strategies and actions, as well as teachers’ roles, characterize the middle. The end considers the consequences in terms of student and teacher learning and the evolution of the mathematical discourse.

Teacher knowledge and decision-making

To teach a subject, according to experts on teacher knowledge, more is required than simply extensive subject content knowledge. Early studies demonstrated that the link between teachers’ mathematical proficiencies and students’ results is weak, at best (e.g., Monk, 1994; Mullens, Murnane, & Willett, 1996). Shulman (1986) proposed that subject matter knowledge and pedagogical knowledge do not necessarily have to be considered separately, calling pedagogical content knowledge (PCK) the missing paradigm. Shulman’s team planned to trace the intellectual biography of pre-service teachers during a year-long preparatory course at a university in California. However, as the subjects’ knowledge of teaching grew during the year, the focus changed. The researchers interest shifted to the domains and categories of content knowledge exhibited by the subjects. The major categories of teacher knowledge that the researchers identified were as follows:

- content knowledge;
• general pedagogical knowledge, with special reference to those broad principles and strategies of classroom management and organization that appear to transcend subject matter;
• curriculum knowledge, with particular grasp of the materials and programs that serve as “tools of the trade” for teachers;
• pedagogic content knowledge, the special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding;
• knowledge of learners and their characteristics;
• knowledge of educational contexts, ranging from the workings of the group or classroom, the governance and financing of school districts, to the character of communities and cultures; and
• knowledge of educational ends, purposes, and values, and their philosophical and historical grounds. (Shulman, 1987, p. 8)

These results had a significant impact on the academic discourse on education, redirecting the focus towards teachers to some extent. Sfard (2005) concluded that, “the last few years have been the era of the teacher as the almost uncontested focus of researchers’ attention”, and the increasing number of contributions on that topic at prominent mathematics education conferences indicated that this trend has continued. Teacher knowledge has been elaborated on within the field of mathematics education research, with researchers taking the idea in different directions (e.g., mathematical knowledge for teaching [MKT; Ball & Bass, 2003] and the knowledge quartet [Rowland, Huckstep, & Thwaites, 2005]).

Ball and Bass (2003) asked the question of what mathematical knowledge teachers need to teach effectively during a conference in 2002. The MKT framework has its foundation in Shulman’s work, and it has relied on extensive empirical data from mathematics classrooms to further develop. Specifically, MKT has two main components, subject matter knowledge and pedagogical content knowledge, and both are equally important for teaching mathematics (Ball, Thames, & Phelps, 2008). Subject matter knowledge comprises of three categories, and each of these deals with mathematics as a subject in a manner dictated by the teaching environment. These categories are common content knowledge (CCK), specialized content knowledge (SCK), and horizon content knowledge. Pedagogical content knowledge concerns numerous perspectives on teaching mathematics, and it is similarly comprised of three classes of knowledge: knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum. This categorization suggests that a flexible and adaptable teacher is one who has profound and specialized knowledge of the subject and the students. This foundation permits him or her to act appropriately in any classroom situation. In contrast to surveys on teachers’ subject matter knowledge that were conducted in the 1990s, weak and strong MKT have
been identified as predictors of low-quality and high-quality instruction, respectively (Hill, Umland, Litke, & Kapitula, 2012). The nature of causality was somewhat unclear, however, since the majority of the teachers assessed had midrange MKT scores and illustrated a much larger variation in teaching quality. Several factors are believed to have an effect on both MKT and instructional quality, and these include professional development, curriculum materials, and teacher beliefs. Thus, the relationship between MKT and instructional quality is much less direct (Hill et al., 2008).

Ma (1999) found that one factor that improved teachers’ practice was the depth of knowledge necessary to break down a mathematical idea into comprehensible knowledge packages. Rowland et al. (2005) agreed that a profound knowledge of mathematical principles influenced teachers’ practice, and they also added that teachers’ beliefs about the philosophy of mathematics comprised one dimension of this teacher knowledge. Just as in the case of MKT, Rowland et al. (2005) argued for knowledge components that extend from pure mathematics, and they arranged these elements into the knowledge quartet. They claimed that a profound knowledge of mathematics should be complemented with knowledge of how to sequence a topic, transform content in a way that suits students, and improvise in situations that could not be anticipated in advance. They thus complemented relatively static notions of teachers’ mathematical knowledge with more dynamic abilities aimed at adapting and improving on the basis of the students and the lesson. Also, incorporating teachers’ beliefs as a component of the framework has added a filter between teachers’ knowledge and their actions in the classroom.

Rowland (2014) has argued that mathematics teachers’ foundational knowledge includes their beliefs about the nature and purpose of mathematics and about the conditions under which students best learn mathematics. Skott (2015) explained that mainstream interpretations of beliefs are mental constructs that sift and shape the process of interpreting prior knowledge into a process of enactment in the classroom. These are relatively stable, value-laden constructs that “significantly influence individuals’ perceptions and interpretations” (p. 6). Beliefs have been thought of as a concealed variable, as indicated by the title of the influential work, Beliefs: a hidden variable in mathematics education? (Leder, Pehkonen, & Törner, 2002), with the potential to explain the limited causality between teacher knowledge and enactment in the classroom. Malmivuori (2006) envisioned beliefs within self-systems as encapsulating the functions of affective response, including content knowledge, beliefs, and habitual behavioural patterns. Self-systems derive from prior engagements with mathematics and becomes activated in mathematical situations, influencing a person’s behaviour. As situated, emergent, and activated in social interactions, self-systems challenges “the assumption of a direct causality between beliefs and behaviour” (Skott, 2015, p. 8). The trend towards highlighting the intricate relationships between these
mental constructs has continued (e.g., Eichler & Ereens, 2015; Schoenfeld, 2011). The book From beliefs to dynamic affect systems in mathematics education (Pepin & Roesken-Winter, 2015) aimed to present the state-of-the-art of the field. The volumes title and contents are aligned with the shift towards considering a more dynamic picture of teachers’ decision-making in the classroom, and that approach is based on a system comprised of different mental constructs.

Another recent line of research that has employed the system lens is the documentational approach of didactics. It stems from an interest in relationships between the use of tools and the process of meaning-making (Gueudet, Buteau, Mesa, & Misfeldt, 2014). According to this school, a document originates from a process carried out by the teacher and his or her set of resources (e.g., the curriculum/text, course materials, and personal resources; Pepin, Xu, Trouche, & Wang, 2017). A document consists of joint resources, their usages, and the knowledge guiding those usages. Here, knowledge refers to content knowledge, PCK, and knowledge of the resource. Pepin et al. (2017) used the documentational approach to develop a deeper understanding of mathematics teaching expertise by looking at teachers’ resource systems. On the basis of their results, they suggested that the concept of mathematics teaching expertise should be reconsidered in light of resource systems, which link teaching to the materials used. They thus created an analytical lens offering a dynamic interpretation of pre-established mathematics teaching expertise.

To summarize, early work on teacher knowledge and beliefs relied heavily on the idea of objectifiable knowledge and belief constructs. The trend since then has been to shift from envisioning a causal relationship between knowledge, beliefs, and practice towards adopting a dynamic, processual view of these concepts (Skott, 2015). However, there might be issues with the core characterizations of knowledge and beliefs (i.e., pre-established and relatively stable reifications), and the expected causal relationship with practice (Skott, 2015). Those core characterizations limit the scope of such approaches from a processual perspective.

**Teacher knowledge in the domain of teaching stochastics.**

Stohl (2005) underlined the importance of CCK for teachers. Without proper training, practising teachers might well rely on their own beliefs and intuitions when teaching probability. According to Batanero, Godino, and Roa (2004), most primary teachers in Spain have not received basic training on probability, a problem previously exhibited in a number of other countries, such as Australia (Watson, 2001). Burgess (2006) and Groth (2013) both added to the teacher knowledge discourse, studying what mathematical knowledge for teaching statistics could be and arguing that there are fundamental differences between statistics and other mathematical topics. They found that mathematics
is a largely deterministic discipline, whereas statistics reasons under uncertainty. Random variation influences inference in statistical reasoning, and therefore teachers’ knowledge for teaching statistics should be considered on its own. These two studies resulted in frameworks building on the tradition of PCK (Shulman, 1986), although they also included knowledge categories that took into account the special nature of teaching statistics.

Kvatinsky and Even (2002) presented a theoretically grounded framework for probability-specific teacher knowledge, and they used it to initiate a discourse about what constitutes adequate subject matter knowledge. Although the framework focuses on teachers’ knowledge and understanding about probability, while not addressing knowledge for teaching probability, it points out fundamental content-related aspects of probability. In particular, the seven aspects that Kvatinsky and Even (2002) pointed out were: essential elements, the strength of probability, different representations and models, alternative ways of approaching the topic, basic repertoire, different forms of knowledge and understanding, and knowledge about mathematics. As this model drew on the work of Shulman (1986), it follows the same tradition as the MKT framework, although it is much more specialized with a focus on teaching probability. Essential features include aspects such as reasoning under uncertainty and the objective and subjective perspectives on probability. This category is in alignment with CCK. The strength of probability category refers to probability theory’s role in other mathematical topics and in everyday life, and this concept is related to what the MKT framework calls the mathematical horizon. Different representations and models concern knowledge of multiple ways of representing probabilities, such as Venn diagrams and tree diagrams, and these constitute a type of KCT. Alternative approaches to probability are also an important aspect of teacher knowledge about probability. This category considers the different computational approaches to probability described earlier, and it represents either SCK or CCK, with the procedural element more closely aligned with CCK, and knowledge regarding the benefits of individual approaches a better fit for SCK. The basic repertoire refers to a broad spectre of useful examples that a teacher could use in a lesson, and this, together with knowledge of different representations, constitutes KCT. Different forms of knowledge and understanding includes awareness of multiple means of solving—and of failing to solve—probability problems. According to this understanding, different types of knowledge and understanding are elements of both SCK and KCS, as they cover alternative ways of solving and explaining probability problems, as well as common student mistakes. The last category of the framework is knowledge about mathematics, and it is concerned with the nature of mathematics as a scientific endeavour. For example, knowledge of inductive and deductive reasoning, proofs, and the axiomatic system would fall into this category and could translate into a layer of every category of the MKT framework’s subject-
matter knowledge domain. The model was later used to analyse the mathematics used by two teachers with contrasting teaching approaches. The researchers concluded that there were differences in the content that the teachers made available for the students, and these depended on the teaching approach analysed with the framework (Even & Kvatinsky, 2010).

Nilsson and Lindström (2013) asked teachers to individually solve probability tasks presented in a questionnaire. The instrument was designed to profile Swedish teachers’ CCK and SCK knowledge bases in probability. Although probability has been a part of the Swedish curriculum for some time, the study revealed a limited understanding of probability concepts amongst Swedish teachers, confirming the results of an earlier study focused on American high-school teachers (Liu & Thompson, 2007). The exception was the very basic use of the classical interpretation of probability when dealing with single events. Particular difficulties highlighted in the study were conjunctions and conditional probabilities, as well as experimental data and random variation.

The connection between the results gained by a classical and a frequentist approach was studied by Mojica (2006), and that analysis centred on practicing middle-school teachers in the US. The four participants in the study were involved in a five-year professional development program. They were all enrolled in a mathematics course on data analysis and probability at the graduate level during the study. One of the findings was that the teachers relied heavily on the classical interpretation as an “indicator of truth” (p. 90), rather than on other estimates of probability. The teachers in the study consequently diminished the role of empirical data. They tended to view probability problems in a deterministic way, expecting a correct answer. That perspective made it difficult for them to transfer between classical and frequentist discourses.

Teachers’ professional identity
To a great extent, teacher knowledge and beliefs rely on the research tradition of constructivist learning, focusing on the individual and his or her mental constructs. In an attempt to offer a fundamentally social alternative, Lave and Wenger (1991), Wenger (1998), and Sfard and Prusak (2005), among others, have conceived of identity as the interplay between the context and the individual. Beliefs touch on the subject insofar as they deal with the teacher’s belief in him- or herself, but the research field on identity goes beyond that. Instead of viewing identity as a personal trait, researchers in this domain treat it as something under constant renegotiation in social practice. It is a processual the view of oneself in the world, much like the self (Mead, Morris, Huebner, & Joas, 2015). The individual’s actions should be understood from the point of view of the practice to which he or she belongs. Sfard and Prusak (2005) operationalized identity as “a set of reifying, significant, endorsable
stories about a person” (p. 14). The self is not housed by the individual but is instead a narrative or a social structure. Moreover, positioning oneself is made possible by the generalizing interpretations of others. Participating in a social practice results in a change in how we view our world. Learning does not teach us about the world; rather, it is how we become a member of the world. Therefore, identity and learning can be viewed as two facets of the same phenomenon.

Empirical studies on identity in mathematics education research seem to put one of two aims in the foreground: (1) understanding how one becomes a teacher or (2) understanding the teacher in a teaching situation. Becoming a teacher focuses on identity development. This can occur within pre-service teachers, whose different prior practices during their teacher training shape and mould their tales of themselves. It is also a process for in-service teachers, whose interactions with students and colleagues, or participation in professional development initiatives, affect how they understand themselves in relation to their teaching practice. The research stream that aims to understand teachers in relation to their practices tries to disentangle how prior practices influence a teacher’s decisions and meaning-making processes. I do not claim that the research presented below offers a complete picture of these trends. However, the below discussion does exemplify these conceptualizations of identity.

In the context of Swedish primary-school teacher education, Palmér (2013) has demonstrated how teacher education promoted the formation of a mathematics teacher’s professional identity in the reform classroom. Moreover, the same teachers later distanced themselves from this school of thought, as their professional identities as reform mathematics teachers were not affirmed by their colleagues once they began work. Palmér (2013) studied identity through the lens of patterns of participation (Skott, Larsen, & Østergaard, 2011), in combination with communities of practice (Wenger, 1998). Moreover, identity was conceptualized, in line with Wenger (1998), as three simultaneous processes, the community of practice, identification and negotiation, and membership in communities of practice and ownership of meaning (Palmér, 2013). The community of practice refers to a set of relationships between people, activities, and the world and is characterized by its participants’ shared repertoires and mutual engagement. Identification and negotiation is an endless process of “becoming”. Participating in a community of practice entails constant re-negotiation of the imagined self via a process of identity development. The process of membership and ownership is associated with a more central membership in a community of practice, where the individual is given extended responsibility for, and ownership of, meaning. It is a study of the social in the individual (Palmér, 2013), placing the individual’s becoming in the foreground and the social context in the
background, while emphasizing that the individual cannot be understood apart from his or her social practices.

It is possible, perhaps even unavoidable, to maintain multiple identities. Multiple identities do not necessarily create synergy, and they can actually be in conflict (Gresalfi & Cobb, 2011). Gresalfi and Cobb (2011) studied how middle-school teachers coped with conflicting identities by analyzing their identity development during a professional development initiative. They found that participating in the professional development program motivated the teachers to change their teaching practices. The teachers placed more value on these new practices than on those aligned with the vision of the institution, and they therefore decided that it was worth their while to change their practices. Gresalfi and Cobb (2011) view this as a case of conflicting normative identities. Normative identity concerns what it means to be competent in a particular context, while personal identity concerns the degree to which teachers affiliate themselves with this normative identity (Gresalfi & Cobb, 2011). Their conceptualization of identity formation centers on a process of recognition, meaning that in a given context, individuals are recognized as acting like a certain “kind of person” (Gee, 2001, p. 99). Thereby, these conflicting normative identities reveal how teachers motivate themselves to abide by another vision of high-quality instruction.

Teaching, as well as other interactions, involves re-engageing in multiple prior practices, which, in turn, means shifting versions of the me (Skott, 2013). Skott (2013) has argued that patterns of participation crosscut concepts of knowledge, beliefs, and identity. Patterns of participation helped develop coherent and dynamic understandings of acts and meaning making of a novice teacher. She was passionate about teaching mathematics in alignment with the reform, but her lessons indicated otherwise. Her seemingly incoherent actions were understood as indicating that her “engagement with mathematics [was] overshadowed by her involvement in other practices” (p. 547). Skott (2012) referred to the identity tradition established by Wenger (1998) and Holland, Skinner, Lachicotte, and Cain (1998), emphasizing the processual nature of identity. That school holds that the self is continuously negotiated in relation to practices or figured worlds, and that there is not one self but multiple situated selves. These shifting versions of the self emerge through interactions (Skott, 2013). The novice teacher’s interactions were interpreted via this lens, with the researcher looking for patterns in her contributions and potential linkages with her shifting tales of herself as a professional.
Teaching and the teacher’s role in classroom mathematical discourse

Walshaw and Anthony (2008) concluded their review of mathematics education research, which focused on the role of the teacher in classroom discourse, by stating that for instructional practices to be effective, students must actively participate. However, encouraging students to actively participate in classroom interactions and mathematical discourse requires inclusive norms (e.g., Wood, 2002; Yackel & Cobb, 1996) and the teacher’s implementation of purposeful strategies (e.g., Ding, Li, Piccolo, & Kulm, 2007; Walshaw & Anthony, 2008; Wood, 2002; Woodward & Irwin, 2005).

