The Four Colour Theorem

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Abstract

This thesis is about of The Four Colour Theorem and its proof which was the first computer aided proof. It will cover the history of the theorem highlighting some of the failed attempts at proving the theorem as well as the first successful proof by Kenneth Appel and Wolfgang Haken in 1977. We will learn what an unavoidable set of configurations and a discharging procedure are, as applied in the proof. The contribution is an algorithm that is used to formulate an unavoidable set. In addition to that, using Java Programming, two computer programs are written; one for discharging a graph and another for performing a four-colouring on a graph.

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Chapter 1

Introduction

It is fascinating to know that you can colour any map using no more than just four colours in a way that no two neighbouring regions are coloured the same. This is explained by the Four Colour Theorem as we will see in this paper.

The purpose of this project is to give a general understanding of the Four Colour Theorem and in more detail, study different discharging procedures used to determine an unavoidable set of configurations that help formulate the proof of the theorem.

In this paper, we will begin off by giving a brief history of the theorem in the first chapter. We will then move onto Chapters 2 and 3 where we will explore in detail what maps and graphs are, as well as how to represent a map by a graph. Chapter 4 will mainly cover Kempe’s strong arguments of the Kempe-chains and how they were helpful in finding the proof. In Chapter 5, we will learn about the actual proof of the Four Colour Theorem by Appel and Haken.

As part of my contribution, in Subsection 5.2.2, we will study an algorithm for a discharging procedure used to construct an unavoidable set of configurations and also write a computer program for discharging. We will also see what conclusions [Chapter 6] can be made in regards to how to choose an ideal discharging procedure. Finally, we will also study a computer program that I have written to perform simple 4-colourings on graphs in Chapter 7.

Figure 1.1: A map of the world coloured with 4 colours[7].
1.1 History

The Four Colour Theorem is attributed to Francis Guthrie. He was colouring the map of England when he noticed that he required only four colours to colour the map without having any two regions that share a boundary having the same colour. He went on to observe that this seemed to be true for any other map.

The Four Colour conjecture was first brought to light by Francis’ brother, Frederick Guthrie on October 23, 1852 [1]. This was when he presented the problem to his lecturer Augustus de Morgan with Francis’ consent. De Morgan could not give an explanation to the problem so he sent a letter to his friend, Sir William Rowan Hamilton, a mathematician in Ireland. In the letter he described the nature of the problem and he hoped Hamilton would make better sense of it. This letter became the first documentation of the theorem that exists today.

For several years to come, the Four Colour problem seemed to be ignored and less talked about by mathematicians. This was until Arthur Cayley brought it up again as an inquiry in a mathematical journal on July 13, 1878, asking if anyone had managed to prove it [1].

In 1879, Alfred Bray Kempe, a lawyer and former student of Cayley, published the first complete proof of the Four Colour conjecture using what is called Kempe Chains [1].

In 1880, another proof to the Four Colour problem was published by physicist Peter Guthrie Tait, even though many still took it that Kempe proved the theorem [3].

For about a decade, the Four Colour theorem was considered proven until in 1889 when Percy John Heawood showed a defect in Kempe’s proof [3]. The proof by Peter Guthrie was also found to be incorrect by Julius Petersen.

It was in 1976 that Kenneth Appel and Wolfgang Haken produced the first valid proof of the Four Colour theorem [3]. The proof was built upon contributions from George David Birkhoff, concepts such as Kempe Chains by Kempe, Heinrich Heesch’s ideas of reducibility and unavoidability.

Appel and Haken’s proof became the first computer aided proof used to proof a major mathematical theorem. The proof raised a lot of criticism and controversy. Many mathematicians did not accept the proof as they wondered if this could be considered a valid proof if it could not be fully checked by hand. Nevertheless, the proof has since then gained more reception.

An improved and simplified version of Appel and Haken’s proof has most recently been published in 1997 by Neil Robertson, Paul Seymour, Daniel Sanders and Robin Thomas collectively. They too used the help of a computer.
Chapter 2

Maps

In this section, we will familiarize ourselves with what a map really is, what its structural elements are and also specify and define what kind of maps are of interest to us as regards the Four Colour Theorem.

A map is a plane divided into several regions separated by borderlines. Two regions are adjacent if they share a borderline. The points at which borderlines meet are called multinational corners [1].

A normal map as defined by Kempe is a map where no country in it completely surrounds another country or countries and if no more than three countries meet at any point [2].

A map is said to be regular if it fulfills the following conditions; it is not empty, it is connected (no islands), it contains no bridges and any two distinct countries have at most one common borderline [1].

While discussing the Four Colour Theorem, we will restrict ourselves to maps that do not have disconnected countries as it possible to have a map of five regions where each one is adjacent to the other four. Such a map is not 4-colourable.

