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On a class of reflected backward stochastic Volterra integral equations and related time-inconsistent optimal stopping problems ☆

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ABSTRACT

We introduce a class of one-dimensional continuous reflected backward stochastic Volterra integral equations driven by Brownian motion, where the reflection keeps the solution above a given stochastic process (lower obstacle). We prove existence and uniqueness by a fixed point argument and derive a comparison result. Moreover, we show how the solution of our problem is related to a time-inconsistent optimal stopping problem and derive an optimal strategy.

1. Introduction

In recent years backward stochastic Volterra integral equations (BSVIEs) have attracted a lot of interest due to their modelling potential in problems related to time-preferences of decision makers, asset allocation and risk management in mathematical finance among other fields, for which the related optimal control problems are time-inconsistent meaning that the associate value-function does not satisfy the dynamic programming principle.

Lin [1] was first to introduce and study a class of BSVIEs driven by Brownian motion. They can be described as follows: For a given Lipschitz driver f and a square integrable terminal condition ξ solving a BSVIE consists in finding an adapted process (Y, Z) satisfying the equation

$$Y(t) = \xi + \int_t^T f(t, s, Y(s), Z(t, s))ds - \int_t^T Z(t, s)dW(s).$$

In a series of papers, Yong [2–4] systematically studied a general class, including the M-solution, of BSVIEs of the form:

$$Y(t) = \xi(t) + \int_t^T f(t, s, Y(s), Z(t, s), Z(s, t))ds - \int_t^T Z(t, s)dW(s).$$

This was followed by a quite extensive list of papers including Djordjević and Janković [5,6], Shi, Wang and Yong [7], Wang and Yong [8], Hu and Øksendal [9], Wang and Zhang [10], Wang, Sun and Yong [11], Popier [12] and Wang, Yong and Zhang [13], to mention a few.

In problems related to mathematical finance, the BSVIE is satisfied by the value function of time-inconsistent optimal control and equilibrium problems related to stochastic differential utility such as in Di Persio [14], dynamic risk measures such as in Yong [3], Wang and Yong [8] and Agram [15] and dynamic capital allocations such as in Kromer and Overbeck [16]. The BSVIE is also satisfied by the adjoint process of a related stochastic maximum principle related to equilibrium recursive utility and equilibrium dynamic risk measures, such as in Djehiche and Huang [17], Wang, Sun and Yong [11] among many more papers (see [13] for further references).

The purpose of the paper is to introduce and study a version of the above BSVIE whose first component of the solution Y is constrained to be greater than or equal to a given obstacle process L. This is achieved by including as a part of the solution a process $K(t, \cdot)$ (parametrized by t), which should be adapted to the filtration generated by the one-dimensional Brownian motion W and increasing for each t, in the sense that the equation considered is of the form

$$Y(t) = \xi(t) + \int_t^T f(t, s, Y(s), Z(t, s))ds + \int_t^T K(t, ds) - \int_t^T Z(t, s)dW(s),$$

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where

- $f(t,\cdot,y,z)$, which (for each $(t,y,z)\in[0,T]\times\mathbb{R}^2$) is an adapted process w.r.t. the filtration $(\mathcal{F}_t)_{0\le t\le T}$ generated by the one-dimensional Brownian motion W,
- $\xi(t)$, which is a random variable (parametrized by t), is $\mathcal{F}_T$-measurable.

Moreover, it is also required the so-called Skorohod flatness condition under which $K(t,\cdot)$ is 'flat' whenever $Y(t)>L(t)$ holds.

Our main results are:

- to introduce the reflected BSVIE,
- to establish existence and uniqueness of the solution,
- to prove a related comparison result,
- to show how the solution to the reflected BSVIE solves a time-inconsistent stopping problem.

We may remark that for simplicity, the driver in the above reflected BSVIE is only allowed to depend on Y and $Z(t,s)$ but the results can be extended to driver depending also on $Z(s,t)$. To the best of our knowledge, this class of reflected BSVIE and its connection to time-inconsistent optimal stopping problems seem new.