When a teacher and students interact, they establish socio-mathematical norms, constituting a sort of rulebook describing how one should participate in, and contribute to, a mathematical discourse (Yackel & Cobb, 1996). These norms can indicate what counts as acceptable contributions and whether one is expected to make a contribution in, for example, whole-class discussions. Moreover, norms regarding what counts as a sophisticated and efficient solution can directly influence the development of students’ reasoning, and it is up to the teacher to establish such norms (McClain & Cobb, 2001). Rystedt, Kilhamn, and Helenius (2016) demonstrated that socio-mathematical norms could also work as a resource permitting the teacher to sustain the dynamic flow of an interaction. A teaching strategy may, for example, entail reliance on an established socio-mathematical norm of what constitutes an acceptable answer when introducing a task. Walshaw and Anthony (2008) reported on teachers’ strategies for creating effective discursive interactions that strengthened students’ reasoning skills. They adopted the view that teachers must take an active role in the mathematical discourse by differentiating between students’ contributions and by supporting their claims. This could mean encouraging students to expand on their ideas and make connections (Manouchehri & Enderson, 1999), or simply noticing reasoning so that students can act knowledgably at an appropriate time (Jaworski, 2004; Stockero & Van Zoest, 2013).

Jaworski (1994) analysed the process of teaching mathematics as a set of practices. The result was a theoretical construct, called the teaching triad, which linked three domains of activities in which teachers engaged. Those domains were the management of learning, sensitivity to students, and mathematical challenges. She used the term management of learning, while others have used different words, including guiding, supporting, tutoring, and orchestrating. Conner et al. (2014) referred to this concept as supporting actions in their analysis of teachers’ roles in collective argumentation in mathematics lessons. They established a framework describing teachers’ direct contributions, questions, and other supporting actions during lessons with collective argumentation. Stein, Engle, Smith, and Hughes (2008), as well as
Staples (2007), have also studied teachers’ actions, in terms of inquiry-based instruction and the orchestration of whole-class discussions, in connection to inquiry-based instruction. Stein et al. (2008) described five practices that teachers can adopt to facilitate productive whole-class discussion, and these are anticipating, monitoring, selecting, sequencing, and making connections between student responses. Whereas Stein et al. (2008) highlighted the importance of preparing for student responses before the lesson after having observed teachers in action, Staples (2007) described the guiding nature of a teacher’s actions in practice. She identified three main characteristics of the teachers’ role: supporting students in making contributions, establishing and monitoring a common ground, and guiding mathematics. Staples (2007) adopted a more interactive approach and emphasised the teacher’s role as a guide in the process of negotiating meaning and for joint enterprises marked by student collaboration, whereas Stein et al. (2008) focused on a teacher’s role as an orchestrator of an activity rather than as a participant.

In the literature on teaching mathematics, teacher actions have been studied in terms of posing questions (Franke et al., 2009), telling (Lobato, Clarke, & Ellis, 2005), exemplifying (Kaminski, Sloutsky, & Heckler, 2008), listening (Davis, 1997), and revoicing (Forman, Larreamendy-Joerns, et al., 1998). Posing questions is a teacher action often used in mathematics classrooms. This approach can be applied in different ways, with teachers making use of, for example, open-ended, closed-ended, leading, and rhetorical questions. Franke et al. (2009) concluded that the way in which a teacher poses questions has an effect on students’ learning process, as it can encourage a higher level of detail and explicitness in their explanations.

Telling is an often-discussed and sometimes-criticized form of teacher action, as it does appear to be consistent with the idea that learning is constructed and closely tied to the learner’s experiences (Lobato et al., 2005). Telling is a form of direct instruction in which the teacher expresses to the students in monologue form what they are expected to learn. Lobato et al. (2005) have suggested that telling should be reformulated to meet these criticisms by focusing on the function and conceptual content, rather than on the form and the procedure. Its value should depend on how it relates to other teacher actions.

Examples constitute a widely-employed technique in the field of teaching mathematics, and they can span from the abstract to the very concrete. Gal (2005) has argued for an extended use of examples connected to everyday-life events, so as to make students probability-literate. In contrast, Kaminski et al. (2008) found that students might benefit more from abstract examples rather than from concrete ones, and their findings were based on teaching experiments in which students were exposed to examples of varying degrees of abstractness and concreteness. The students exposed to the abstract examples performed significantly better on transfer tasks than did the other
students. Eckert and Nilsson (2013) also pointed out some pitfalls of using examples from everyday life, as the teacher in their study struggled to keep the students focused on the mathematical content of the example.

Revoicing has the potential to direct students’ participation in classroom discussions (O’Connor & Michaels, 1993). Revoicing involves re-uttering students’ mathematical explanations, and Forman, Mccormick, and Donato (1998) have defined it as a strategy for “shar[ing] the responsibility and authority for explaining and evaluating mathematical problems” (p.313). They reported how teachers overlapped students’ solutions by revoicing when they did not concur with them. Over time, the students came to espouse the same strategies for finding solutions as did the teacher. Forman, Larreamendy-Joerns, et al. (1998) proposed that the purpose of revoicing could be to align or contrast students’ arguments by highlighting certain aspects of their reasoning through expansion, rephrasing, or pure repetition. Adding words or sentences to a student’s utterances is called expansion, and changing some of the words constitutes rephrasing. Finally, repetition is when a teacher articulates an idea in exactly the same words that the student used. Herbel-Eisenmann, Drake, and Cirillo (2009) demonstrated that the aim of revoicing within a lesson context might be significantly more multi-faceted from the teacher perspective than previously reported. The goal of revoicing might be related to how teachers use it to influence classroom interactions. By highlighting certain aspects of students’ arguments and sometimes clarifying them, teachers position students in relation to the content.

Jaworski (2006) pointed out that within this praxis for characterizing teachers’ actions and strategies as a means of understanding teaching, theories of teaching are underdeveloped, especially relative to theories of learning. There is no theory of teaching that compares with the “big” theories of learning, such as constructivism or sociocultural theory. To make sense of teachers’ actions and their roles, one should first address the issue of teaching. She wrote that, “Teaching develops through a learning process in which teachers and others grow into the practices in which they engage” (p. 187). Furthermore, by viewing teachers as critical professionals, she described teaching as a process of learning to develop learning, and learning as teachers’ ongoing critical alignment. This perspective has been used to research teaching communities through collaborative action research (e.g., Edwards & Hensien, 1999; Raymond & Leinebah, 2000), and more recently, to research professional development communities (e.g., Potari, Sakonidis, Chatzigoula, & Manaridis, 2010). The literature has argued for a suitable frame to study a teacher’s own development, as it has regarded both teachers and researchers as members of a community, either a community of practice (Wenger, 1998) or a community of inquiry (Jaworski, 2006). Teachers’ engagements in different communities became the focal point of the analysis of this line of research. As a result, researchers were able to gain insight into how collaboration,
awareness, and reflection on their own practices enabled teachers to develop their teaching practice.

Preciado (2011) has argued that the community perspective on learning to develop learning has mainly focused on interactions amongst that community’s participants—teachers and researchers—rather than on how interactions influence classroom practice. An interesting next step could be to see what could be gained by interpreting teachers’ contributions to classroom interactions as continuous professional development. Meanwhile, research on the teacher’s role is somewhat lacking a perspective on how teaching influences mathematical practices. Questions remain as to the role of the teacher in terms of the development of content in interactions and the subsequent meaning-making processes. Walshaw and Anthony’s (2008) review of studies on the role of the teacher indicated how teachers guide and structure students’ thinking and reasoning. However, these works did not take into account teachers’ own development as they contributed to the negotiation of meaning of mathematical objects.

Teaching consequences

Students learning mathematics is probably the most desired consequence of teaching mathematics. As this thesis argues for the inclusion of teachers’ learning in the conceptualization of teaching mathematics, this section includes previous research on how teachers learn in, and through, their own practices and issues of students learning probability.

Students’ intuition of probabilistic principles

Research on probability education has its roots in educational psychology, and so this section covers relevant results from both the educational psychology literature and the mathematics education literature. The educational psychology tradition has lived on, and most of the prominent studies applicable to this thesis adopted a strong focus on the individual. Learning viewed as individuals acquiring knowledge, has strong ties to the motivation of teaching probability as probability literacy. Gal (2005) has argued that probability education is critical in preparing students for real life, as students are thought to be capable of transferring the knowledge acquired in the classroom to the multitude of situations outside of school that require reasoning under uncertainty.

There are a number of personal heuristics in the field of probability theory that earlier research on educational psychology has identified. The following paragraphs describe a select few that have had an impact on educational research. Researchers believe that they are intuitively formed through experiences of situations involving random events, for example, during games that involve chance (Borovcnik, Bentz, & Kapadia, 1991). Some of these
heuristics might foster a false view of probability estimations, and which is why they have been labelled as misconceptions. It is possible—and indeed quite likely—to possess more than one probability heuristic. These heuristics can even be in conflict with each other if applicable to different situations (Batanero & Sanchez, 2005).

Representativeness. Kahneman and Tversky (1972) set out to explore misleading heuristics in probability. They used a questionnaire comprised of multiple-choice questions, and this survey was administered to a sample of approximately 1,5000 individuals. The sample was comprised of Israeli students, aged 15- to- 18-years old, who were attending college-preparatory high schools. The major finding was that the respondents often judged the probability of an event based on how well the samples in the tasks reflected the characteristics of the entire population. The respondents even expected outcomes in small samples to reflect the distribution of the population. For example, let us use two coin-flip series as our small samples: TTTHTTTT and HTHHHTTHH. In line with Kahneman and Tverski’s (1972) findings, the first outcome would be deemed less likely, because the respondents believed that even a small sample should resemble a larger one. In this case, a more sizable sample would probably display a relatively even distribution of heads and tails. Kahnemann, Slovic, and Tversky (1982) also found that the degree to which the process by which the sample is generated fulfils the subject’s view of a random event was of importance. Again, this finding can be illustrated via coin-flip series: HTHTHTHT and HHTHTTTTH. The first would be deemed less likely, because it looks too ordered and does not represent the unpredictability of the coin-flip process (Kahnemann et al., 1982). An estimation of probability based on either of these two grounds, if it reflects the characteristics of the population or is deemed to have a suitable amount of unpredictability, is called representativeness.

Base-rate fallacy. The base-rate fallacy refers to a tendency to disregard information about the base rate in connected events that will alter the current chance situation, and researchers have demonstrated its prevalence in the wider population (Borovcnik et al., 1991). An example of a task that might generate thinking characterised by the base-rate fallacy is the following:

“A cab was involved in a hit and run accident at night. Two cab companies, the green and the blue, operate in the city. You are given the following data: (a) 85% of the cabs in the city are Green and 15% are Blue. (b) a witness identified the cab as Blue. The court tested the reliability of the witness under the same circumstances that existed on the night of the accident and concluded that the witness correctly identified each one of the two colours 80% of the time and failed 20% of the time. What is the probability that the cab involved in the accident was Blue rather than Green?” (Kahnemann et al., 1982, pp. 156-157)
Among Kahnemann et al.’s (1982) respondents, the most frequent (80%) answer indicated that the participant had neglected the information regarding the number of each type of taxi in the city. This prevalence of this kind of oversight—an example of the base-rate fallacy—depended on how the information was presented. Fewer answers indicated complete neglect of the base-rate when the same problem included base-rate information with a causal quality, for example, when the text read, “85% of cab accidents in the city involve Green cabs and 15% involve Blue cabs” (Kahnemann et al., 1982, p. 157).

The results found that this heuristic played different roles, depending on how the information in the assignment was presented. Cosmides and Tooby (1996) exposed subjects from Stanford University to tasks regarding probability within the context of a medical diagnosis. They used three different scenarios, each involving 25 respondents, in which the data was presented either as probability measures (i.e., percentages or fractions) or as absolute frequencies. The results indicated that the respondents were more successful at solving the absolute-frequency questions than the questions posed with probability measures. The researchers suggested that our ability to properly use our intuition in uncertainty situations and to arrive at correct assumptions is stronger than Kahneman and Tversky (1972) had previously reported. They conclude that the way in which a task is presented plays an important role, and that an absolute frequency representation is more likely to encourage an accurate intuitive approach.

**Equiprobability bias.** When every possible outcome is equally likely, the event is labelled equiprobable. The classical apriori model of probability is based on the notion that equiprobability is the case if nothing indicates otherwise (Borovcnik et al., 1991). An understanding of equiprobability is necessary to comprehend the likely outcomes in many situations. For example, when an individual rolls a single die, each possible outcome is viewed as equally likely on account of the apparent geometric uniformity. Most situations are not equiprobable, however. For example, when we cross a street, we do not have the same chance of crossing safely or being run over by a car. Even though there are only two possible outcomes—being hit by a car or not—we should not consider them as equally likely. To view every event as equally likely and to disregard information that could potentially refute such claim is to hold an equiprobability bias (Lecoutre, 1992).

Equiprobability might be an obstacle to developing one’s intuitive probability awareness. To evaluate risks is to weigh probabilities against each other to make decisions. Fischbein, Nello, and Marino (1991) conducted a study to better understand the nature of such intuitive obstacles. They analysed responses to multiple-choice questionnaires that provided participants with additional space for each item to justify their choice. A total of 618 students, aged 9-to 14-years-old, participated. Of these respondents, 489 had not
completed any prior instruction on probability, and the remaining 130 participants have received some training on the topic. The researchers found that the respondents tended to view “chance” as an equalizing factor instead of correctly evaluating the probability of different outcomes. In questions represented on a specified form, this tendency was more pronounced. Questions asking the students to, for example, specify the likely outcome of a dice roll proved difficult for the students:

"Let us consider the rolling of two dice. Is it more likely to obtain 5 with one die and 6 with the other, or 6 with both dice? Or is the probability the same in both cases?" (Fischbein et al., 1991, p. 532)

This case was viewed as equiprobable by 68% of the students. All of those answers were deemed as incorrect, since none of the students noted in the additional space provided that the pairs were equally likely when the order was considered. In tasks adopting a more generalized form and not including an example, the results were different, however:

"One rolls two dice. Which is more probable: to obtain the same number with both dice, or different numbers?" (Fischbein et al., 1991, p. 536)

When the question was presented without the numbered pair, as in this second example, only 47% of the students gave an answer demonstrating an equiprobability bias. The students appeared to find it easier to imagine a more complete sample space and thus answered accordingly.

Nilsson (2007) encountered a similar behaviour amongst students playing a game involving sums. The students began by viewing all possible sums of two dice as equally probable, their views changed as the game proceeded. This finding undermined the strength of the equiprobability bias, which researchers have deemed to be a very rigid construct that is difficult to influence (Lecoutre, 1992). Nilsson (2007) explained it as a case of individuals struggle to create a complete sample space rather than as a case of equiprobability bias, a distinction not made by Lecoutre (1992) or Fischbein et al. (1991).

Availability. Tversky and Kahneman (1982) identified a heuristic they called availability. It arises when an event that involves elements of uncertainty is interpreted in the terms of a previous event that person has encountered. Availability relies on the strength of our capacity to associate a specific event with prior knowledge of related occasions. The ease with which these outcomes are recalled then represent a basis for judging probabilities (Kahneman & Tversky, 1972). For example, when meeting a new person, if he has glasses and a brown leather belt, I might find it probable that he is a mathematician, since I have a strong memory of a mathematics teacher with these features.