Figure 2.1: A map with a disconnected country R that requires 5 colours.
A cubic map is a map where there are strictly three borderlines at every meeting point of countries. Any map can be turned into a cubic map. This is done by replacing a point where more than three countries meet by a new region. This region will have a borderline with all the other regions. If this new graph can be coloured with 4 colours, then so can the original map since the original regions are still non-adjacent in the new cubic map. Removing the new region will not alter the colouring of the map. So proving that the theorem is consistent with cubic maps proves that it is consistent for all maps.

Figure 2.2: Forming a cubic map.

Through stereographic projection, a map on a sphere can be transformed into a planar map. If the planar map can be coloured with four colours, then you are safe to say that the theorem also applies to spherical surfaces.
Chapter 3

Graph Theory

In this chapter, we move away from the topological understanding of maps to a more combinatorial one. We will use graphs to represent special kinds of maps and also help present the Four Colour Theorem in a much clearer manner.

A graph \( G \) is a set of vertices and edges which we will represent as \( V(G) \) and \( E(G) \) respectively.

A vertex is the end point of an edge. Two vertices are adjacent if there exists an edge between them.

A loop is an edge that has both of its ends at a single vertex. Parallel or Multiple are edges that are incident to the same two vertices. A simple graph is a graph without loops or parallel edges.

![Figure 3.1: A loop (left) and parallel edges (right).](image)

A face of a graph is a region bounded by edges. The outer region of a graph is also a face of infinite size.

The degree of a vertex \( v \) is the number of edges that are incident at that vertex, denoted \( \deg(v) \). A vertex of degree \( i \) is denoted as an \( i \)-vertex.

A connected graph is a graph in which there is a path between any pair of vertices.

![Figure 3.2: Connected graph (left) and non-connected graph (right).](image)
A planar graph is one that can be drawn on a plane in such a way that edges only meet at vertices (edges do not intersect).

![Figure 3.3: A non-planar connected map.](image1)

A maximal planar graph is a planar graph that loses its planarity if any more edges are added to it.

3.1 Duality

Using the concept of duality, a map can be turned into a planar graph called a dual. This is done by placing a vertex inside every region of the map and connecting any pair of vertices with an edge if the regions they represent share a boundary.

If the map is a normal map, then the edges form triangular faces. This dual graph is called a triangulation. Note that a dual graph has one vertex more than its map has regions. This is because the outer area of the map is regarded as a region of infinite size.

![Figure 3.5: A dual graph of a map (in red).](image2)
In this case, the idea of colouring shifts from regions of a map to vertices of the dual graph in such a way that no two adjacent vertices have the same colour.

### 3.2 Euler’s Formula

**Theorem 3.1** For any connected planar graph $G$, $V - E + F = 2$ where $V$ is the number of vertices, $E$ is the number of edges and $F$ is the number of faces of $G$.

**Proof** [10, Theorem 3] The proof is by induction on the number of vertices. Suppose that $V = 1$. The number of faces $F$ is given by $F = E + 1$. Then by substitution we have that $V - E + F = 1 - E + (E + 1) = 2$. The formula holds.

Now consider a case where $V > 1$. If we erase an edge from $G$, we are left with $(E - 1)$ edges. In one case, if this edge was incident to a 1-degree vertex, we will be left with $(V - 1)$ vertices maintaining the number of faces. Substituting these into the formula; $(V - 1) - (E - 1) + F = 2$, which is $V - E + F = 2$.

On the other hand if the erased edge was incident to two vertices both of degree at least 2, then we have $(F - 1)$ faces and the same number of vertices. By substitution, $V - (E - 1) - (F - 1) = 2$, giving $V - E + F = 2$. In both cases the formula holds.

**Corollary 3.1** For a simple connected planar graph $G$, $E \leq 3V - 6$.

**Proof** [9] The sum of degrees of all faces of $G$ is twice the number of edges in $G$, $3F = 2E$. This is specially true for a triangulation where each face is bounded by three edges. We also know that an edge bounds two faces. However a face can be bounded by 3 or more edges which means

$$3F \leq 2E.$$  

From this we have that

$$F \leq \frac{2}{3}E.$$  

Substituting this into Euler’s formula;

$$V - E + \frac{2}{3}E \geq 2$$

which simplifies to

$$E \leq 3V - 6.$$  

Note that if a graph is a triangulation, then $E = 3V - 6$.

**Theorem 3.2** Every simple connected planar graph has a vertex $v$, where $\deg(v) \leq 5$.

**Proof** [9] The proof follows by contradiction. Assume that there is a graph $G$ with all of its vertices of degree 6 or more. Then the sum of degrees of all vertices is at most twice the number of edges in $G$;

$$6V \leq 2E$$
which simplifies to

\[ E \geq 3V. \]

This is clearly in contradiction to Corollary 2.1 that \( E \leq 3V - 6 \). Hence there must be a vertex of degree five or less in a connected planar graph.
Chapter 4

Kempe Chains

The Kempe Chains named after Alfred B. Kempe played an important role in the attempt to proving the Four Colour Theorem, even though it turned out that there was a flaw in the Kempe chains as was pointed out by John Percy Heawood a decade later.