The content of our paper is as follows. After some preliminaries in Section 2, we give a formulation of the class of continuous reflected BSVIEs with lower obstacle. In Section 3 we derive existence, uniqueness and comparison results. Finally, in Section 4, we give an application to time-inconsistent optimal stopping problems.

Notation and preliminaries

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space on which is defined a standard one-dimensional Brownian motion $W=(W(t))_{0\le t\le T}$. We denote by $\mathbb{F}:=(\mathcal{F}_t)_{0\le t\le T}$ its natural filtration augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$.

Let $\mathcal{B}(G)$ be the Borel σ-field of the metric space G. In the sequel, $C>0$ represents a generic constant which can be different from line to line.

In this paper we only consider reflected BSVIEs driven by a one-dimensional Brownian motion. Extension to higher dimensions being straightforward.

We define the following spaces for the solution.

- $\mathcal{S}^2$ is the set of $\mathbb{R}$-valued $\mathbb{F}$-adapted processes $(Y(u))_{u\in[0,T]}$ such that
$$\|Y\|^2_{\mathcal{S}^2}:=\mathbb{E}[\sup_{t\in[0,T]}|Y(t)|^2]<\infty.$$
- $\mathcal{H}^2$ is the space of progressively measurable processes $(v(u))_{u\in[0,T]}$ such that
$$\|v\|^2_{\mathcal{H}^2}:=\mathbb{E}\left[\int_0^T|v(s)|^2ds\right]<\infty.$$
- $\mathbb{L}^2$ is the set of $\mathbb{R}$-valued processes $(Z(t,s))_{(t,s)\in[0,T]\times[0,T]}$ such that for almost all $t\in[0,T]$ $Z(t,\cdot)\in\mathcal{H}^2$ and satisfy
$$\|Z\|^2_{\mathbb{L}^2}:=\mathbb{E}\left[\int_0^T\int_t^T|Z(t,s)|^2dsdt\right]<\infty.$$
- $\mathcal{K}^2$ is the space of processes K which satisfy
  - for each $t\in[0,T]$, $u\mapsto K(t,u)$ is an $\mathbb{F}$-adapted and increasing process with $K(t,0)=0$;
  - $(t,u)\mapsto K(t,u)$ is continuous and $K(\cdot,T)\in\mathcal{H}^2$.

The spaces $(\mathcal{H}^2,\|\cdot\|_{\mathcal{H}^2})$ and $(\mathbb{L}^2,\|\cdot\|_{\mathbb{L}^2})$ are Hilbert spaces.

2. Formulation of the problem

We investigate existence of a unique triple (Y,Z,K) of processes taking values in $\mathbb{R}\times\mathbb{R}\times\mathbb{R}^+$ which satisfy the following reflected BSVIE with one obstacle associated with (f,ξ,L):

$$Y(t)=\xi(t)+\int_t^T f(t,s,Y(s),Z(t,s))ds+\int_t^T K(t,ds)-\int_t^T Z(t,s)dW(s), \tag{1}$$

(where $K(t,ds)$ is the Lebesgue–Stieltjes measure induced by the function $s\mapsto K(t,s)$), such that

(a) $Y\in\mathcal{H}^2$, $t\mapsto Y(t)$ is continuous and $Z\in\mathbb{L}^2$;
(b) $Y(t)\ge L(t)$ $\mathbb{P}$-a.s. , $0\le t\le T$;
(c) The process K enjoys the following properties:
  (c1) $K\in\mathcal{K}^2$;
  (c2) The Skorohod flatness condition holds: for each $0\le\alpha<\beta\le T$,
$$K(t,\alpha)=K(t,\beta)\quad\text{whenever}\quad Y(u)>L(u)\quad\text{for each}\quad u\in[\alpha,\beta]\quad\mathbb{P}\text{-a.s.} ,$$

Definition 1. A solution of (1) is a triple $(Y,Z,K)\in\mathcal{H}^2\times\mathbb{L}^2\times\mathcal{K}^2$ satisfying (1) for every $t\in[0,T]$ $\mathbb{P}$-a.s. , and satisfying the obstacle condition (b) and the Skorohod flatness condition (c2).