Kahneman and Tversky’s (1973) results suggested that individuals apply this heuristic, even when they have undergone training on probability. Even when the subject has the knowledge to accurately compute the probability of
the outcome, he or she is still likely to give an incorrect answer when asked to intuitively evaluate the situation. Jones, Langrall, Thornton, and Mogill (1999) placed this method of judgment into the lowest category of probabilistic thinking, or level-one thinking (see next paragraph), which is characterized by its highly subjective nature. This approach is associated with students who do not yet have the capability to evaluate uncertain events in a meaningful way. Labelling this technique level-one thinking could be viewed as a criticism of the original studies on the phenomena, acknowledging availability as a knowledge gap on the part of the subjects rather than the result of intuition. Nonetheless, treating subjectivity in this limited way could also be viewed as a shortcoming in the construction of the taxonomy.

**Probability concepts in a school environment**

Jones, Langrall, Thornton, and Mogill (1997) conducted a laboratory study of third-grade pupils who had not received any prior probability instruction. Before the study took place, they created a cognitive framework of probabilistic thinking after synthesizing earlier research on the topic. This framework served to assess different teaching approaches by evaluating students before and after receiving that type of instruction. It could also act as a tool for teachers and others constructing different tasks for use in teaching. Eight children were selected from the group to take part in a longitudinal study seeking to evaluate and further develop this framework. The underlying idea was that understandings of four key concepts (i.e., the sample space, the probability of an event, probability comparisons, and conditional probability) take shape at four distinct levels. These stages were numbered 1-4 and were (in ascending order) the subjective, transitional, informal quantitative, and numerical levels.

Students that exhibit level-one thinking typically make judgments of probability problems based on subjective beliefs, such as lucky numbers, or the misconceptions mentioned above. This type of reasoning should not be mistaken with the formalized version of the subjective perspective, such as Bayesian reasoning. When constructing sample spaces, level-one thinking might result in an incomplete set of outcomes, even in simple one-stage experiments. On the transitional level (level two), students are expected to have a firmer grasp of randomness and to be able to objectively determine more likely and less likely scenarios. Students at this stage correctly identify that they are more likely to choose a white ball at random from a bag containing more white balls than black ones. They should also grasp that the probability of picking a second white ball decreases after the first one has been selected, but they might not realize that the probability of choosing a black ball increases as well. Students that exhibit level-three thinking use quantitative strategies when determining the probability of an event. Instead of evaluating outcomes as more likely or less likely, they quantify the sample
space to handle the problem. Students are expected to successfully generate a sample space for two-stage experiments and to then quantify it. At the most advance level of thinking (level one), students use systematic strategies that enable them to generate sample spaces for up to three-stage experiments, and they are capable of assigning numerical probabilities for both equally likely events and other events. When handling conditional probabilities, level-four students can assign numerical probabilities in both replacement and non-replacement situations.

Effect of instruction. Pratt (2000) studied how students construct internal resources concerning probability when working with a computer-based resource called the Chance-Maker. The students’ internal resources regarding probability appeared to belong to two distinct categories, local and global resources. Global resources provide a broad view of a probabilistic concept, such as predictability, or the fact that some—although not all—cases are equiprobable. A local resource is often the inverse of a global resources. Thus, in this case, it might constitute unpredictability or an equiprobable bias. Constructing global internal resources is the intention of classroom tasks. In short, the goal is to promote knowledge of probability that is generalizable and generally accepted within the realm of mathematics.

The students use their pre-existing internal resources, both local and global, to engage with the computer software, investigating what happened when the repeatedly threw two dice or spun one or two spinners. Encouraging the students to repeat the trials with small variations challenged their local resources and create new global resources. For example, the students had a preconception of fairness (or equiprobability) when encountering chance events. By encouraging them to create spinners with an unequal distribution of fields, Pratt (2000) set the stage for the creation of new global resources, implying that not all chance events can be considered “fair”.

Pratt (2005) suggested four possibilities for the pedagogical implementation of probability in the classroom setting: (1) purpose and utility, (2) tests of personal conjectures, (3) large-scale experiments, and (4) systematic variation. These approaches are meant to challenge students’ local resources and to create new global resources. The below paragraphs describe them in greater detail.

Purpose and utility. One approach to teaching probability is to maintain a strong connection to tasks that are meaningful for the students and to give them an opportunity to grasp the purpose of the theory (Pratt, 2005). Here, each task is set within a context to which students can relate. For example, students are not simply asked to explore probability through dice but are instead asked to create a meaningful context tied to those dice. Nilsson (2007) used the game of totals with two dice to motivate and challenge the students. In this game, the students were asked to place markers on different sums and to compete with each other by removing one marker every time a particular
The competition was used to create a meaningful context for the exercise, and it kept the students’ interest up during the several iterations of the task. Games and gambling approaches have been employed in several settings (e.g., Batanero et al., 2004; Jones et al., 1999; Lannin & Tarr, 2005; Lecoutre, 1992), as they often contain a natural, unmasked element of probability and a motivating setting for students.

However, there are also pitfalls when using a well-known context. Students tend to use local resources, such as representativeness, to a wide extent when they face tasks that have a familiar context (Lannin & Tarr, 2005; Nilsson, 2007; Pratt, 2005; Steinbring, 1991). Moreover, creating a working model that corresponds to reality, and justifying this model mathematically, is also complex from a mathematical point of view. There is no empirical way to definitively prove that a model is a complete representation of reality. Thus, another probability model is needed to prove that the original is likely to produce the most relevant result (Batanero, Henry, & Parzysz, 2005; Borovcnik et al., 1991). Take, for example, a dropped sandwich, and the probability of it landing with the buttered side up. At first, many students might intuitively answer that it would be more likely for the sandwich to land with the buttered side facing down, since that occurrence would stimulate more emotions and be more memorable. To model the event, one might choose to drop coins. One could then assume that it would be equally likely for the coin to land heads-up as for the sandwich to land buttered-side up. To create this probability model, one must make assumptions regarding, for example, the geometric uniformity of the sandwich and the coin. Justifying the model would imply the use of another probability estimation about the underlying suppositions.

Testing personal conjectures. To create opportunities for students to develop global resources, instruction should deliberately challenge students’ preconceptions, creating a cognitive conflict and—by extension—accommodation (Greer, 2001; Shaughnessy, 1992; von Glaserfeld, 1995). One way of creating such opportunities is to manufacture tasks like the game of totals, in which the dice were numbered differently to challenge the students’ conceptions of the sample space (Nilsson, 2007). Another suitable task might employ spinners with uneven field divisions to challenge the concept of equiprobability (Jones et al., 1999; Pratt, 2000), as earlier discussed.

Jones et al. (1999) concluded that challenging students’ misconceptions does not always lead to them accommodating more feasible notions. In fact, 41% of the students in their study exhibited a sample space misconception prior to the 8-week instructional program that provided them with individual mentoring. Moreover, 14% of the participants did not benefit from the instructional program and still demonstrated the same misconceptions after completing it. Similar conclusions, although more qualitative in nature, were
reached by Greer (2001) during his review of the complete works of Efraim Fischbein.

*Large-scale experiments.* Pratt (2000) has advocated for the use of sizable quantities of empirical data from large-scale experiments to teach probability. In Pratt’s (2000) study, the students used a software program that utilized a computer’s ability to generate numerous repetitions of a random event. Students’ local resources were then challenged by the empirical data, and to some extent, the students exchanged their local resources for global ones. Greer (2001), along with Batanero et al. (2005), concurred that using a frequentist approach to teach probability has benefits, but he claimed this technique risks strengthening students’ preconceptions if the instructor asks the students to handle the data on their own. It is beneficial if the analysis of the empirical data takes place after the students have already studied a classical approach to probability theory, as that knowledge allows them to accommodate the conflicting impressions between the data and their preconceptions (Batanero et al., 2005).

*Systematic variation.* Systematically varying the set context of a task or its prerequisites is a way to address probability and to create global resources amongst students (Pratt, 2005). Pratt (2000) used interventions in his work with Chance-Maker, encouraging the students to re-investigate a phenomenon after altering the conditions. Likewise, Nilsson (2007, 2009) used systematic variation, introducing different set of dice into the iterations of the game of totals. In the first round the dice was marked as 111 222 and 111 222, and in the next round, they instead contained the following numbers 222 444 and 333 555. The use of a different sample space in each round stimulated the students to transfer their acquired knowledge between sets and to continuously make adjustments.

**Professional development in teaching mathematics**

The professional development of mathematics teachers can be researched from many different perspectives. Teacher change can mean, for example, a growth in knowledge, a belief change, or identity development. Professional identity development has already been discussed in the text. The following paragraphs thus focus on change in terms of knowledge and beliefs. Sowder (2007) set out to summarize the relevant literature on teachers’ professional development in mathematics, and that analysis focused on 10 questions. Three of these are of special interest in this study, namely: “How do teachers learn what they need to know for teaching mathematics?”, “How do teachers learn from their professional communities about teaching mathematics?”, and “What can be learned from research on teacher change?” (p. 158).

A multitude of studies have addressed formal professional development for mathematics teachers. Sowder (2007) pointed out trends in this regard, emphasizing students’ thinking, the curriculum, and relevant cases as ‘good
examples.’ In the context of probability, Batanero et al. (2004) suggested that a specialized approach to teacher training courses can be appropriate. Such courses should teach teacher candidates how to carry out didactic analyses in the context of probability instruction. Extra emphasis on this type of training could be helpful for teachers at every level, since students must grapple with counterintuitive results when faced with probability tasks, even at elementary levels. If a coin toss, for example, results in tails five times in a row, that outcome does not, contrary to what intuition would suggest, change the probability of obtaining yet another tails in the next flip. Results from Mojica (2006) have indicated that one mathematics course at the graduate level containing probability theory is not sufficient to provide teachers with the necessary teacher knowledge to instruct students in probability. Additional training, or a different type of training, was needed to produce sufficiently strong conceptual knowledge in the practicing teachers, allowing them to navigate the complex relationship between the classical and the frequentist approaches. Sowder (2007) has argued that the academic discourse has emphasized acquiring knowledge-for-practice, or knowledge of mathematics in some form that relates to teaching. This knowledge is thought of as belonging to the mathematical community, and teachers need to search out external resources, such as universities of professional development programs, to acquire it.

In contrast to knowledge-for-practice, knowledge-in-practice is situated within the professional community (Sowder, 2007). It entails creating a shared repertoire with fellow mathematics teachers, as well as developing both practice and professional identity as a community. Examples include establishing communities of practices (Wenger, 1998) for sharing within or between schools and creating lesson study groups, in which groups of teachers develop lessons together by engaging in discussions and classroom observations. Communities of inquiry, discussed earlier, also play a role in how teachers learn how to teach mathematics from their professional communities. Teachers’ inquiries into their practices have the potential to foster meaningful shifts in teaching methods (Butler & Schnellert, 2012). For example, Kazemi and Franke (2004) investigated teachers’ collective examinations of students’ work. They found that the teachers shifted their participation in the collective work as they developed a deeper understanding of students’ thinking. The results demonstrated the significance of using students’ work as a focal point for teachers seeking to better understand their own practices. Knowledge-in-practice puts less focus is on teachers acquiring knowledge and more focus is on social aspects of mathematics teachers’ continuous professional development. Also of note is that knowledge-in-practice is more closely connected to teachers’ own practices. It thus has the potential to be a continuous and integral part of that practice.
In the research literature, teacher change has been described as a process rather than as an event (Sowder, 2007). Liljedahl (2010) presented arguments and examples of rapid and profound change through targeted actions. While Sowder (2007) somewhat contradicted those views, she argued that teacher change should be viewed in terms of growth over time. As the teaching climate shifts, the teacher comes to understand that he or she is not necessarily the sole authority in the classroom, his or her teacher identity develops gradually. It is a transformation of knowing and of ways of acting. As Liljedahl (2010) has pointed out, this change can be the product of targeted and organized professional development. However, it can also result from professional action (Bromme & Tillema, 1995) that contributes to the mathematical classroom discourse. Connecting the academic discourse on professional development back to the teaching metaphor developed in this thesis, teacher change induced by teacher-student interaction is viewed as slow and incremental in contrast to Liljedahl’s (2010) idea of rapid and profound change.

Way forward

Summarizing the thesis so far, we note that there are several frameworks propose that we look towards cognitive aspects, such as teachers’ knowledge, beliefs and goals, to understand teachers’ role and teaching actions in the classroom (e.g., Rowland, 2014; Schoenfeld, 2011; Stahnke et al., 2016). Looking at a teacher’s role and actions in a probability classroom would entail interpreting the teacher’s knowledge of established interpretations and of possible student misconceptions of for instance chance, and how they act on them. Such approach identifies and take into account the special nature of probability concepts in teaching probability (e.g., Burgess, 2006 & Groth, 2013). However, Skott (2013) and Mason (2016) argues that the underpinning assumption, that teacher competence is a ‘thing’, and the search for that thing as an explanatory principle of a situated action is problematic. Instead, recent literature suggests more processual oriented perspectives that takes into account the complexity of acting in the situation without separating the act from the situation.

One way forward is a processual view, where teaching is thought of in terms of arising in the classroom rather than a result of pre-conceived mental structures of the teacher. The research literature on teaching probability is particularly lacking such social, processual perspectives and the literature on teaching mathematics in general is under developed in terms of theory. Taking inspiration from the professional development discourse, there are processual views on teacher change. Conceptualizations of teachers as professionals who develops in their practices in terms of systems of knowledge and beliefs, as well as professional identity, draws the field towards seeking explanations in
processes. Professionals in their practice are thought of as engaged in an endless process of becoming a teacher.

But, as Goodchild (2014) questions, is “a theory of practice that does not clearly embrace concepts of cognition and concept formation is sufficient for the task of analysing the teaching of mathematics” (p.180)? Hence, to understand the teacher’s role in a social, discursive, teaching practice one needs to appreciate the complexity of social interaction in the classroom while maintaining the view of the teacher as an individual who contributes and develops in its own way. The suggested way forward in this thesis is to seek explanations in the practice, taking care not to separate teachers’ acts from their situations. It promises to offer a complementary perspective on teaching compared to the dominant body of academic literature on teaching mathematics.
METHODOLOGY

This qualitative study sought to arrive at a deep understanding of a mathematics teacher’s role and actions. To that end, it explored and investigated the empirical world (Blumer, 1986). I argue in the following sections that the research design enabled me to construct a theoretical framework with the potential to explain a teacher’s role and actions in negotiating the meaning of mathematical objects in experiment-based mathematics teaching and learning practices. The methodology was inspired by a constructed grounded theory approach (Charmaz, 2006), with data from a case study. Video recordings of actual lessons, along with earlier research findings, generated the rich data needed for such this approach, and its implications are discussed in the following sections.

Case studies

To generate data about teachers’ roles and actions in negotiating the meaning of mathematical objects in experiment-based discursive practices in mathematics, active practitioners were examined via case studies. This study defined a case study as an in-depth investigation of a single social phenomenon—teaching probability—using qualitative research methods (Platt, 2007). This technique has the potential to yield meaningful insights on the complexity of human behaviour and a nuanced picture of reality, incorporating the wealth of detailed relationships between the researcher and his or her subjects (Flyvbjerg, 2006). Moreover, this approach also offers a suitable opportunity to develop the skills needed to perform high-quality research (Flyvbjerg, 2006).

Flyvbjerg (2006) has argued that a strategy for selecting a case should coincide with one’s ideas and expectations of the generalizable knowledge to be produced through that case. Instead of relying on the representativeness of a case, one should select a case that would be likely to produce rich data (Clarke, 2007) capturing the complexity of interacting with students in the
classroom. In this case, the goal was to create a generalizable initial theory of what it means to teach mathematics, focusing on teacher-student interactions and classroom meaning-making. Flyvbjerg (2006) identified different approaches of information-oriented case selection to reach this, and the method employed in this study bore the nearest resemblance to the critical case approach. Probability theory, in its applied form in school mathematics, is closely connected to our decision-making heuristics and biases (Kahnemann et al., 1982). I argue that teaching probability constitutes a critical and potentially rich case of teaching mathematics, when viewing meaning-making as a social phenomenon, since it evokes numerous interpretations, decisions, and contributions in the classroom. In this study, the heuristics and biases connected to one’s understanding of probability and randomness came to the foreground, putting to the test teachers’ strategies for making use of their own contributions and students’ contributions to shape the meaning of mathematical concepts. The case was critical in the sense that the wide range of mathematics teachers’ decision-making process, and the dynamics at play within them, were revealed in the data. Specifically, the teachers’ actions in regard to the students highlighted these factors.