4.1 Minimal Criminals

Suppose that there are maps that are not 4-colourable, maps that require five or more colours. Among these maps, those that have the least number of regions are what we call minimal criminals. They are also referred to as minimal counterexamples. Minimal criminals have more than four countries since a map of four or less regions is 4-colourable.

Kempe used the approach that since minimal criminals have the least number of regions, any map with fewer countries had to be 4-colourable. Showing that minimal criminals cannot exist is enough to prove the Four Colour Theorem.

4.2 Kempe’s Proof

Kempe’s proof [9] of the Four Colour Theorem is by induction. Consider a connected planar graph $G$. Assume that $G$ is a minimal criminal. From Theorem 2.1, we know that $G$ must have a vertex $v$ such $\deg(v) \leq 5$. We exclude vertices of degree one (like islands) since they do not constitute a connected map.

Consider $\deg(v) \leq 3$;
If $G$ contains $v$, remove $v$ and the edges incident to it. By induction, the new graph is 4-colourable. When you colour the graph and put back $v$, the vertices adjacent to $v$ have used at most three colours leaving at least one colour for $v$. Hence a 4-colouring of $G$.

Consider $\deg(v) = 4$;
If $G$ contains $v$, remove $v$ and the edges incident to it. By induction, the new graph is 4-colourable. After colouring the new graph and replacing $v$, if the four vertices adjacent to $v$ have used less than four colours, we can use the unused
colour for \( v \) to give \( G \) a 4-colouring.

On the other hand, if the four vertices adjacent to \( v \) have used up all four colours, we introduce the Kempe Chains method. To illustrate this, take a look at Figure 4.1 below.

Consider two vertices adjacent to \( v \) that are not adjacent to each other for example the Green and Yellow vertices. Create a subgraph from the Green vertex, moving edge by edge connecting all vertices coloured Green or Yellow. This subgraph is what is called a Kempe Chain (a Green-Yellow Kempe chain).

If the Green-Yellow chain from the Green vertex adjacent to \( v \) does not connect to the Yellow vertex adjacent to \( v \), we can interchange the colours of the vertices in this chain, Green vertices to Yellow and the Yellow ones to Green.

This will not interfere with the colouring of the rest of the graph since any edges coming out of this subgraph are incident to vertices coloured Blue or Red. This way we free up the colour Green for \( v \) giving a proper 4-colouring of the graph.

![Figure 4.1: Chain does not connect](image)

If the Green-Yellow chain connects to the Yellow vertex adjacent to \( v \) (see Figure 4.2), then we may abandon this chain and try the Blue-Red chain starting from the Blue vertex adjacent to \( v \). This chain will not connect to the Red vertex adjacent \( v \) because it is enclosed within the Green-Yellow Kempe chain. So we can go ahead and interchange the colouring in the Blue-Red chain, Blue vertices to Red and Red ones to Blue thus freeing up colour Blue for \( v \) to give a 4-colouring of the graph.

![Figure 4.2: Chain connects](image)
Consider $\text{deg}(v) = 5$.
If the planar graph contains a vertex $v$, remove it and colour the new graph which by induction is 4-colourable. Put back $v$. If the colours of the vertices adjacent to $v$ are coloured with less than 4 colours, we colour $v$ with the unused colour. If the adjacent vertices use up all four colours, then we go back to the Kempe chains. See Figure 4.3.

![Figure 4.3: A 5-degree vertex.](image)

Take the Blue-Yellow chain starting at the Blue vertex adjacent to $v$. If this chain does not connect to the Yellow vertex adjacent to $v$, we can interchange the colouring of this chain and free up the colour Blue for $v$.

If the chain connects to the Yellow vertex, then we can go to the Blue-Red Kempe chain still starting at the Blue vertex adjacent to $v$. If this chain does not connect to the Red vertex, we can interchange the colouring of the chain and free up the Blue for $v$.

If the Blue-Red Kempe chain connects to the Red vertex, then we have two other Kempe chains to consider. These are the Green-Yellow Kempe chain starting at $g1$ and the Green-Red Kempe chain starting at $g2$. Neither of these two chains will connect to a vertex adjacent to $v$ since they have already been enclosed by the Blue-Yellow and Blue-Red Kempe chains. As a result we may interchange the colouring of these Kempe chains and leave the colour Red for $v$, and hence a 4-colouring of the planar graph.

### 4.3 Flaw in Kempe’s Proof

Kempe’s proof of the Four Colour Theorem was so convincing that it took a period of about 10 years before it was debunked. We look at an example to illustrate the inconsistency in the method of Kempe chains.