The present study used lesson sequences involving experiment-based instruction in probability, resulting in ample opportunities for teachers and students to discuss mathematics. The reform emphasises mathematics as a collective undertaking, in which individuals work together to investigate, explore, and solve problems. The purpose of the classroom observations was to shed light on the object of study (i.e., teachers’ roles and actions) and the unit of analysis (i.e., teachers’ social interactions within their classroom practice). The data was generated through examining a case comprised of three lesson sequences involving experiment-based probability instruction, as described above. The first two lesson series were conducted by two teachers who had little experience in teaching probability, while the second was facilitated by a teacher who had taught probability theory several times. All three teachers shared in common that they organized their lessons in a manner aligned with the reform. The two teachers who conducted the first two lesson series worked at the same school in a medium-sized city in Sweden. This institution was a primary school that provided instruction for students through the sixth grade, which is the seventh year of schooling for most Swedish students. They had both taught there for several years and were experienced teachers. Neither had any experience in teaching probability, as this topic is traditionally covered in high grades, although in this school’s curriculum younger students also explored the subject (Skolverket, 2011). They were both generalists, meaning that they worked with most subjects, including mathematics, with their respective students. Moreover, they were both highly motivated and had initially contacted my supervisor for ideas about teaching probability, and this contact became the starting point of the collaboration.
Their cohorts of students were in fifth and sixth grade, 22 and 24 students in total. This project was these students’ first explicit encounter with probability theory within a school context.

The case was extended, with the intention of theoretical sampling (Strauss & Corbin, 1997), and an additional teacher was chosen because he had several years of experience with teaching probability in primary schools. The intention was to generate data to further develop the framework, as well as to shed additional light on the already-coded data. The primary criterion for extending the case was whether the teacher had similar traits as the participants in terms of continuous professional development. This teacher worked at a primary school in a small city in Sweden with students in grades six through nine. He was a specialist, meaning that he only taught mathematics and physics. Like the other teachers, he was highly motivated and interested in new teaching ideas, and he attended professional development courses as often as possible. He recommended that our study focus on an eighth-grade class that he was also mentoring, as he had close contact with these students and knew them quite well. In previous years, these students had encountered the classical model of probability and applied it in gaming situations, such as those involving dice.

Data was generated from video-recorded classroom observations of the three teachers, each of whom conducted 5–6 experiment-based probability lessons. The data from the classroom observations captured the complexity of the classroom and the phenomenon of teaching mathematics. To triangulate the phenomena in the classroom, the researchers together with the teachers also held meetings between lessons to discuss their outcomes and plan for the next one. These discussions between the lessons were primarily viewed as background information, since they did not involve the classroom’s social setting and interactions. However, they did add the practitioners’ perspectives on the interactions, and they also enriched the researchers’ interpretations. The practitioners and the researchers worked together to triangulate the study’s object, as each analyse the situation based on their experience. In 2013, these meetings included two teachers and the research team (my supervisor and myself), while in 2015, only one teacher and myself were involved. All of the meetings were video-recorded for easy recollection, but this material was not transcribed in full or submitted to in-depth analysis, as was the case for the materials from the lessons themselves.

The lessons

The lesson materials were designed to promote teacher-student interactions and teacher decision-making, and they were inspired by a task created by Brousseau, Brousseau, and Warfield (2001). They had used a bottle filled with coloured marbles to encourage a discussion on probability. In the present
study, a similar bottle activity formed the basis of a lesson sequence offering opportunities to discuss chance and the bi-directional relationship between theoretical and experimental probability. Since the lesson sequence was developed during the project, this section begins by describing those elements of the three teachers’ lesson sequences that were identical. It then considers the differences among them.

A bottle containing a number of small, colourful marbles was used as a random generator. In the first lesson, the bottle was opaque. Neither the students nor the teachers knew the contents of the bottles (which was 1 red marble, 4 white ones, and 5 blue ones). When someone turned the bottle, the colour of one ball was revealed, although the ball remained inside the bottle. In this way, we created a constant, unknown sample space. The activity was presented as a competitive race in the first lesson, with three contestants (i.e., the blue, white, and red marbles) on a six-step track on the whiteboard. As one of the three colours was observed on each bottle turn, that colour advanced one step down the track. The race finished when one colour reached the end of the race. The students were asked to predict which colour would be the first to appear six times during each race. Between each race, the teachers provided space for reflection and discussion, and the results of each race remained on the board. The distribution of the sample space was constructed such to result in small differences—or even conflicting results in small samples—between the blue and the white marbles and in a large difference between the red marble and the other colours. The topics that the students discussed in the first lessons led us to structure the upcoming three lessons around a transparent bottle with a known sample space.

When working with the transparent bottle, the students were challenged to reflect on issues of randomness and the role of the sample size, connecting their findings to the outcomes for the transparent bottle. We wanted the students to be able to map the empirical results with the now-known sample space while simultaneously experiencing the effects of chance in both small and large samples. The transparent bottle contained a uniform distribution of 2 red marbles, 2 white ones, and 2 blue ones, and each working group had its own bottle and worksheet (Appendix I). The second lesson’s activities mainly focused on issues of randomness, asking the students to predict results based on the now-visible sample space. The third and fourth lessons used the same transparent bottle and distribution and focused on issues related to the sample size, aiming to establish the bi-directional relationship between the sample space and experimental probability. The students continued to work in small groups of 2-3 students, each with its own bottle and worksheet (Appendix II). The students returned to the opaque bottle in the last lesson(s).

In the last lesson(s), the students were supposed to use their new understanding of the bi-directional relationship between the sample space and experimental probability to infer the contents (the sample space) of the opaque...
bottle. The same unknown distribution from the first lesson was again used. Now, however, each working group had its own opaque bottle, enabling the class to create a large, joint sample from which it could infer the contents of the bottle.

There were three main differences in the three lesson sequences. Firstly, in fifth grade, the last lesson—the one in which the students returned to the opaque bottle—was split into two lessons for reasons related to time management. Secondly, in grade eight, the teacher inserted an extra, self-developed lesson between lessons 1 and 2. It focused on important concepts within probability. The students were asked to pair concepts with statements. As the students used preconceived explanations, which the teacher had produced, it is reasonable to assume that his interpretations had a major impact on the students’ future contributions to discussions. From an outside perspective, it seemed as if the teacher intended for the students to use and interpret the concepts in a certain way. Thirdly, in grades five and six, the teachers worked with graphic representations during lessons 3 and 4, creating bar charts of different samples and joint samples. In contrast, the eighth-grade teacher worked exclusively with numerical representations. Again, it is reasonable to believe that the eighth-grade teacher’s choices had a major impact on how the students negotiated meaning, as they were not exposed to the same symbols as the students in grades five and six (i.e., the bar charts). Perhaps their previous experiences of mathematics education could have enabled the eighth-grade students to interpret the ratios in the small and large samples similarly to the fifth and sixth grade students and teachers, who used the symbols in their reasoning.

**Paradigm**

This study is positioned within the interpretative paradigm, which assumes that “human behaviour and human learning are responsive to a context that is interpreted by participants” (M. A. Eisenhart, 1988, p. 101). This positioning influenced both the study design and the research questions. The basic assumption underlying this work is that our reality is socially constructed, influenced by both culture and history (Carr & Kemmis, 1986). It makes no sense to study meanings, actions, contexts, or situations separately, since they are intrinsically linked and cannot be isolated (M. A. Eisenhart, 1988). This stance was embedded in the research questions of the included papers, in which the focus was on the complexity of teaching practices. Symbolic interactionism framed the underlying theoretical assumptions and provided insight into this complexity. However, the research design was not merely interpretative, since it also intended to further develop practice (Goodchild, 2014). Even though the scientific objective was not to explicitly develop practice, it was indeed the goal of the participating teachers. As their
interactions with the researchers made them co-constructors of the reality presented in this thesis, their interpretations and aims also influenced the results.

In this view of socially constructed reality, knowledge is subjective, and we cannot separate ourselves from what we know (Ernest, 1991). This means that there cannot be a real separation between the researcher, the object of research, and reality and our knowledge of it. Thus, during data analysis, the researcher and her or his understanding of the world play a vital role in assigning meaning to others’ actions as he or she interacts with the world. We should take into consideration that the outcome—the interpretation of the theoretical framework—was also subjective. According to the symbolic interactionism perspective, objective truths do not exist. Rather, personal meanings are assigned to objects through dialogue and interaction. When making statements about the empirical world, “human beings must see it from their perspective; they must depict it as it appears to them” (Blumer, 1986, p. 22). The analysis of observations thus took the form of a dialogue between the researcher and the object of study. Likewise, conclusions were viewed as subjective truths negotiated within that dialogue. All interpretations of teachers’ roles and actions were therefore viewed as subjective constructs created by the researcher and the subjects, with the cultures and histories of the involved parties influencing the meaning-making process.

There is also an epistemological consideration as regards the written report as a means of communication. This paper seeks to communicate the outcomes of the dialogue between the researcher and the subjects to unknown readers with diverse cultural and historical backgrounds. For this communication to be effective, the reader must take the role of the other (Blumer, 1986). It becomes the writer’s responsibility to convey his or her intentions with the text. For the interaction to be productive, the writer must also be clear about his or her expectations. The reader then forms meanings according to his or her interpretations of the objects and actions pointed out by the writer.

**Analysis**

When developing theory based on empirical data, one is engaged in studying the processes of the social world (Charmaz, 2006). Blumer (1986) described our engagement with the empirical world with the goal of understanding it as science inquiry. He has proposed that we should explore the social lives of humans and that we should investigate the empirical social world. We gain familiarity with the object of study by exploring, shifting perspectives as we proceed so that we can narrow and sharpen our inquiry. When familiar with the object of study, we can directly examine the empirical social world. In this study, that meant exploring teaching and teachers’ symbolic interactions in the classroom. While comparing different interactions, the goal is to allow the
“why” to emerge from our consideration of the “what” and the “how” (Charmaz, 2008b). By constructing theories regarding the meanings that teachers assign to key concepts within probability theory, the ways in which they interpret that process while teaching, and the manner in which they influence meaning-making in the classroom, this analysis moved theory beyond a descriptive framework. The goal was to create a theoretical framework with explanatory properties, a construct of categories and their relations capable of explaining a teacher’s roles and actions in negotiating the meaning of mathematical objects in discursive mathematics practices (the “why”).

This study employed an inductive approach to develop a theory of teachers’ roles and actions in negotiating the meaning of mathematical objects in experiment-based discursive practices in mathematics. It drew on principles from grounded theory (often referred to as a grounded approach). The complexity of the classroom practices provided rich data suitable for generating theory. Although the design of this study did not make use of the extensive, grounded-theory data collection methodologies described by Glaser and Strauss (1967), the principles underlying the data analysis process were based on social interaction and were inspired by Strauss and Corbin (1998). As part of the analytical process, a set of sensitizing concepts (Charmaz, 2008a) was introduced. These sensitizing concepts were developed into a guiding conceptual framework in Paper 2 as a partial step towards the final theoretical framework. This initial conceptual framework was comprised of three main constructs:

- Symbolic interactionism (Blumer, 1986) was used to conceptualize social interactions and meaning-making.
- Stetsenko’s (2008) transformative activist stance was used to conceptualize learning as contributing.
- Previous research on probability education was used to distinguish key mathematical concepts within the case under investigation and the expected interpretations of such principles.

Together, these three constructs guided the analysis, providing a structure for the coding. As such, the methodology relied more on constructivist grounded theory (Charmaz, 2006) and informed grounded theory (Thornberg, 2012) than on the original iteration of grounded theory (Glaser & Strauss, 1967). However, the point is not to label the analysis as a particular type of grounded theory. Rather, the study pragmatically used this systematic approach and its tools. The process of constructing theory from sensitizing concepts (Charmaz, 2008a) is further described in the following paragraphs. The basic idea of this grounded approach was to allow a theory to emerge from the data by starting the analysis processes without pre-constructed coding categories. Instead, open coding sought to capture the richness of the data. This coding created data fragments that could later be connected and further developed in terms of
their properties and dimensions. Ultimately, they were used to generate theory through different techniques (Bryman, 2008). The data generation and analysis process consisted of five main steps: first- and second-level summarizing, transcribing, coding, and generating/refining theory.

Both symbolic interactionism and grounded theory rely on the principles of open (interpretative) inquiry (Blumer, 1986; Charmaz, 2006). The aim was to engage with the data while remaining open as to what could emerge from it. That said, it was not a passive process in which results were expected to emerge by themselves. Rather, the process was interpretative, in line with the research paradigm, with the researcher moving as close to the data as possible during the analysis. The initial interaction with the data, elaborated upon in terms of the first- and second-level summaries (below), was exploratory, and the goal was to remain open to the twists and turns of the analysis. The first-level summary constituted the first step towards reducing the data, and the goal was to create an overview of the data. Together with the second-level summary, that stage sought to reduce the amount data into a more manageable quantity (Miles & Huberman, 1994). Video segments from the lessons were defined by their start and end times. Each clip contained a social interaction, namely, a teacher-student interaction, a teacher-class interaction, or a non-teacher interaction. In these final instances, the teacher appeared to be passive (e.g., student-student interactions, students or teachers fetching material, etc.).

During the second-level summary, the reduction process continued. The next step was to go back and identify potentially interesting segments, and to then isolate them and become familiar with them. These segments were then described in more detail. The segments from the second-level summary were transcribed word-for-word. I transcribed the data myself, and while this process was time-consuming, it brought me closer to the data and was therefore another step in terms of exploring the data (Bryman, 2008).

The transcripts were analysed line-by-line through open coding, and this approach was a suitable choice for the beginning of the study (Strauss & Corbin, 1998). Open coding is a tool within the school of thought of open inquiry. The first step was conceptualizing, and this meant reviewing all of the data, labelling the phenomena, and creating concepts. A phenomenon could take the form of an object or an interaction. They were not predefined but were allowed to emerge from the data. Open inquiry gives the researcher an opportunity to experiment with different ideas. It positions the researcher close to the data but leaves him or her vulnerable at the beginning of the process due to its limited guidance. To help narrow my scientific inquiry, I introduced a set of sensitizing concepts (Charmaz, 2008a) after an initial round of coding. It offered suggestions on where to begin inspecting the data and the emergent concepts, and the existing literature thus guided the analysis without constricting it. The use of sensitizing concepts should not be confused with the use of preconceived analytical categories, however. This tool is in line with
the principles of open inquiry on which grounded theory was founded (Charmaz, 2008a). In this approach, previous research also becomes data that the researcher then interprets in light of the sensitizing concepts to further enrich the results. By drawing on our own ideas, as well as on previous research and the empirical world, we actively interact with the theorizing practice.

The sensitizing concepts of this project helped to conceptualize key concepts, such as interaction, teaching, meaning-making, and probability. Initially, the analysis considered concepts from symbolic interactionism (Blumer, 1986), and these were intended to underscore fine points of teacher-student interactions and probability theory with the goal of better understanding contributions and meaning-making. As work on the second dataset began, concepts from the transformative school (Stetsenko, 2008) were also incorporated to draw attention to aspects of teaching and learning in the classroom. These sensitizing concepts are further discussed in the theory section of this thesis, as well as in Paper 2’s results section.

After concepts had been created through the open-coding of the data, the analysis continued by identifying similarities between the concepts via the process of classification. This created fewer concepts, as ideas were grouped together, meaning that some concepts could be viewed as belonging to a higher order. The goal was to discover categories amongst the concepts that could answer the question of “What is going on here?” The categories were more abstract and explanatory in nature than were the concepts, and they were created on the basis of their distinguishing properties and dimensions. As needed, these categories were broken down into subcategories, properties, and dimensions to answer the questions of “what”, “where”, “how”, and “why”.

The last step of the analysis was a continuous process of generating and refining theory. The first step was to identify the central category (Strauss & Corbin, 1998). All other main categories were related to this central category, and it represented what the researcher interpreted as the core of the study. The first version of this central category was presented in Paper 2 as the CDC teaching metaphor, and it resulted from interpreting the existing research in light of the empirical data. When the central category was identified, the theoretical framework evolved around it, with the main categories, subcategories, and relations all forming. As the theoretical framework took shape, I began to refine the theory. When reviewing the theoretical framework and the relations within it, I began to look for gaps within the logic. I asked myself if categories and relations were missing that would have added to the explanatory power but that had not emerged from the data. The data gathered during the second stage of the data collection process proved useful when refining the theory. During that phase, the main goal was to look for gaps in the framework, to add density to the categories, and to exemplify the theoretical framework in Paper 3.
The methodological approach enabled the researcher to come close to the data during the analysis. The CDC framework is the result of a deep and explorative investigation of the empirical world. The results stayed true to the data by moving back and forth between pieces of data and the theorizing practice. This has ensured the relevance of the framework as a tool for describing and understanding teaching and teacher learning as reflexive processes related to in-the-moment teacher-student interactions.

**Trustworthiness**

Lincoln and Guba (1985) have suggested that trustworthiness can serve as an alternative to reliability and validity when evaluating qualitative research. This is because the aim of social research is not objective truths; rather, it is possible for a case to have multiple interpretations (Miles & Huberman, 1994). Trustworthiness can be judged by four criteria: credibility, transferability, dependability, and confirmability. Confirmability is closely connected to the constant comparison method. By continuously checking and re-checking each emerging category against all of the data, looking for similarities and differences, and trying to confirm or debunk the proposed categories, researchers can achieve confirmability.

Determining the theoretical framework’s credibility was a matter of internally comparing the newly created abstract world of concepts and categories with the raw data to judge the degree of fit (Strauss & Corbin, 1998). In contrast, transferability pertains to the density of the categories and the details of a report’s accounts (Miles & Huberman, 1994). Rich data and descriptions allow readers to take on the role of the researcher and to evaluate the level of transferability in the results. The ability to make a study transparent is also connected to its dependability, which ensures the reader that the context have been carefully described. Credibility, transferability, and dependability are all connected to trustworthiness, as the results are presented to readers in written form, permitting them to interpret them on their own. Erickson (1986) has suggested that writers should invite readers to adopt the role of a co-analyst. In this case, the reader is invited to validate the theoretical framework alongside the writer. The question of trustworthiness then depends on whether the reader judges the paper’s interpretations as plausible and on whether he or she assesses that the theoretical framework as an appropriate fit for the raw data. In a sense, the reader is invited to determine the trustworthiness of the study for himself or herself.


**Ethical considerations**

Conducting research entails the responsibility of ensuring that the work complies with laws, rules, and recommendations for both research in general and the specific discipline. Even if external ethical approval of the research design is granted, it still is the researcher’s own responsibility to enforce internal and external ethical considerations during its execution. It is therefore important for those personal considerations to be based on familiarity with the law, codices, and moral judgements (Hermerén, 2011). I divide ethical considerations into two categories, researcher ethics and research ethics.

Researcher ethics are meant to protect the research community, and they extend beyond this project. Good practice is formed and maintained in large part by the research community itself (Hermerén, 2011). It is through our actions as researchers in mathematics education that we negotiate our ethical norms. It is therefore important to ensure that our research does not challenge those norms by moving them in a negative direction. This project used video recordings to capture classroom interactions. Even though this method represents a norm within mathematics education research, it entails ethical drawbacks that researchers must consider. Hermerén (2011) has argued that this approach should be avoided because of the difficulties of anonymizing the material afterwards. Smedler, Bohlin, Heimann, and Preissler (1996) have also pointed out that the captured material tends to be very personal for the participants because of the many intentional and unintentional details recorded. These factors must be weighed against the necessity of capturing the fine details of classroom interactions to fulfil a study’s goals. This study’s approach sought to ensure minimal access to the data, only using it for research purposes and then storing it safely to minimize any potential negative effects.

Another researcher-based ethical consideration of relevance to this study pertained to the form of publication. Hermerén (2011) has highlighted ‘communism’ as one of the basic principles of sound researcher ethics. It means that the research community and the general public have a right to access the research results. To comply with this principle, one must publish results so that that study’s findings add to the joint efforts of the research community. Both the form and the language should be considered in the context of who might find the results useful. It was against this background that a compilation thesis approach was selected, and the included papers have been (or will soon be) published in relevant international journals.

Research ethics, as opposed to researcher ethics, are meant to address issues regarding the participants of the study. Floyd and Arthur (2012) have further divided this category into internal and external ethical considerations. Internal factors are those related to the relationship between the participants and the researcher, while external factors address technical issues, such as

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anonymity and written consent. This study was situated close to the participants to gain insight into their practices, and it have challenged them to reflect on their practices. Close relationships developed during the course of the study, and these may have provided me with more intimate information than other methods. However, these relationships entailed the risk that I might influence the participants’ practices, both intentionally and unintentionally (Floyd & Arthur, 2012). This posed the risk of influencing my research practice as well. It was therefore important to analyse my own judgements and to ask myself about the reasons behind my interpretations. I thus considered whether my interpretations were logical consequences of the empirical data or whether my relationships with the teachers had affected them. I weighed these issues against the strength of the methods, reaching deeper understandings of the teachers’ practices by immersing myself in the data. In addition to preventing false interpretations, this approach allowed me to honestly consider how the participants and myself might have influenced each other. This paper thus addresses these issues so that readers can judge the results’ trustworthiness on their own.

All participants signified their informed consent prior to the study. The students and their legal guardians gave their written consent, while the teachers gave their oral consent, in line with the Swedish research council’s recommendations (Hermerén, 2011). They were informed about the purpose and scope of the study and could then choose whether they wanted to participate. Miles and Huberman (1994) have highlighted that true informed consent is impossible in a qualitative study because of the nature of the research process. There is no guarantee that the research focus will remain exactly the same during the whole study. Even if it did, there is no way of ensuring that the participants’ consent is based on their complete understanding of the study’s implications or whether it is based on, for example, a sense of obligation towards the researcher. Hermerén (2011) has suggested that the best approach is to offer participants both written and oral information, but Alderson, Morrow, and Alderson (2011) have claimed that children and their parents still find it difficult to grasp the extent of their consent, even with this knowledge. There is no way around this issue but to remember that observations and interviews might be affected if the participants perceive themselves as under-informed and thus seek to safeguard their actions (Miles & Huberman, 1994). Another failsafe approach is to ensure the participants that they have the opportunity to withdraw their consent at any time, no questions asked.

When asking for the participants’ consent, I promised them anonymity. Miles and Huberman (1994) have argued that full anonymity is impossible to achieve. Since it was difficult to anonymize the raw data (i.e., the video recordings), in this study, anonymization was achieved at the level of the written report. The schools, teachers, and students were ensured that they
would not be named or described in a way that could enable readers to identify them. Nonetheless, other teachers working at these schools who knew that their colleagues participated might read the research reports and recognize these individuals. Hence, true anonymization might not be possible (Miles & Huberman, 1994). Instead, this issue was addressed via the practice of positive reporting. The Swedish research council holds that positive reporting might be damaging in some disciples, as with research in which commercial profits might be tied to positive results (Hermerén, 2011). Within educational research, one could view positive reporting as mainly focusing on instances that advance a lesson. In this case, the goal was not to evaluate a teacher’s actions but to try to understand his or her role and actions in context. The description of the practice becomes something that the teacher can recognize and with which he or she can feel comfortable, and that factor balanced the fact that full anonymity and informed consent were impossible to guarantee.
THEORY

This section further elaborates on the theoretical discussion contained within Paper 2 that frames the whole thesis. The section connects to the thesis’ overall objectives by discussing perspectives on teaching and learning in mathematics education research and meaning-making in terms of symbolic interactionism. Jaworski (2006) has placed learning at the heart of teaching by framing teaching as learning to develop learning, and this thesis’ framework has employed the same concept. This section also summarizes the most important mathematical concepts for this work, thus creating a deeper understanding of the teacher’s role in making sense of probability concepts. This background information is essential for understanding both the analysis of the negotiation of meaning in the classroom and the design of the classroom tasks. Towards the end of the section, the conceptualization of teaching that is presented in Paper 2 is contextualized within the overarching methodology of the thesis, situating the results in relation the project’s goals. This section starts by exploring a fundamental concept related to teaching, namely, the question of what constitutes learning.

Learning and development are the primary objectives of teaching. I initially use two metaphors for learning to structure this discussion of the theoretical discourse on learning, namely, learning as acquisition and learning as participation (Sfard, 1998). In essence, these descriptors mean that learning refers either to the acquisition of knowledge by accumulating basic building blocks or to participation in a certain community by becoming able to interact according to the language and norms of said community. The following paragraphs further elaborate on these ideas.

The acquisition of knowledge is perhaps the most common view of learning in everyday discourse. It has also been a dominant view in research, building on the legacy of Piaget. The most prominent frameworks from this school represent different versions of constructivism (e.g., trivial, radical, and social constructivism). Von Glasersfeld (1995), for example, described the acquisition of knowledge from the point of view of radical constructivism, referring to it as a process of assimilating experiences from an activity and
accommodating earlier knowledge constructs to harmonize with new and contradictory experiences. He stated that knowledge resides in the heads of people and that thinking persons have no other alternative other than to construct knowledge based on experiences. Knowledge is viewed as one’s private possession, and it consists of well-defined building blocks, such as concepts, that can be carried over into different contexts (Sfard, 2008). The danger of using acquisition as a metaphor is that it may induce colloquial connotations. If learning is understood as acquisition, knowledge can be thought of as a sort of material, and the human mind is thus reduced to a container that the individual is meant to fill during his or her years of schooling. Sfard (2008b) has argued that this metaphor serves its purpose well, despite the risk that one might place too much emphasis on it, since it captures a broad perspective within a simple concept, while also highlighting its strengths and weakness in relation to the alternative, namely, learning as participation.

Learning as participation means becoming able to take part in existing forms of activity (Sfard, 2008b). Instead of conceiving knowledge as a well-defined entity, the focus shifts towards treating it as an action. This action involves being a member of the community of practitioners, such as the community of mathematicians, while learning is the process of becoming a member of this group. Membership requires the ability to communicate in a fashion accepted by the community and to act according to established and constantly negotiated norms. Sfard (1998) has argued that viewing learning as participation eradicates the issue of how to acquire new knowledge, due the refusal to objectify knowledge in that fashion. It also circumvents the issue of personal knowledge versus community knowledge by opting not to distinguish between internal and external knowledge. However, the issue of dis-objectifying knowledge raises the question of what learning really is. If nothing is carried from one situation to another, it remains unclear what is the motivation for learning. Greeno (1997) has argued in favour of redefining the concept of transfer within the participation metaphor, since this transmission implies that knowledge is objectifiable. Within the participation metaphor, transfer could be rephrased as re-engagement in prior practices, rather than as a matter of participation in various past and present practices. Greeno (1997) has strengthened the concept of knowing as an action by shifting the focus from the transfer of knowledge to improving participation as a means of understanding learning.

Sfard (1998) has highlighted three questions relevant to the acquisition metaphor discourse in relation to participation: (1) How can we want to acquire knowledge of something that is not yet known to us? (2) Why do people construct personal constructs that seems to be fully compatible with others’ constructs? and (3) How do we view research in terms of knowledge as a personal possession? She has argued that the participation metaphor is not
the answer to these questions and that it instead represents an alternative that makes these questions irrelevant. That said, the way forward is not, according to Sfard (1998), to choose one metaphor over the other and to eradicate the other one. Rather, we should make use of both metaphors’ advantages. By only applying the acquisition metaphor in terms of well-defined subject matter in teaching and learning while studying the processes of knowing in terms of participation, we keep the opportunity to explain learning as participation as if there exist elements of objectified knowledge. Since then, many scholars who initially supported finding a balance between the two metaphors have instead signalled their backing of the participation metaphor (e.g., Sfard, 2008b). However, before the discussion moves in front of itself, Cobb and Bauersfeld (1995a) first suggested the use of symbolic interactionism (Blumer, 1986) to coordinate both psychological and social aspects of classroom interactions. This approach developed into the emergent perspective that one could describe as “keeping one leg in each metaphor”.

Meaning – emerging in interaction

Mead (1934) identified two different forms of social interaction, now termed non-symbolic interaction and symbolic interaction. Non-symbolic interaction is when actors respond to actions and gestures (indications of action) without interpretation. Our reflexes are an example of a non-symbolic interaction. Symbolic interactions are the compliment to this idea, and they encompass all social interactions that involve a response shaped by an interpretation of others’ actions and gestures. In this context, others can be individuals, but the term can also take a more generalizable form, such as when interacting with a group or an imagined other (such as when reading a book). Symbolic interactions are the ones of interest to this thesis, as they comprise all classroom interactions that are guided by teachers’ interpretations and that have the potential to influence the negotiation of meaning. Symbolic interactionism relies on three premises (Blumer, 1986):

- “human beings act toward things on the basis of the meanings that the things have for them” (p. 2)
- “the meaning of such things is derived from, or arise out of, the social interaction that one has with one’s fellows” (p. 2)
- “these meanings are handled in, and modified through, an interpretative process” (p. 2)

According to Blumer (1986), people’s actions towards objects depend on how they interpret those objects. An object refers to anything—concrete or abstract—that the mind can name and on which it can focus. Meaning is derived from, modified through, and handled in an interpretative process (Blumer, 1986). People assign and renegotiate meaning by interpreting the actions of other actors. Students assign meaning to mathematical objects by
interpreting their teacher’s and classmates’ actions towards them. These actors, in turn, manage their assigned meanings by constantly re-interpreting them in the interaction. They interpret these assigned meanings by engaging with the object in an internalized symbolic interaction with themselves, and the outcomes become a guide for action. In the case of instruction, the teacher acts toward a mathematical object by engaging in explanatory examples together with the class. The teacher first engages with the mathematical content in a self-interaction in which meanings—perhaps assigned during teacher education—are interpreted within the context of the classroom. The meanings of objects are therefore negotiated through social interaction (Voigt, 1996). The teacher also interprets the other actors (students) in the interaction by taking the role of the other. This role-taking involves an interpretation of how the others might construe the interaction. The result of the self-interaction and the role-taking then determines how the teacher behaves towards this object throughout the explanatory example.

Self-interaction is made possible by the self (Mead, 1934). The separation of the I and the self is the idea of oneself, viewed from by imagined other person, and how others perceive us. Cooley introduced the looking-glass self, and this broadened view of the self includes not only cognition but also emotions (Scheff, 2005). Two examples of this emotional dimension of the self are pride and shame. These emotions seem to be exclusively related to how we imagine others’ perceptions of us. In the process of negotiating meaning, the self plays a role in how we interpret others’ actions and the meanings assigned to objects in relation to those actions. However, it also influences our role in the negotiation and the actions we do and do not take. The looking-glass self provides a frame for explaining this type of positioning (Scheff, 2005). It also influenced this study’s methodology, as it had an effect on the type of data gathered. Namely, the teachers’ actions depended on the individuals with whom they had interacted, implying that the study needed to focus on classroom-based data on teacher-student interactions rather than on, for example, interview data, since the purpose was to better understand teachers’ actions.

Voigt (1996) proposed that classroom interactions, when considered in terms of the negotiation of meaning, represent ongoing joint action. Blumer (1986) has defined joint action as a “societal organization of conduct of different acts of diverse participants” (p. 17). Joint action can be viewed in its own right without breaking it down into separate acts or the group that engages in the joint action. Moreover, joint action includes a formational process, and its participants determine its purpose. It is important to recognize teachers’ contributions to the formation of joint action, the negotiation of meaning, and the fact that their actions and intentions guide the process.

A teacher’s intentions can be attributed to different hierarchical levels, with the curriculum governing one of the highest forms of intentions for classroom
interaction, all the way down to the teacher’s goals for single lessons and student interactions. These intentions—or, as Mead (1934) phrased it, stimuli—are communicated in social acts through gestures. If a teacher seeks confirmation from a particular student regarding his or her interpretation of a concept, that would represent the intention of the interaction. The teacher acts by using language, this becomes a question through the use of vocal gestures (i.e., moving up the tone register at the end of the sentence) and serves as a stimulus for an answer in the interaction and so the negotiation continues. Mead (1934) highlighted the significance of vocal gestures (e.g., raising one’s tone at the end of a sentence to indicate a question). Moreover, in social interactions, gestures can either be physical or embedded in the phrasing of an utterance. Small details give participants in an interaction insight into each other’s intentions and interpretations.