![Figure 4.4: Counterexample to Kempe chains.](image)
The problem appears in the case where we have \( \text{deg}(v) = 5 \). As we can see in Figure 4.4 above, \( b \) is adjacent to \( y \) and \( r \). Vertices \( y \) and \( r \) are adjacent to each other and also a part of the Blue-Yellow and Blue-Red Kempe chains (both starting at \( b \)) respectively. When we interchange the colouring of the Green-Yellow and Green-Red Kempe chains, both \( y \) and \( r \) will be coloured Green which is an illegal colouring.

Kempe was unable to fix this problem which resulted to the disqualification of his attempted proof.

4.4 Five Colour Theorem

**Theorem 4.1** Every planar graph can be coloured with at most five colours in such a way that no two adjacent vertices have the same colour.

**Proof** [11, Theorem 8] The proof is by induction similar to what we did in Kempe’s proof. Assume that \( G \) is planar graph that requires more than five colours. Once again for the base cases, \( G \) will be 5-colourable if it is made up of five or less vertices.

From Theorem 2.1, there must be a vertex of degree at most five in any planar graph. If \( G \) has a vertex \( v \) such that \( \text{deg}(v) \leq 4 \), we remove it from the planar graph. By induction the new graph will be 5-colourable. When we replace \( v \) into the graph, there will always be at least one colour for \( v \). Hence a 5-colouring of \( G \).

If \( v \) is such that \( \text{deg}(v) = 5 \) (see Figure 4.5), take out \( v \) and colour the new graph which by induction is 5-colourable. When you replace \( v \), the vertices adjacent to it may have used up less than five colours. This means there is one unused colour available for \( v \) to give a 5-colouring of the graph.

In case we replace \( v \) and the vertices adjacent to it have used up all 5 colours then we bring in the Kempe chains. We can start by creating a Green-Yellow Kempe chain from the Green vertex adjacent to \( v \). If this chain does not connect to the Yellow vertex adjacent to \( v \), we interchange the colours of the chain. Green vertices turn yellow and Yellow ones turn green. This frees up the colour Green for \( v \) to give a 5-colouring.

![Figure 4.5: A five degree vertex.](image)
Should the Green-Yellow chain connect with the Yellow vertex adjacent to \( v \), we turn to the Blue-Red Kempe chain starting from the Blue vertex adjacent to \( v \). There is no way the Blue-Red chain will connect to the Red vertex adjacent to \( v \) because it will be locked within the Green-Yellow Kempe chain. Therefore we may interchange the colouring of the chain turning Blue vertices red and Red vertices to blue. This frees up Blue for \( v \) to give a 5-colouring of the graph.
Chapter 5

Proof of the Four Colour Theorem

In this chapter, we will explore the work of Appel and Haken which led to the actual proof of the Four Colour Theorem. The construction of their proof was basically divided into two main parts: finding a set of subgraphs from which at least a member must be contained within every connected planar graph and secondly showing that these subgraphs are 4-colourable.

5.1 Unavoidability and Reducibility

We have shown that every connected planar map must contain at least one country with five or less neighbours. These countries form an example of an unavoidable set of configurations.

An unavoidable set is a set of configurations (or subgraphs) from which any planar graph has at least one member of the set as a subgraph. All Kempe would have had to do to prove the theorem was to show that this unavoidable set of configurations was reducible.

A reducible configuration is a subgraph one if contained in a graph, any colouring of the rest of the graph can be extended into a colouring of the entire graph. Unfortunately, he failed to show that the 5-vertex configuration was reducible.

Similarly, Appel and Haken’s proof involved a construction of an unavoidable set of initially 1936 reducible configurations [3]. The unavoidable configurations were found by the method of discharging. This discharging method is credited to Heinrich Heesch. Heesch had previously applied the method to the movement of charge in an electrical network before associating to the Four Colour problem.
5.2 Discharging Procedure

Discharging is used to identify an unavoidable set of configurations (which might not necessarily be reducible) by reallocating positive charge amongst vertices of a graph. From Kempe’s proof, we now know that a minimal criminal cannot contain a vertex \( v \) such that \( \text{deg}(v) \leq 4 \). We will therefore be limiting ourselves to graphs of vertices with a minimum of degree five.

We assign a charge of \((6 - i)\) to every vertex where \( i \) is the degree of that particular vertex. For instance a 5-vertex will have a charge of +1, a 6-vertex will have zero or no charge and a vertex of higher degree will have a corresponding negative charge. An illustration of this can be seen in Figure 5.1.

![Figure 5.1: Charge on vertices.](image)

The total charge on a graph is given by \( \sum_{i=1}^{\infty} (6 - i)v_i \) where \( v_i \) is the number of vertices of degree \( i \) and always sums up to 12. This can be illustrated with the help of Corollary 3.1 from which we can know that for a triangulation, \( E = 3V - 6 \).