**Meaning of what?**

An object does not have inherent meaning; rather, meaning arises in social interactions (Blumer, 1986). Hence, an object cannot be “mathematical” on its own accord. The term mathematical object is used in this thesis, because I interpret some objects as mathematical and others as not mathematical within certain contexts. Nevertheless, mathematical objects were central to the research project, since when participants interpreted each other’s actions, they also interpreted the meaning of the (mathematical) object on which the interaction was focused (Blumer, 1986). Mathematical concepts are often viewed as objective truths, but Voigt (1996) has suggested that this is not the case in school mathematics. Rather, school mathematics leave room for individual interpretations of mathematical concepts based on, for example, students’ previous experiences. The concept of chance illustrates this ambiguity in school mathematics, since it can be interpreted in many ways. It thus provided numerous opportunities to discuss the various negotiations taking place in the classrooms on which this study focused. Chance can be defined as an informal concept, with chance and randomness essentially synonymous. In Swedish (the native language of the study’s participants) chance and randomness are actually the same word (Ordbok över svenska språket, 1893), which might have further complicated the participants’ attempts to define the concept. This informal concept describes chance as an effect or as an accident of which we do not know the cause—or even if it had a cause (Steinbring, 1993). Chance event is informally defined as the opposite of deterministic events, due to their unpredictable nature. Explanations relying on chance can also substitute for more deterministic justifications when these are deemed too complex. The fact that an outcome is predicted by a measure of probability, rather than by deterministic law, creates a link between the informal concept of chance and probability (Steinbring, 1991). The probability
measure marks the difference between statistical reasoning and reasoning in other domains of mathematics (Groth, 2013). Steinbring (1991) has claimed that this connection means that everyday definitions of chance are synonymous with improbable events and commonplace notions, such as “bad luck”. When chance is treated as a formal concept, part of this informal definition holds true. When it comes to chance as a formal concept, mathematicians have placed much emphasis on the random sequence produced by a series of chance events (Steinbring, 1993). The randomness of a sequence has been defined mathematically based on a general idea from the field of information theory (Shannon, 1948). Steinbring (1993) has argued against engaging in formalized discussions in classroom contexts, instead claiming that the formal concept should be viewed as implicit and open, leaving room for interpretation in exploratory lesson settings.

If we, at least for the duration of this text, accept the existence of chance, events of interest are governed by probability distributions. Every possible outcome, also known as the sample space, then has a natural probability of between 0 and 1. The sum probability of all outcomes in the sample space always equals 1, meaning that the event will surely produce one of the outcomes from the sample space. This concept is formally expressed via the Kolmogorov axioms, and it forms the axiomatic foundation of modern-day probability theory (Borovcnik, Bentz, & Kapadia, 1991). The actual basis of probability theory lies in measure theory, where the probability measure space is constructed out of a triple \((\Omega, \mathcal{F}, P)\) that satisfies the Kolmogorov axioms (Gut, 2005). Specifically, \(\Omega\) is a set called the sample space; \(\mathcal{F}\) is a \(\sigma\)-algebra of sets, namely, the measurable subsets of \(\Omega\) called events; and \(P\) is a probability measure on \(\mathcal{F}\).

The actual probability of an event may be estimated through different approaches, and a method’s suitability depends on the nature of that event (Stohl, 2005). The three main approaches to calculating probability measures encountered in the literature are the classical approach (also known as the theoretical or Laplacean approach), the frequentist approach (also known as the experimental or empirical approach), and the Bayesian approach. They are presented, along with their strong points and shortcomings, in the upcoming paragraphs.

**Frequentist interpretation**

Focusing on the known history of an event is one way of approaching probability estimations. Dividing the absolute frequency of each outcome by the total number of iterations results in a measure of probability called the relative frequency (Borovcnik et al., 1991). Let us assume, for example, that a buttered sandwich is dropped 1,000 times, and in 340 of these instances, it lands with the buttered side up. We could conclude that the relative frequency at which a buttered sandwich lands buttered-side up is 340/1,000, and this
figure can be translated into a measure of probability. Working with relative frequencies as an estimate of probability takes into account possible irregularities but requires a large amount of repeated trials to become reliable.

The law of large numbers provides the theoretical foundation for the frequentist view. It states that the probability of a difference between an event’s experimental probability and actual probability declines to 0 as the number of trials increases (Stohl, 2005). However, a simple computer simulation of a coin toss illustrates that the empirical probability does not necessarily strictly converge towards the expected result (0.5) in large samples every time. Stohl (2005) has provided two examples, each using 7,000 simulated coin throws. In one case, the empirical probability stabilized around 0.5 after approximately 3,000 iterations. However, in the other instance, the empirical probability settled at 0.48 after the same number of tosses. It was not until approximately 6,000 throws that the probability approached 0.5 in the second example, and this case demonstrates some of the complexity involved in understanding and applying the law of large numbers. The frequentist interpretation is often used in cases featuring easy access to large quantity of data, and weather forecasts are one such example.

**Classical interpretation**

The traditional (or classical) way of approaching a probability distribution is by analysing the sample space. According to Laplace, the probability of an event is obtained by dividing the number of favourable events by the total number of events in the sample space (Borovcnik et al., 1991). The classical definition enables one to estimate probability a priori, but it relies on the assumption that all events are equally likely. For example, when using a fair, six-sided die, we could conclude that the probability of obtaining a 4 would be 1/6, since only 1 of the 6 sides displays a 4. The classical interpretation is often applicable in gambling situations, such as the card or dice games mentioned in the example.

**Bayesian interpretation**

The Bayesian approach situates itself as a subjective means of calculating probabilities, as it regards probability as a measure of uncertainty in a person’s degree of belief (Borovcnik & Kapadia, 2014). Therefore, all Bayesian probabilities are conditional, and the probability measure is constantly updated as new information about the event becomes known. The basic tool of Bayesian inference is Bayes’ theorem, which is a rearrangement of conditional probabilities (Borovcnik et al., 1991).

An example of a situation in which Bayes’ theorem is applicable is the famous Monty Hall problem. In that case, a game-show contestant is presented with three doors, but only one of the doors leads to the prize (Selvin et al., 1975). After the contestant has chosen a door, the host opens one of the
remaining doors, revealing that it does not contain the prize. At that point, the contestant is offered the chance to switch doors. With the new knowledge about the option that did not lead to a prize, the probabilities for the remaining two doors have changed. By applying Bayes’ theorem, it becomes clear that it is better to switch doors, since the probability of receiving the prize has increased to 2/3, while the original door only had a 1/3 chance of containing the prize. The Bayesian interpretation is often used in programming self-learning software, as it allows the software to adjust itself according to new information.

The mathematical objects in focus differ in the different papers of this thesis, but all of them played an essential role in student-teacher interactions, and therefore, in interpretations of those interactions. When aiming to explain mathematics teachers’ roles and actions as concerns the negotiation of meaning, one cannot engage in interpreting the teacher’s actions without also interpreting the mathematical meaning indicated by those actions. Thus, mathematics represents a vital component of this study’s theoretical frame, as well as of the analysis and the results.

**Symbolic interactionism in mathematics education research**

Drawing on von Glasersfeld’s definition of learning as self-organization when operationalizing learning in symbolic interactionism, Cobb and Bauersfeld (1995) developed the notion of the emergent perspective. From that perspective, learning is a constructive activity comprising processes of assimilation and accommodation as the individual interacts with others. In the language of symbolic interaction, personal meanings of symbols are formed through the process of interpreting others’ actions in interactions (Yackel, 2001). The concept of personal meanings represents knowledge, and the fact that it is formed through a personal process indicates that the type of knowledge referred to is knowledge-objects that can be acquired. Effective interaction requires participants to utilize common symbols, and all participants must assign these symbols the same meanings and then communicate these through gestures and indications (Blumer & Morrione, 2003). In the case of mathematics education, this could mean that a teacher talks about right-sided triangles while simultaneously roughly sketching a triangle in the air. This interaction would remain effective, as long as the students interpreted the subject to be a generic triangle with one 90° angle.

A complementary interpretation of the concept of common symbols—and therefore, of effective interactions—is the shared definition of the situation. Knowledge is not viewed as shared per-se, but as “taken-to-be-shared”, meaning that teachers and students interact as if they shared a common
definition of the situation (Voigt, 1996). As meaning is negotiated in the classroom, effective interaction can occur when participants act as if the meanings evoked by the symbols in the interaction are shared with the other participants. Consequently, “[l]earning is characterized by the subjective reconstruction of societal means and models through negotiation of meaning in social interaction” (Bauersfeld, 1988, p. 39). Participation in the negotiation of meaning is viewed as crucial for the learning process. Krummheuer (2007) emphasized this facet even more strongly, arguing that learning mathematics depends on students’ participation in communicative processes, such as collective argumentation. The researcher aims at separating analytically between the dynamics of the learning process and that of the interaction process. This approach clarifies the connections between social interaction and mathematical learning.

Skott (2013) has argued the opposite, claiming that social processes and learning processes should not be separated. He stressed the second premise of symbolic interactionism in trying to disentangle shifts in participation in different social practices. Together, symbolic interactionism and a participation perspective on learning mathematics form the patterns of the participation perspective. Instead of differentiating learning processes and participation in the classroom community, Skott (2013) viewed learning mathematics as a process of becoming able to participate in mathematical practices. Thus, learning constitutes a process of self-development in which the learner advances from peripheral participation to fuller types of participation. An example is students developing both mathematical language and an alignment with the norms of the practice. From this viewpoint, learning is a transformation of human action rather than a transformation of humans themselves (Sfard, 2015). The focus is on actions rather than on states of mind, as is the case with symbolic interactionism.

Mead et al. (2015) discussed participation from a purely symbolic interactionist view, emphasizing participation in interactions and participation in the other. For effective interaction to occur, participants need to envision the perspectives of the others, which means participating in the other by visualising that actor’s possible intentions and interpretations. When taking the role of the other, we can establish patterns (Blumer, 1986), which allow us to develop strategies for potentially effective interactions. By envisioning others’ possible interpretations and intentions, we can act in ways that make sense in the eyes of the other participants. If one desires, for example, to interact with a student in a mathematics classroom and to discuss the areas of different geometrical shapes, one can reason according to different geometrical theories. Effective interaction requires the actor to participate in the other to anticipate how that individual’s use of symbols might evoke different interpretations amongst the participants of the interaction. For elementary-school classroom settings, perhaps the most common
interpretation of geometry is the Euclidean one. It would thus make the most sense to reason by using common symbols, such as the triangle, rectangle, and circle. Interaction would probably become less efficient if one participant reasoned with symbols from, for example spherical geometry, without considering that those symbols might evoke other interpretations than those intended.

Voigt (1995) has argued that patterns of interaction are methods of structuring an interaction into themes. They are routines that minimize the risk of the interaction collapsing. Wood (1994), for example, has identified what she calls the focusing pattern. This pattern of asking questions seeks to encourage the student to focus on critical aspects of a problem to help him or her solve it. Skott (2013), on the other hand, viewed patterns of interaction as patterns of participation and as methodological tools to understand interactions in the classroom. Patterns of participation are a way to “understand how a teacher’s interpretations of and contributions to immediate social interaction relate dynamically to her prior engagement in a range of other social practices” (Skott, 2013, p. 549). Whether the focus lies on patterns of actions or patterns of prior engagements, both perspectives highlight the teacher’s contribution to the interaction. Contribution, as a form of active participation, become the centre of attention when trying to understand interactions in the mathematics classroom.

Eckert and Nilsson (2015) used patterns of interaction in a similar fashion as did Voigt (1995), emphasizing the content matter and the role of the teacher in negotiating meaning. They used revoicing as an example of how a teacher’s actions might be understood in the context of negotiation of meaning. As regard patterns of interaction, Eckert and Nilsson (2015) differed from scholars such as Voigt (1995), as they ascribed more authority to teachers’ professionalism as actors in an ever-changing classroom. Instead of describing revoicing as a theme, they focused on the details present in the negotiation of meaning, allowing for different types of revoicing depending on their contributions to the negotiation. The identified differences were connected to the extent to which the teacher’s own intentions and interpretations were made available to the students. According to Mead et al.’s (2015) notion of taking the role of the other, effective interaction depends on the participants’ ability to envision others’ intentions and interpretations. The window to a participant’s intentions and interpretations lies in his or her actions and in the extent to which those factors are clearly visible in those actions.

To summarize, symbolic interactionism has a history of enabling mathematics education researchers to study teacher-student interactions and to gain rich understandings of that complex phenomenon. Whether the study object is viewed as representative of the mathematical community or as a member of a classroom micro-culture, the analysis extends beyond the purely individual into the realm of the social. Previous studies on symbolic
interactionism differ in terms of goals and theoretical foundations Blumer (1986) did not explicitly address the issue of learning. Studies in mathematics education research have therefore, combined Blumer’s (1986) basic concept of meaning-making with a learning perspective, either explicitly or implicitly. These works have employed both metaphors, that of acquisition and that of participation, and some have also sought to combine the two. Each metaphor has a profound impact on how classroom interactions are conceptualized and analysed.

**An alternative metaphor**

The symbolic interactionism perspective views social interaction as a continuous flow of actions and interpretations of these actions (Blumer, 1986), and meanings are negotiated in the process. Stetsenko (2008) has argued that learning can be understood as contributing to the continuous flow of actions as part of a collaborative purposeful transformation. That is, by contributing to the negotiation of meaning, an individual is transforming both the collective understanding of the negotiated objects and his or her own conceptualization of them, and this perspective is in alignment with the third premise of symbolic interactionism. The transformative activist stance suggests that we can view learning as contributing to collaborative practices (Stetsenko, 2008). This reading implies an alternative metaphor of learning, one of contribution, rather than of participation or acquisition. She has argued that this approach is an alternative that combines and transcends individual and social aspects of learning, without the historical connotations of the mainstream metaphors (see, for example, Figure 1 for a side-by-side comparison of the three key points). Stetsenko (2010) has argued that this perspective does not adhere to the dichotomy of the individual and the social, since “contribution is something that individuals do but only as members of their communities who are fully immersed in social collaborative practices” (p. 9). Individuals play an active role, since their actions transform their world, just as the world transforms them (Stetsenko, 2008). The key goal of learning is to adjust our interpretations of previous interactions in light of both the present and the intended future. This means that if we perceive all outcomes as equally likely (e.g., assuming that all totals possible after rolling two dice are equally probable), then interacting with an expert, such as a mathematics teacher, could encourage us to beginning adjusting that initial idea. A teacher might contribute the notion that some sums (e.g., 7) are more probable than others (e.g., 12), because of the number of possible combinations, and this suggestion might enable a novice to interpret past and future interactions with dice in a different manner.

The mathematical content is what makes the symbolic interactions in the mathematics classroom unique. It directs the flow of actions and constitutes a
As Figure 1 demonstrates, and as this section argues, the three different learning metaphors each conceptualize key ideas in their own way. Looking closely, there are also similarities across them, however. Using the last three ideas as an example, one can see how the contribution perspective might represent a middle ground between the acquisitionist and participationist perspectives, although it does not use the vocabularies of either of these two approaches. The question of “Where is the mind?” is the answer neither in the head or in patterns of participation but in continuous flow of transformative action. Actions are representations of the individual and his or her interpretations of the world. The individual’s actions must be viewed in relation to the continuous flow of actions, which represents a contribution to the collaboration. Hence, collaborative practices are enacted and embodied through the contributions of individuals who are fully immersed in that social, collaborative practice (Stetsenko, 2010). In my interpretation and use of the learning metaphor, the goal is to understand learning from both a social and an individual perspective, without separating them as the more traditional metaphors have done.

Theorizing practice Wedege (2010) has postulated that if we acknowledge that the research field of mathematics education studies the relationship between humans and mathematics, and that this relationship has both social and individual dimensions, then multidisciplinarity is at the heart of research. Connecting theories both within the field and outside of it is an important tool for developing new theory. The concept of theory is used in the broadest sense here, as a guide for acting and thinking (Wedege, 2010). Theory represents a dynamic construct in constant flux, as it is shaped by “its core ideas, concepts and norms on the one hand and the practice of researchers—aand mathematics educators in practice—on the other hand” (Prediger, Bikner-Ahsbahs, & Arzarello, 2008, p. 176).

Prediger et al. (2008) have also wrote that connecting theoretical approaches is an appropriate starting point when “developing empirical studies, which allow connecting theoretical approaches in order to gain an increasing explanatory, descriptive, or prescriptive power” (p. 169). In line with the aim of the present project (i.e., to explain a mathematics teacher’s roles and actions), a productive way forward is to connect symbolic interactionism with the transformative activist stance in light of the ongoing empirical analysis.