We have that \( v_1 + 2v_2 + 3v_3 + ... = 2E \) and \( v_1 + v_2 + v_3 + ... = V \). By substituting these equations into

\[
2E = 2(3V - 6)
\]

we get

\[
v_1 + 2v_2 + 3v_3 + ... = 6(v_1 + v_2 + v_3 + ...) - 12.
\]

This simplifies to

\[
5v_1 + 4v_2 + 3v_3 + 2v_4 + v_5 - v_7 - 2v_8 - ... = 12
\]

which can be expressed as

\[
\sum_{i=1}^{\infty} (6 - i)v_i = 12.
\]

Discharging does not affect the overall charge of the graph but rather that of the individual vertices that either gain or lose charge in the process.
5.2.1 Wernicke’s Discharging Procedure

Consider a discharging procedure [2, page 121] on an arbitrary minimal criminal with vertices of degree at least 5, where a charge of \(+\frac{5}{2}\) is drawn from positively charged vertices (5-vertices) to their adjacent negatively charged neighbours.

Since the total charge of the graph is always positive, there will always be some positively charged vertices after discharging. It is these vertices that we focus on as they will be either the unavoidable configurations we are looking for or be adjacent to one.

A 5-vertex (●) will retain a positive charge if it is adjacent to at least one 5-vertex or 6-vertex (○) since there is no charge transfer to these two vertices as they are non-negatively charged. A 5-vertex does not discharge in the following configuration;

![Figure 5.2:](image)

A 6-vertex initially has zero charge and it stays this way because it is neither negatively charged to receive charge nor is it positively charged to give out any.

A 7-vertex (●) starts with a charge of \(-1\) and will become positively charged only if it is adjacent to at least six 5-vertices, of which at least two are adjacent to each other. We can skip this configuration since it contains a smaller configuration of two adjacent 5-vertices that we have already mentioned above.

![Figure 5.3:](image)

An 8-vertex with an initial charge of \(-2\) can at most receive a total charge of \(+\frac{5}{2}\) which is not sufficient to make it positively charged. This goes for all other \(i\)-vertices where \(i > 8\).

As a result, this discharging procedure yields an unavoidable set that contains (in addition to a 2-vertex, 3-vertex and 3-vertex) a 5-vertex adjacent to 5-vertex and a 5-vertex adjacent to a 6-vertex.

\[
\{ 2, 3, 4, \bullet--\bullet, \bullet--\circ \}
\]

![Figure 5.4:](image)

This example gives a good idea of how to look for unavoidable configurations. However, this particular set of configurations is not reducible[2]. An adjustment of the discharging procedure (by altering \(\frac{5}{2}\)) can be done to produce a better set with all configurations reducible.
5.2.2 Discharging Procedure II

In this next example we will construct an unavoidable set using a different discharging rule in which a 5-vertex transfers a charge of $+\frac{1}{2}$ to its negatively charged neighbours. Once again we will assume that the unavoidable set contains a vertex of degree 2, 3 and 4.

A 5-vertex will only be discharged if it is adjacent to at least two negatively charged vertices.

The obstacles in trying to discharge a 5-vertex arise in the presence of the subgraphs shown in the figures below.

Figure 5.5: A 5-vertex adjacent to three vertices of degree 5.

Figure 5.6: A 5-vertex adjacent to three vertices of degree 6.

Figure 5.7: A 5-vertex adjacent to three vertices of degree 5 and 6.

A vertex of degree 6 remains with no charge just like in the previous discharging method.

A 7-vertex needs to be adjacent to at least three vertices of degree 5 to acquire a positive charge ($+\frac{3}{2}$). This can be achieved with or without having any of the neighbouring vertices of degree 5 being adjacent.

Vertices of degrees 8, 9, 10 and 11 need to be adjacent to at least five, seven, nine and eleven vertices of degree 5 respectively to acquire a positive charge. In all four cases, at least two neighbouring vertices of degree 5 will be adjacent.

All vertices of degree higher than 11 possess a negative charge so high that they cannot be adjacent to enough vertices of degree 5 to gain positive charge.
The obstacles shown in the figures above together with the 7-vertex form our unavoidable set of configurations. We can narrow this set down by taking out the common subgraphs which are; a 5-vertex adjacent to a 5-vertex and a 5-vertex adjacent to three vertices of degree 6 in the case of discharging a 5-vertex. The unavoidable set becomes as shown in figure below.

\[ U = \{ \begin{array}{c} \bullet - \bullet, \end{array} \} \]

Figure 5.8: An unavoidable set, \( U \).

**Proof that \( U \) is an unavoidable set.**

We will now prove that the set \( U \) is an unavoidable set. This is a proof by contradiction. Suppose that a triangulated graph \( G \) contains none of the configurations in \( U \), then no 5-vertex is adjacent to another 5-vertex or three vertices of degree 6.

This means that every 5-vertex has at least two neighbours of degree greater than 6 except vertices of degree 7 which are absent from \( G \). As a result, all vertices of degree 5 will be discharged.