There are multiple strategies for connecting theoretical approaches, and these range from ignoring other theories to aiming for a grand and globally unifying theory (Prediger et al., 2008). The most common strategies for obtaining a rich understanding of an empirical phenomenon are those of tool within the social practice (Stetsenko, 2010). Agency for social change in the mathematics classroom co-evolves on both the individual level and collaborative level from the foundation of mathematical practices. The transformative activist stance emphasizes human action, and it relies on this co-evolving agency to enable action towards purposeful transformation. The transformative activist stance represents a shift from viewing change as an adaptation to viewing it as a transformation, thus putting an end to the predominant theory that biology is the driving force of human evolution. Rather, biological factors are replaced with history, society, and culture (Stetsenko, 2008). Stetsenko (2010) has argued that in “contrast to the notion of participation, the concept of contribution highlights the fact that human development and learning are contingent on agency and deliberation, purposes and goals, responsibility and commitment” (p. 9). Teachers’ and students’ learning, when seen over time, transforms how they view mathematics in light of the present and the intended future. As the transformation is both meaningful and purposeful, it is seen as directional, and the aim towards which an individual’s mathematical pursuits are moving must be understood in the context of mathematics. The change is purposeful; in the future, it will enable students to contribute in a meaningful way to the established mathematical discourse outside the classroom. Learning mathematics viewed according to the transformative activist stance is a process of becoming human, where activity defined by a goal is the anchoring for development and learning.

<table>
<thead>
<tr>
<th>Key definition of learning</th>
<th>Acquisition</th>
<th>Participation</th>
<th>Contribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Information processing; obtaining knowledge; individual process ‘in the head’</td>
<td>Information processing; obtaining knowledge; individual process ‘in the head’</td>
<td>Participation, i.e. becoming a member of community; the permanence of having gives way to the constant flux of doing</td>
<td>Contributing to collaborative practices of humanity: continuing, while simultaneously transforming them</td>
</tr>
<tr>
<td>Knowledge, concepts, meaning, fact, content; acquisition, internalization, transmission, attainment, accumulation</td>
<td>Knowledge, concepts, meaning, fact, content; acquisition, internalization, transmission, attainment, accumulation</td>
<td>Apprenticeship, situatedness, commonality, cultural embeddedness, discourse, communication, social constructivism, cooperation</td>
<td>Contribution, transformation, history as collaborative practices, cultural tools, vision and directionality, activism and commitment</td>
</tr>
<tr>
<td>The individual mind and what goes into it; test and control of acquisition outcomes</td>
<td>The evolving mind and what goes into it; test and control of acquisition outcomes</td>
<td>The evolving mind and what goes into it; test and control of acquisition outcomes</td>
<td>Diachrosis of continuity and transformation, tradition and innovation, knowledge for and as action; learning-for-change</td>
</tr>
<tr>
<td>Individualized learning</td>
<td>Individualized learning</td>
<td>Individualized learning</td>
<td>Individualized learning</td>
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<tr>
<td>Mentality and community building</td>
<td>Mentality and community building</td>
<td>Mentality and community building</td>
<td>Mentality and community building</td>
</tr>
<tr>
<td>Activity open to collaboration and dialogue; agent of a collaborative change</td>
<td>Activity open to collaboration and dialogue; agent of a collaborative change</td>
<td>Activity open to collaboration and dialogue; agent of a collaborative change</td>
<td>Activity open to collaboration and dialogue; agent of a collaborative change</td>
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<tr>
<td>Collaboratively transforming the past in view of present conditions and future goals</td>
<td>Collaboratively transforming the past in view of present conditions and future goals</td>
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<td>Collaboratively transforming the past in view of present conditions and future goals</td>
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<tr>
<td>Interface of the past, the present, and the future; the past and present are known through positioning vis-a-vis the future</td>
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<td>Interface of the past, the present, and the future; the past and present are known through positioning vis-a-vis the future</td>
<td>Interface of the past, the present, and the future; the past and present are known through positioning vis-a-vis the future</td>
</tr>
<tr>
<td>No agency for social change</td>
<td>Collaborative agency</td>
<td>Co-evolving individual and collaborative agency</td>
<td>Co-evolving individual and collaborative agency</td>
</tr>
<tr>
<td>Individual learner</td>
<td>Community</td>
<td>Learners-through-humanity and humanity-through-learners</td>
<td>Learners-through-humanity and humanity-through-learners</td>
</tr>
<tr>
<td>In the head</td>
<td>In patterns of participation</td>
<td>In continuous flow of transformative action</td>
<td>In continuous flow of transformative action</td>
</tr>
<tr>
<td>Key goals of learning</td>
<td>Key goals of learning</td>
<td>Key goals of learning</td>
<td>Key goals of learning</td>
</tr>
<tr>
<td>Knowledge of facts and skills</td>
<td>Knowledge of facts and skills</td>
<td>Knowledge of facts and skills</td>
<td>Knowledge of facts and skills</td>
</tr>
<tr>
<td>Ability to communicate in the language of community and act according to its norms</td>
<td>Ability to communicate in the language of community and act according to its norms</td>
<td>Ability to communicate in the language of community and act according to its norms</td>
<td>Ability to communicate in the language of community and act according to its norms</td>
</tr>
</tbody>
</table>

Figure 1 Descriptions of the acquisition-based, participation-based, and contribution-based metaphors of learning (Stetsenko, 2008 p.489).
As Figure 1 demonstrates, and as this section argues, the three different learning metaphors each conceptualize key ideas in their own way. Looking closely, there are also similarities across them, however. Using the last three ideas as an example, one can see how the contribution perspective might represent a middle ground between the acquisitionist and participationist perspectives, although it does not use the vocabularies of either of these two approaches. The question of “Where is the mind?” is the answer neither in the head or in patterns of participation but in continuous flow of transformative action. Actions are representations of the individual and his or her interpretations of the world. The individual’s actions must be viewed in relation to the continuous flow of actions, which represents a contribution to the collaboration. Hence, collaborative practices are enacted and embodied through the contributions of individuals who are fully immersed in that social, collaborative practice (Stetsenko, 2010). In my interpretation and use of the learning metaphor, the goal is to understand learning from both a social and an individual perspective, without separating them as the more traditional metaphors have done.

**Theorizing practice**

Wedge (2010) has postulated that if we acknowledge that the research field of mathematics education studies the relationship between humans and mathematics, and that this relationship has both social and individual dimensions, then multi-disciplinarity is at the heart of research. Connecting theories both within the field and outside of it is an important tool for developing new theory. The concept of theory is used in the broadest of sense here, as a guide for acting and thinking (Wedge, 2010). Theory represents a dynamic construct in constant flux, as it is shaped by “its core ideas, concepts and norms on the one hand and the practice of researchers – and mathematics educators in practice – on the other hand” (Prediger, Bikner-Ahsbahs, & Arzarello, 2008, p. 176). Prediger et al. (2008) have also wrote that connecting theoretical approaches is an appropriate starting point when “developing empirical studies, which allow connecting theoretical approaches in order to gain an increasing explanatory, descriptive, or prescriptive power” (p. 169). In line with the aim of the present project (i.e., to explain a mathematics teacher’s roles and actions), a productive way forward is to connect symbolic interactionism with the transformative activist stance in light of the ongoing empirical analysis.

There are multiple strategies for connecting theoretical approaches, and these range from ignoring other theories to aiming for a grand and globally unifying theory (Prediger et al., 2008). The most common strategies for obtaining a rich understanding of an empirical phenomenon are those of
combining and coordinating. Wedge (2010) has explained the difference in terms of whether the constituent theories’ chosen elements are coherent and well-aligned (i.e., coordinating theories) or not (i.e., combining theories). Coordinating theories typically result in conceptual frameworks with sensitizing concepts that guide the analytical process (Eisenhart, 1991; Prediger et al., 2008). Wedge (2010) has claimed that coordinating theories can be the starting point for a research process whose goals extend beyond gaining a better understanding a phenomenon and instead also seek to develop new theoretical frameworks². Integrating symbolic interactionism with the idea of learning as a contribution (see Paper 2) represents an approach from this school.

Wedge (2010) has also highlighted the need for a common core—or basic principles—when coordinating theories. In Paper 2 and the extended analysis that stemmed from that work, the common core consisted of pragmatism (Blumer, 1986; Stetsenko, 2008) and the assumption that meaning arises in social interactions. The concepts that are coordinated are the three premises of symbolic interaction, resulting in the negotiation of meaning of mathematical objects, and the notion of learning as contributions to the continuous flow of transformative action. Symbolic interactionism provides a structure for focusing on individual and social aspects of teacher-student interactions, while simultaneously maintaining the mathematical objects under negotiation as a central and inseparable unit. The transformative activist stance provides a frame for explaining a learning process in the classroom, and separates the individual and social aspects from the connotations of the acquisition and participation metaphors.

Contributing to develop contributions

When coordinating symbolic interactionism and the transformative activist stance, there is a risk of blurring the line between learning and teaching. If learning means contributing to the mathematical discourse, the question arises of what is teaching. Eckert (in press), in line with Jaworski (2006), has argued that teaching is a practice under constant development and that it constitutes learning to develop learning. In this conceptualization, both teachers and students are viewed as learners, but the teacher’s role is to purposefully contribute to the students’ learning. Jaworski (2006) has highlighted the process of becoming members of a community of inquiry through critical alignment but has tended to focus on interactions between community members (teachers and researchers) rather than on those between teachers and students (Preciado, 2011). Eckert (in press) has suggested that a metaphor

² In this context, the term theoretical framework is used in place of the term theory to indicate the lack of methodology tied to the theoretical principles (Wedege, 2010).
more oriented towards the mathematics classroom could be developed around the idea of learning as contributing to the negotiation of meaning. Hence, the metaphor of teaching as learning to develop learning (Jaworski, 2006) can be reformulated in terms of the CDC concept. This metaphor is layered (Eckert, in press); the first tier focuses on the teacher’s processes, indicating that teaching involves both contributing to the negotiation of mathematics in the classroom and transforming the teacher’s own understanding of teaching mathematics. The second layer explains how the teacher influences how students contribute to the negotiation, and so it considers how students interpret and contribute to the continuous flow of actions. During a lesson, a teacher might contribute to the discussion by asking students a question. If the answer is an unexpected one, the teacher might try again after reformulating the question, perhaps making it more direct or providing an example. These small modifications of the teacher’s contribution are treated as related to that teacher’s learning process. Seen over the course of several lessons, the teacher might change his or her way of perceiving the students, his or her approach to contributing to their discussion, or his or her patterns of interactions. These shifts are connected to a transformation in how that teacher views such situations. However, the change is not “one-sided”; as the teacher modifies the pattern of interaction, the collective transforms as well. The process can be thought of as a sort of push/pull between the individual and the collective.
CONCLUDING DISCUSSION

The aim of the thesis was to explain a teacher’s roles and actions when negotiating the meaning of mathematical objects in discursive transformative practices in mathematics. The goal was two-fold: (1) to provide insights from classroom practice on how teaching affects the mathematical discourse in the classroom, and (2) to explain teacher learning as a process of teaching. Three analytical categories describing teachers’ means of contributing and their attributes form the CDC framework and capture the dynamic nature of teaching and teacher learning in in-the-moment interactions. The framework also illustrates how such interactions support teachers in transforming their understandings of teaching mathematics. Figure 2 depicts these analytical categories positions relative to one another. The three small bubbles list the three ways of contributing, and the bullets within their attributes. These categories together position the teacher’s role in the negotiation. The two larger bubbles depict how the processes of negotiation and transformation positions relative to one another.
CONCLUDING DISCUSSION

The aim of the thesis was to explain a teacher's roles and actions when negotiating the meaning of mathematical objects in discursive transformative practices in mathematics. The goal was to provide insights from classroom practice on how teaching affects the mathematical discourse in the classroom, and to explain teacher learning as a process of teaching. Three analytical categories describing teachers' means of contributing and their attributes form the CDC framework and capture the dynamic nature of teaching and teacher learning in in-the-moment interactions. The framework also illustrates how such interactions support teachers in transforming their understandings of teaching mathematics.

Figure 2 The CDC framework and its subcategories, including the relationship between the local process of negotiation in the classroom and the more global process of transformation.
The transcripts presented in Paper 3 illustrate the core categories of how teachers are contributing to develop contributions (Figure, 2). The analysis revealed how teachers constantly adjusted their ways of contributing in relation to the in-the-moment interactions. Constantly reviewing feedback when interacting with students, in line with the reform approach to teaching mathematics, enables the teacher to adapt while working towards a collaborative purpose. The teachers aimed at developing students’ contributions so that they would become aligned with established interpretations of mathematical objects. When teachers contributed their own interpretations, these offerings had a substantial impact on the negotiation of meaning. While eliciting contributions, the teacher created an opportunity to broaden the negotiation via the consideration of multiple perspectives. Eliciting contributions also provided an opportunity for the teacher to reinterpret his or her own contribution in light of the present and the intended future interaction. The three ways of contributing illustrated the reflexive relationship between the individual’s understanding and the collective’s understanding, as the teacher’s contributions emerged in relation to the students’ interpretations and the negotiation of meaning. By contributing to the negotiation, one develops one’s own understanding, as well as that of the group.

Each way of contributing was found to have a set of attributes (Figure 2). However, this list of attributes should not be considered exhaustive. That said, those that were found in the empirical material illustrated the relation between the teacher, the students, and the negotiation. For meaning-making to progress, teachers and students need to contribute to the negotiation of that meaning. The analysis demonstrated that not all contributions had the same impact. The teachers’ interpretations, for example, were more or less explicit when they were contributing a student’s perspective. Repeating a student’s contribution word-by-word provided little insight into why the teacher chose to highlight that response, while actively altering and paraphrasing the student’s contribution generated more insight. The mathematical content was developed through the individuals’ contributions and positioned in the negotiation of meaning through the attributes. Together with the different ways of contributing, these attributes demonstrated that teaching must be understood as a dynamic interplay between the teacher, the students, and the mathematical content.

Contributing to the negotiation of meaning was conceptualized as learning in the framework. Figure 2 illustrates the position of such local learning processes in relation to the global process of transforming one’s understanding of teaching mathematics. As a teacher engage with students and contributes to the negotiation of meaning, he or she receives instant feedback, in terms of interpreting the students’ contributions (or the lack thereof). The analysis
revealed that by teaching, a teacher gains insights into the dynamics of this situation. The teacher also learns how to proceed towards an intended future. These minute adjustments seem to add up over time, and they are connected to how a teacher understands the process of teaching. As such, accounting for processes of negotiation and transformation adds to our understanding of the reflexive relation between teaching, teacher learning, and in-the-moment interaction. Namely, this reading implies that teaching represents a transformative practice, with teaching and learning as two sides of the same coin.

**Implications for research**

The overarching aim of the thesis was to add to the theoretical discourse on teaching mathematics. This section discusses how the thesis’ findings are connected to, and extend, previous research in this field. This evaluation seeks to deepen our understanding of the teacher’s roles and actions when negotiating the meaning of mathematical objects in discursive and transformative mathematical practices. This section also touches on teacher learning, and that discussion is connected to the chapter on theory and the literature review on teacher knowledge, beliefs and professional identity. Moreover, the individual papers did not address this concept, and so it represents a unique contribution of this framing thesis.

**Contributing to the mathematical discourse**

As it currently stands, the CDC framework is comprised of contributions, attributes and processes. Each sub-category of contributions emerged from the data as I sought to make sense of the teacher’s role in meaning-making. These sub-categories can be understood in the context of teachers’ interactional strategies, as indicated in Paper 1. Contributing ones’ own interpretations takes the form of interactional strategies, such as telling (Lobato et al., 2005) and exemplifying (Kaminski et al., 2008). Telling and exemplifying both entail expressing one’s own interpretation of a mathematical object. Telling and exemplifying are transparent in the sense that the teacher is explicit about his or her interpretations. These contributions also tend to have a high level of authority, as they originate from the teacher. Thus, they often have a substantial impact on the negotiation. The analysis presented in the three papers demonstrated that students often latch onto these ideas and develop their own contributions accordingly. This development of contributions is the essence of student learning. The relationship between teachers’ contributions and the negotiation of meaning is not causal, however. All teacher contributions do not necessarily have a significant impact, nor is their effect always the one that the teacher had intended. Instead, the relationship between these variables is reflexive, as the researcher considered how the teacher’s
contribution was interpreted by the collective. The interplay between attributes is one way of more holistically understanding how telling and exemplifying impact a negotiation. The strategies are transparent in the sense that the teacher reveals his or her interpretation, although this categorization raises the question of for whom they are transparent. By taking the role of the other, the teacher envisions the students’ view of the world and formulates his or her contribution accordingly. Since teachers and students negotiate meaning together, and since the negotiation is dependent on both contributions and interpretations, it is beneficial to contribute in a way that makes sense to the others.