Vertices of degree \( d \geq 8 \) will have at most \( \frac{d}{2} \) neighbours of degree 5 since there can be no 5-vertex adjacent to another 5-vertex. This means they will receive charge of at most \( (6 - d) + \frac{d}{2} \cdot \frac{1}{2} = 6 - \frac{3d}{4} \leq 6 - \frac{3(8)}{4} = 0 \). Therefore, the absence of all members of set \( U \) from \( G \) yields the contradiction that the total charge on \( G \) will be negative which proves that \( U \) is an unavoidable set.

### 5.2.3 Computer Program for Discharging Procedure II

Below is a computer program that has been used to discharge the graph in Figure 5.9. The graph is made up of twenty two vertices and their corresponding degrees have been labelled in black. In red are the positions of each vertex as represented in the adjacency matrix \( A \) in the program.
public class discharging {
    
    public static void main(String[] args) {
        //Adjacency matrix; 1(adjacent), 0(non-adjacent), a vertex is not adjacent to itself
        int[][] A = {
            {0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0},
            {1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
            {0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0},
            {0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
            {0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
            {0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
            {0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
            {0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
            {0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
            {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
            {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
            {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0},
            {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0},
            {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0},
            {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0},
            {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0},
            {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0},
            {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0},
            {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0},
            {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0},
            {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1},
            {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
            {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
        };

        //array of degrees of the vertices 0, 1, 2, 3, ...., (in that order)
        int[] degrees = {3, 4, 4, 3, 4, 3, 4, 3, 4, 3, 4, 4, 6, 5, 7, 7, 6, 7, 4, 5};
        
        //weight of charge to be transferred
        float weight = 0.5f;
        
        //array of charges on the vertices before discharging
        int[] charges = new int[degrees.length];
        int c = 0;

        //flexible list of charges of vertices
        List<Float> newCharges = new ArrayList<Float>();
        for(int i = 0; i < charges.length; i++)
            { //calculating charge value
                c = 6-degrees[i]; //adding value to 'charges'
                newCharges.add((float)c); //adding value to flexible list 'newCharges'
            }

        //Iterating row by column
        for(int row = 0; row < charges.length; row++)
        { //if row value is +ve and adjacent to column value and column value is
Charges after discharging:

\[3.0, 2.0, 1.5, 2.5, 2.5, 1.0, 2.5, 1.5, 3.0, 3.0, 1.5, 2.5, 1.5, 2.0, 0.0, 0.5, 2.0, 1.0, 0.0, 1.0, 1.5, 0.0\]

Positively charged vertices:

\[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17, 19, 20\]

We have learned that every connected planar graph must contain a vertex of degree 5 or less, and that Kempe was able to prove that among those, vertices of degree 2, 3 and 4 are reducible. These vertices become the first elements of our unavoidable set. We will thus treat the graph as a minimal criminal and ignore these vertices in the rest of the unavoidable configurations.
From the output given above, we notice that the only other positively charged vertices with degrees greater than 4 are vertices 15, 16, 17 and 19 shown in Figure 5.10. It is at these vertices that we may find more elements for the unavoidable set.

*Figure 5.10:*

Vertex 15 keeps half of its positive charge due to having four neighbouring vertices (1, 2, 21 and 14) all of which are non-negatively charged and therefore to which it cannot pass charge. It only gives out $+\frac{1}{2}$ to vertex 16. This gives us the next element which is a 5-vertex adjacent to four vertices of degree either 5 or 6. These can be seen in Figure 5.11.

*Figure 5.11:*

Vertices 16, 17 and 19 are all of degree 7. They acquire a positive charge when adjacent to at least three positively charged neighbours. Our final configuration to the unavoidable set here is a 7-vertex adjacent to three vertices of degree 5. This configuration is shown in Figure 5.12.

*Figure 5.12:*

These elements together with Kempe’s unavoidable set form our unavoidable set of configurations for the graph in Figure 5.9.
5.3 Reducible Configurations

A configuration is said to be reducible if any colouring of the rest of the graph can be extended onto the configuration. If a map contains a reducible configuration, then the map can be reduced to a smaller map. This smaller map has the condition that if it can be coloured with four colours, then the original map can also [8]. Such a configuration cannot be contained in a minimal criminal.

Reducible configurations may be D-reducible or C-reducible. D-reducible configurations are those that always result into a good colouring of the rest of the graph or by performing Kempe-chain changes to the colouring such as a digon, triangle and a square. C-reducible configurations are those in which not all their colourings can be extended to the main graph and so require some modifications or use contractions to acquire a proper colouring.

A good example of a C-reducible configuration is the Birkhoff diamond (see Figure 5.13) named after George David Birkhoff. It is made up of a 5-vertex that is adjacent to three other 5-vertices, \(y_1, y_2, y_3, y_4\) in the figure below. The Birkhoff diamond was the first found reducible configuration. We will show that a Birkhoff diamond is reducible by proving the following theorem.

![Figure 5.13: A Birkhoff Diamond.](image)

**Theorem 5.1** A minimal criminal cannot contain a Birkhoff diamond.

**Proof** [1, page 180] Assume that there is a minimal criminal \(G\) that contains a Birkhoff diamond. The first step is to compress the Birkhoff diamond into a smaller graph, \(G'\) in Figure 5.14.

We begin by eliminating all the interior vertices \((y_1, y_2, y_3, y_4)\) along with all edges incident to them. This leaves a hollow ring of six vertices \((x_1, x_2, x_3, x_4, x_5\) and \(x_6\)). We then adjoin the vertices \(x_2\) and \(x_4\) to form the vertex \(u\). This is acceptable since they are not adjacent in \(G\) (the original graph). Vertices \(x_4\) and \(x_2\) will have the same colour as \(u\). Lastly, we add an edge between \(x_6\) and \(u\). We can add this edge since \(x_6\) is not adjacent to \(x_2\) or \(x_4\).
By induction, $G'$ is 4-colourable since it has fewer vertices than $G$. From this compression, we are able to get six different possible colourings ($c_1$, $c_2$, $c_3$, $c_4$, $c_5$, $c_6$) of the exterior vertices (see Figure 5.15).

![Figure 5.14: A compressed Birkhoff Diamind, $G'$.](image)

We can now colour the interior vertices. We have direct colourings of the interior with all except for the colouring $c_6$ which requires extra operations. In this case we may consider the Kempe chains. Take the Red-Yellow Kempe chain starting from $x_3$. If this chain does not connect with $x_1$ and $x_5$, then we can interchange the colouring of the chain, leaving $x_3$ coloured Yellow. On completion of the colouring, we will have the same colouring as $c_5$.

If this Kempe chain connects to $x_1$ and $x_5$, then we can consider the Blue-Green Kempe chain starting from $x_2$. This chain will surely not connect to $x_6$. We may then interchange the colouring of the chain leaving $x_2$ coloured Green, $y_3$ Yellow, $y_2$ Blue, $y_1$ Yellow and $y_4$ Green (see Figure 5.16).
We have shown that all possible colourings of $G$ can be extended to the Birkhoff diamond, hence a Birkhoff diamond is reducible and cannot be contained in a minimal criminal.
Chapter 6
Conclusion

The discharging procedures we have studied in subsections 5.2.1 and 5.2.2 are nearly equivalent in that they lead to very similar unavoidable sets. This similarity can also be seen below in Table 6.1 (values from the program in Subsection 5.2.3) where they give the same number of positively charged vertices after discharging. Even though both discharging procedures follow the basic concept of the actual proof, their unavoidable sets of configurations are not reducible and they are very simplistic compared to the actual proof by Appel and Haken.

Appel and Haken’s discharging algorithm involved over 300 different rules starting with one where a 5-vertex gives out charge equally to all its neighbours of degree at least 7, then repeatedly modifying it to exclude particular configurations. Their work initially produced 1936 reducible configurations before managing to reduce it to 1482 and eventually 1408 reducible configurations [3].

A newer proof by Neil Robertson, Daniel Sanders, Robin Thomas and Paul Seymour following similar principles as Appel and Haken used a simpler discharging procedure that involved 32 rules [4]. They managed to come up with 633 reducible configurations. There is yet to be a proof of the theorem that can be fully hand checked.

Another observation is the range within which an ideal weight may be selected. On one end we have +1 as the highest weight that can be used. This is because it is the highest charge any vertex can have in a minimal criminal. On the other end, $+\frac{1}{5}$ is the lowest ideal weight. Choosing a lower weight $l < +\frac{1}{5}$ implies that a 5-vertex can at most give out a total charge of $5l < +1$, which means that a 5-vertex will always retain its positive charge after discharging. In addition to that, a very small weight makes it hard for negatively charged vertices to gain enough charge to become positively charged after discharging. As a result, we can expect to get similar results in which vertices retain relatively the same charge after discharging. This can be seen in the table below.
<table>
<thead>
<tr>
<th>Weight (Charge)</th>
<th>Number of positively charged vertices</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>17</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>19</td>
</tr>
<tr>
<td>$\frac{1}{3}$</td>
<td>19</td>
</tr>
<tr>
<td>$\frac{1}{4}$</td>
<td>19</td>
</tr>
<tr>
<td>$\frac{1}{5}$</td>
<td>19</td>
</tr>
<tr>
<td>$\frac{1}{6}$</td>
<td>17</td>
</tr>
<tr>
<td>$\frac{1}{7}$</td>
<td>17</td>
</tr>
<tr>
<td>$\frac{1}{8}$</td>
<td>17</td>
</tr>
<tr>
<td>$\frac{1}{9}$</td>
<td>17</td>
</tr>
<tr>
<td>$\frac{1}{10}$</td>
<td>17</td>
</tr>
<tr>
<td>0</td>
<td>17</td>
</tr>
</tbody>
</table>

Table 6.1: Number of positively charged vertices after discharging Figure 5.9.
Chapter 7

Computer Program for the Four Colour Theorem

Once a potential reducible configuration has been found, it is essential to determine whether this configuration is 4-colourable in the first place. The following program performs 4-colourings on basic graphs or configurations.