Contributing with others’ interpretations of mathematical objects can be understood in the context of the interactional strategy of revoicing. By repeating or paraphrasing a student’s contribution, the teacher re-uses the students’ interpretation in the negotiation (Forman, Mccormick, et al., 1998; O'Connor & Michaels, 1993). The teacher and students share responsibility and authority when revoicing. By revoicing a student’s contribution, the teacher gives that idea authority in the negotiation, and as a result, the subsequent contributions draw on it. O'Connor and Michaels (1993) described this process as “credit[ing] students with teachers’ warranted inferences” (p. 318). When using students’ words and expressions, the teacher assumes the role of the other to contribute in a way that makes sense to them. Revoicing strategies, such as expanding or word-by-word repetition, are suitable examples of role-taking but not necessarily of transparency. Word-by-word repetition reveals very little of the teacher’s interpretations and intentions, whereas expanding means that the teacher adjusts the revoicing according to his or her own interpretations. Hence, this latter approach is more transparent (Eckert & Nilsson, 2015). Paper 1 indicated that the more transparent the revoicing, the more likely it is to have an impact on the negotiation. This result is in line with findings by Forman, Mccormick, et al. (1998), who demonstrated that when a teacher continuously overlapped the students’ contributions with her own, the students eventually employed her explanation.

Contributing by eliciting contributions can be understood in the context of asking questions and probing students’ explanations. The teachers in this study were observed to contribute to the whole-class discussion without mentioning any explicit mathematical interpretations. Rather, they encouraged the students to contribute their interpretations. Probing and asking questions are instructional strategies to support students in formulating more explicit and detailed contributions (Franke et al., 2009). Eliciting contributions represents a negotiation of authority, as the goal it to evenly distribute agency to collect diverse contributions and to encourage students to feel a sense of ownership for the mathematics. This approach attempts to influence the negotiation of meaning through active, interpretative listening (Davis, 1997). It seeks contributions that the teacher could interpret and re-use to develop further.
contributions. The teachers in this study transformed in terms of how they elicited contributions during the course of the lessons. Specifically, they re-interpreted how their way of asking questions were interpreted by their students in a process of role-taking, to elicit contributions that add to their purposeful collaborative transformation.

**Implications of the theoretical framework**

The notion of CDC derived from the idea that teaching is a continuous process of learning to develop students’ learning (Eckert, in press; Jaworski, 2006). In the past, this concept has been pursued in the context of professional development (e.g., Kazemi & Franke, 2004), but this paper employed it as a basic principle for understanding a teaching practice. This interpretation implies that when we teach, we interact with our students and discover ways of helping each individual. Interaction means constantly interpreting others’ actions to figure out their intentions and interpretations (Blumer, 1986). Teaching entails understanding the students and being able to develop their in-the-moment reasoning. Other frameworks (e.g., Ball et al., 2008; Rowland, 2014; Schoenfeld, 2011) have acknowledged this issue and have discussed various knowledge categories that might be capable of capturing this phenomenon. Paper 1 highlighted the interactive nature of teaching by analysing teacher-student interactions and revealing that teaching is not simply the outcome of relatively stable knowledge constructs of teachers’ MKT (Ball et al., 2008). The findings of Paper 1 highlighted the importance and uniqueness of each interaction for a teacher’s decision-making, and other scholars have argued the same (e.g., Skott, 2013). I proposed a dynamic interpretation of what it means to teach, extending Jaworski’s (2006) ideas in Papers 2 and 3. Specifically, I posited that the discrepancy between pre-established knowledge constructs and the situatedness of teacher-student interactions can be understood in terms of teaching as learning to develop learning.

When it comes to teaching mathematics, mathematics education research trends have shifted towards more dynamic interpretations of teaching and the teacher. Frameworks with an acquisitionist approach discuss the interplay among systems of knowledge, categories, and beliefs (e.g., Ball et al., 2008; Rowland, 2014; Schoenfeld, 2011). A full understanding of teachers’ pre-established knowledge and beliefs, the interplay between them, and the possibility of contingency provides one perspective on the teacher’s role in meaning-making. Frameworks with a participationist approach (e.g., Skott, 2013) converge around the processual idea of professional identity. Knowledge of how shifting participation in a multitude of practices affects a teacher’s actions in the classroom has the potential to create a holistic picture of the teacher’s role in meaning-making. This thesis seeks to contribute to that trend and the related theoretical discussion, as it has proposed an alternative
way of understanding this dynamic. The study’s rationale was based on the hypothesis that combining symbolic interactionism (Blumer, 1986) and the transformative stance proposed by Stetsenko (2008) could provide a theoretical frame for an empirically grounded theorization of the dynamics of teaching. The transformative activist stance suggests that we do not merely participate but also contribute to social practices (Stetsenko, 2008). Focusing on contribution rather than on participation highlights that “human development and learning are contingent on agency and deliberation, purposes and goals, responsibility and commitment” (Stetsenko, 2010, p. 9). The transformative activist stance sees participants as active agents who contribute to purposeful collaborative transformation (Stetsenko, 2008).

Placing learning at the core of teaching is not a new idea. The participationist approach suggests a similar interpretation (Lave, 1996). It is one way of considering and explaining the dynamics of the art of teaching. What the CDC framework adds, apart from the proposed contribution and attribute categories, is how to manage direction in the flow of actions in the idea of teaching as learning. Purposeful and collaborative transformation means altering one’s interpretation of the past in view of the present conditions and one’s future goals. It means transforming both one’s individual view of the world and the collective view of the world, moving towards established forms of mathematics—and thus, towards contributions representing accepted interpretations of mathematical objects.

**Teaching – a transformative practice**

As a metaphor, CDC is layered (Eckert, in press). Paper 2 has argued for the layering between the teacher’s learning and the students’ learning, as the first and second part of the metaphor. Paper 3 further developed this notion, suggesting an additional layer—development over time, a process captured by the concept of transformation. Together, negotiation and transformation speaks to the idea of teaching as a transformative practice. By contributing to the negotiation of meaning, one modifies both the collective understanding of the negotiated objects and one’s own understanding of them. Over time, individuals’ actions transform their world, just as the world transforms them (Stetsenko, 2008). As a teacher contributes to the negotiation of a mathematical object, he or she modifies the class’ collective understanding of that object and thereby changes the way in which students contribute. Paper 3 revealed how a teacher develops, over the course of several lessons, as a result of learning from his or her experience in teacher-student interactions.

Teacher development can also be viewed from different perspectives. As discussed in the literature review, mathematics education research has put forward several acquisitionist perspectives on teaching and teacher development. Contingency knowledge (Rowland, 2014) is a potential answer to the question of what do teachers learn as they contribute to develop
contributions. Contingency knowledge means knowing how to respond to the unforeseen and how to react to unfolding situations when working with students. It is reasonable to believe that this type of knowledge, at least to some extent, comes from experience with many of these interactions. Knowledge of content, knowledge of students, and SCK (Ball et al., 2008) are another potential answer, as these forms of knowledge are closely connected to understanding and acting on students’ ideas as they arise in classroom interactions. However, epistemologically speaking, the fit is not perfect. Specifically, CDC relies on the idea that the mind resides in a continuous flow of transformative action (Stetsenko, 2008). Thus, the act of knowing is not viewed in terms of the possession of pre-established and relatively stable knowledge constructs; rather, it constitutes an action.

Stetsenko (2010, p. 12) has claimed that “education is not about acquiring knowledge for the sake of knowing, but an active project of creating one’s identity”. The reflexive relationship between the self and the collective harmonizes with the ideas of Wenger (1998) and Holland et al. (1998), as it emphasizes identity as a process. Identity represents a leading activity in which the self is created and enacted through our actions (Stetsenko & Arievitch, 2004). This conceptualization speaks to the idea that transformation in framework could be viewed in terms of identity development over time. As teachers contribute to the negotiation of meaning, they influence their own views, as well as those of the group. These incremental teacher changes in direct interactions lead to a purposeful transformation over time as regards how the teacher views and understands his or her own practice. Teaching mathematics entails the never-ending, purposeful project of becoming a mathematics teacher.

Implications for practice

This section begins by discussing the framework’s potential applications for practicing mathematics teachers. Then, I address the implications for pre-service and in-service teachers’ professional development.

Teaching implications

The results presented in this thesis are not meant to fundamentally shift how we teach mathematics. Rather, the goal is to influence how we talk about professional competence in terms of the art of teaching mathematics. As Jaworski (2006) has argued, theorization is lacking when it comes to teaching this subject. As compared to learning, conceptualizations of teaching remain relatively informal, as this latter concept has received less attention when it comes to theoretical foundation. I propose that it is equally important to understand the process of teaching as it is to develop a coherent vocabulary for doing so. The development of a shared language is an important piece of the
puzzle in developing our professional identities as mathematics teachers (Wenger, 1998).

One initial aspiration of this project was to develop a vocabulary underscoring teachers’ important role in shaping mathematics in the classroom. I strongly believe that the CDC metaphor, even when understood in a colloquial sense, implies that teachers have an active role in both students’ learning and the process of meaning-making in the classroom. This metaphor has the potential to strengthen the professional identities of mathematics teachers within mathematics education research, as well as to provide insights into the intricate art of teaching mathematics in a classroom. Contributing one’s own interpretations of mathematical objects, as well contributing those of others, highlights the important role of subject matter, as well the teacher’s pivotal role in shaping how mathematics is approached in the classroom.

Reflections on the quality of the thesis

All genuine research strives for quality. Whether this quality is judged according to positivist measures, such as validity and reliability; qualitative
measures, such as trustworthiness; or some other quality criteria, the ultimate goal is to produce high-quality work. I, too, have strived to completely synthesize prior research, to coherently and systematically analyse the results, to present the findings as accurately as possible, and to employ these criteria to produce the best possible thesis. Nonetheless, it is all too easy to find imperfections, especially according to these rigorous, predetermined quality criteria. It is also possible to argue that my work is of high-quality and to defend my systematic approach. Simon (2004) has stated that “[t]he strengthening of research in mathematics education rests not on acceptance of a set of criteria, but rather on a dynamic and ongoing discussion of research quality”. In my view, this quote supports a more holistic view of quality, rather than one that divides the research into easily comparable subcategories. That said, I do not claim that sets of quality criteria are unnecessary in research. Rather, they are put to more use as tools during the research process rather than as a means of judging the final result. It is the researcher’s responsibility to make sure that all aspects of the study have been justified and to provide a coherent chain of reasoning throughout the report. In the end, the question that matters is whether the study has advanced our knowledge of teaching mathematics in mathematics education research.

To attend to the question of quality, I will consider this research project from the perspective of a learning process. Adopting this lens, one might interpret the research via the theoretical perspective on learning used in the thesis, namely, that learning means contributing to collaborative and purposeful transformation. Thus, as I have contributed to the field of mathematics education research, I have advanced my own view of teaching mathematics, as well as that of the community. This endeavour has truly resulted in a purposeful and collaborative transformation in the sense that the thesis’ purpose is subject to continuous negotiation within the research community. The study began with an authentic inquiry (Simon, 2004), the goal of disentangling the sometimes seemingly incoherent acts of mathematics teachers in the classroom. This aim has since then been re-negotiated in interaction with my peers, both in person and through their academic work. As I collected and analysed the data, I attempted to justify my results through transparent descriptions of the reflexive process of categorization that led to the final framework. The choice fell on writing a compilation, making my ideas explicit and open to interpretation and criticism in my process. I invite the reader to take the role of the other, the role of the researcher, as well as to agree or disagree with my results, and to that end, I have included as many transcripts as possible. My ultimate goal has to been contribute to the field as transparently as possible. As concerns the results, I have argued that the principles presented in thesis are generalizable. To generate findings with relevance within the field, I have sought to go beyond mere descriptions of the situations that arose during the study (Simon, 2004). I now invite my readers,
after having reviewed this work in full, to make a final judgement regarding the quality of this study’s contributions to the theoretical discourse on teaching mathematics.

Further research

During the time it takes to complete a PhD programme, several interesting side-tracks can emerge. Even as I write these final lines of the thesis, I wish that I had more time to explore. This section elaborates on some of these ideas for additional research.

First and foremost, a data-driven analysis, such as that presented in this thesis, does not come to an end. The CDC framework is not a finished product; rather, it is part of the process of theorization. In this thesis, we have viewed the meaning-making process in the classroom as a negotiation, and I suggest that the CDC framework should be treated in the same manner. It is now up to future projects, including those of my own, to re-negotiate the interpretations of the framework’s categories, digging deeper into the role of the teacher in classroom meaning-making.

One idea that this thesis was unable to explore is the role of different types of symbols (Blumer, 1986) or cultural tools (Stetsenko, 2008) in theorization. The CDC framework’s current categories emerged from an analysis in which language constituted the main tool. The question remains open as to what would happen if a different perspective were adopted, such as examining interactions via a focus on concrete objects. This idea sprung out of observations of the teachers using bottles in experiments as a tool to negotiate meaning. This thesis treats those bottles in terms of its verbalized representation, treating it similar to words and in most cases through the fact that the teachers were talking while indicating something with the bottle. Thus, future research should adopt a different perspective, examining whether complementary attributes might emerge.

Another idea, one inspired by the literature on beliefs and belief change, its critique (e.g., Skott, 2015), and particularly by Liljedahl (2010), is the concept of noticing rapid and profound mathematics teacher changes. Thus, researchers could apply the CDC framework to mathematics teachers’ professional development, exploring its potential to identify rapid and (seemingly) trivial mathematics teacher changes in terms of contributions. This thesis discussed teaching in terms of learning and purposeful, collaborative transformation. It would be interesting, in my opinion, to explore the connections between teachers’ in-the-moment learning and the purposeful and collaborative transformation of their professional identities through a professional development program.

Lastly, this study developed a framework for understanding the dynamic and processual nature of teaching mathematics: the CDC framework (Figure
This raises the question of how the application of this model might contribute to future research on the mathematics classroom. This thesis represents a lengthy journey of theorizing and justifying concepts. As I wrote the constituent articles, I argued at length for these ideas’ explanatory potential. I thus propose—and am, in fact, planning—to examine these concepts through empirical research. The end goal is not simply to further develop them, although that would be a welcome outcome; rather, the objective is to apply the framework to understand complex teaching situations from this contribution-based perspective.
REFERENCES

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Appendix II

*Stickprov 1 (25 observationer)*

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*Stickprov 2 (25 observationer)*

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*Freqvastabell för stickprov 1 + stickprov 2*

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Instruktioner

1. Ta fram reflektionsboken och sortera in de första 25 observationerna där det står stickprov 1. Använd färgeois och rita in i rutorna, så att denna gång allt av samma färgr bredvid varandra

Ex: 

2. Fyll i frekvenstabellen för stickprov 1

3. Sortera in de andra 25 observationerna där det står stickprov 2. Använd färgeois och rita in i rutorna, så att även denna gång allt av samma färgr bredvid varandra

4. Fyll i frekvenstabellen för stickprov 2

5. Klipp ut era staplar

Ex: 

6. Ställ staplarna för stickprov 1 bredvid varandra och likadant för stickprov 2

Fråga A) Ser de båda stapediagrammen likadana ut?

Fråga B) Tycker ni att något av stapediagrammen stämmar bättre överens med hur det ser ut i flaskan?

Diskutera båda frågorna i gruppen och skriv era svar samt motivera i era reflektionsböcker

7. Ställ staplarna av respektive färgr för stickprov 1 och stickprov 2 ovanpå varandra

8. Fyll i den sista frekvenstabellen

Fråga C) Jämför det stora stapel diagrammet med de två små diagram, tycker ni att det stora diagrammet stämma bättre eller sämre överens med flaskan innehåll? (Tips: jämför både höjd på staplar, absolut frekvens och relativ frekvens) Diskutera i gruppen och skriv era svar samt motivera i era reflektionsböcker