The four colours ('colours' in the code) at our disposal (say red, blue, green and yellow) are represented by 4 integers 1, 2, 3 and 4 respectively. Vertices are accounted for in the rows and columns of an adjacency matrix \(A\) in which a '1' represents adjacency and '0' represents non-adjacency. Note that a vertex is adjacent to itself.

It contains a help method \(\text{unusedColor}()\) that arranges the unused colours in an ascending order and always returns the first one when called. This approach ensures that we use our colours with caution to not run into problems where we need a fifth colour. In other words, we only use a new colour if the previously used colours cannot be used for the next vertex to colour.

As an example, below is a 4-colouring of a dodecahedron as well as an image of what the result.
public class Color {

    // input is a list of used colours
    static int unusedColor(List<Integer> used) {

        // list of the four colours to use
        List<Integer> colors = Arrays.asList(new Integer[]{1, 2, 3, 4});

        // will store unused colours
        List<Integer> unUsed = new ArrayList<Integer>();

        // iterates through 'colours' to find unused colours
        for (Integer i : colors)
            if (!used.contains(i))
                unUsed.add(i); // adds unused colours to 'unUsed'

        // finds the smallest of the unused colours
        int min = unUsed.get(0);
        for (Integer i : unUsed)
            if (i < min)
                min = i;

        return min; // returns the smallest unused colour
    }

    public static void main(String args[]) {

        // Adjacency matrix; 1(adjacent), 0(non-adjacent)
        int[][] A = {
            {1, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
            {1, 1, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
            {0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
            {1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
            {0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
            {0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
            {0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
            {0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
            {0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
            {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
            {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
            {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0},
            {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0},
            {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0},
            {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0},
            {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0},
            {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0},
            {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0},
            {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1},
            {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1}};

        // Graph that will store the colour for each vertex(index)
        int[] G = new int[A.length];

        // iterates through the graph (equivalent to rows of A)
        for (int i = 0; i < G.length; i++)
        {
            // a ring of colours of vertices adjacent to vertex 'i'
        }
    }
}
List<Integer> ring = new ArrayList<Integer>{};

// j represents columns of A
for(int j = 0; j < G.length; j++)
{
    if(A[i][j] == 1) //if 'i' is adjacent to 'j'
    {
        ring.add(G[j]); // add colour of vertex j
    }
}
G[i] = unusedColor(ring); //colour 'i' with smallest unused colour
System.out.println(Arrays.toString(G)); //prints out the graph

Output: [1, 2, 1, 2, 3, 1, 2, 3, 1, 3, 1, 2, 1, 3, 2, 1, 3, 2, 3, 2]

1 - Red
2 - Blue
3 - Green
4 - Yellow
Chapter 8

Discussion

Being the first ever proof to be achieved with substantial help of a computer, it has raised questions to what a proof really is. Many mathematicians remain sceptical about the nature of this proof due to the involvement of a computer. With the possibility of a computing error, they do not feel comfortable relying on a machine to do their work as they would be if it were a simple pen-and-paper proof.

The controversy lies not so much on whether or not the proof is valid but rather whether the proof is a valid proof. To mathematicians, it is as important to understand why something is correct as it is finding the solution. They hate that there is no way of knowing how a computer reasons. Since a computer runs programs as they are fed into it, designed to tackle a problem in a particular way, it is likely they will return what the programmer wants to find leaving out any other possible outcomes outside the bracket.

Many mathematicians continue to search for a better proof to the problem. They prefer to think that the Four Colour problem has not been solved and that one day someone will come up with a simple completely hand checkable proof to the problem.

You will not find many useful applications of the theorem but it is worth noting that the theorem broadened our knowledge and understanding of some branches of mathematics like Combinatorics and Graph Theory as these were extensively explored during the course of the proof.

The journey of the Four Colour Theorem teaches discipline, persistence and patience as it took a collective effort of mathematicians such as Kempe, Heawood, Birkhoff and many more over a long stretch of time and several failed attempts to finally reach the correct proof.

The Four Colour Theorem has changed the course of mathematics and paved the way for mathematicians to embrace the use of computers when it comes to proofs. There have been many computer aided proofs since then and many more to come in the future.
I believe that times have changed. Computers have become part of our lives in all aspects. With computers, proofs and solutions to problems that would otherwise take lifetimes or centuries to solve can now be solved more efficiently and in a much shorter period of time. It is something the mathematical world will have to open hands to.
References