Remark 2.

(1) The Skorohod flatness condition (c2) implies in fact that
$$\int_t^T K(t,ds)=0 \text{ whenever } Y(t)>L(t) \text{ for each } t\in[0,T]\quad\mathbb{P}\text{-a.s.} \tag{2}$$
This form is more natural for the reflected BSVIE (1), since only $(K(t,s),\ s\ge t)$ is involved in the definition of $Y(t)$.

(2) In the case of a standard reflected BSDEs, the condition (c2) is analogous to
$$\int_0^T(Y(s)-L(s))\,dK(s)=0,\quad\mathbb{P}\text{-a.s.}$$
where the integral is to be interpreted in the Lebesgue–Stieltjes sense. It would be interesting to have a similar characterization for the Volterra type reflected equations.

We make the following assumptions on (f,ξ,L).

2.1. Assumptions on (f,ξ,L)

(A1) The terminal condition $\xi(t)$, parametrized by t is a $\mathcal{B}([0,T])\otimes\mathcal{F}_T$-measurable map $\xi:\Omega\times[0,T]\longrightarrow\mathbb{R}$ which satisfies
$$\sup_{0\le t\le T}\mathbb{E}[|\xi(t)|^2]<\infty;$$

(A2) The driver f is a map from $\Omega\times[0,T]\times[0,T]\times\mathbb{R}\times\mathbb{R}$ onto $\mathbb{R}$ which satisfies, for any fixed $(t,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}$, the process $f(t,\cdot,y,z)$ is progressively measurable. Moreover,

  (i) $$\sup_{0\le t\le T}\mathbb{E}\left[\left(\int_t^T|f(t,s,0,0)|ds\right)^2\right]<\infty,$$
  (ii) There exists a positive constant c_f such that $\mathbb{P}$-a.s. , for all $(t,s)\in[0,T]^2$ and $y,y',z,z'\in\mathbb{R}$,
$$|f(t,s,y,z)-f(t,s,y',z')|\le c_f\left(|y-y'|+|z-z'|\right).$$

(iii) For some $\alpha \in (0, 1/2]$ and $c_1 > 0$, for all $(y, z) \in \mathbb{R} \times \mathbb{R}$ and all $0 \le t, t' \le s \le T$,

$$|f(t', s, y, z) - f(t, s, y, z)| \le c_1 |t' - t|^{\alpha},$$

and for some $\beta > 1/\alpha$ and $c_2 > 0$,

$$\mathbb{E}[|\xi(t) - \xi(t')|^{\beta}] \le c_2 |t' - t|^{\alpha\beta}$$

and

$$\mathbb{E}\left[\left(\int_0^T |f(0, s, 0, 0)|^2 ds\right)^{\beta/2}\right] < \infty.$$

(A3) The obstacle $(L(u),\ 0 \le u \le T)$ is a real-valued and $\mathbb{F}$-adapted continuous process satisfying

$$L(T) \le \xi(t), \quad t \in [0, T] \quad \text{and} \quad \mathbb{E}[\sup_{0 \le u \le T} (L(u))^2] < \infty. \tag{3}$$

Remark 3.

(1) Assumption (iii) is similar to $(\widetilde{H1})$ in [10] and implies that

$$\mathbb{E}\left[\left(\sup_{0 \le t \le T} \int_t^T |f(t, s, 0, 0)|^2 ds\right)^{\beta/2}\right] < \infty \tag{4}$$

which is stronger than (i) since $\beta > \frac{1}{\alpha} > 2$.

(2) As we will see it below, the assumptions (A1), (i) and (A3) yield existence of a unique process (Y, Z, K) which satisfies (1) along with $(Y, Z) \in \mathcal{H}^2 \times \mathbb{L}^2$, Property (b) and K satisfying (c1) except the continuity of $(t, u) \mapsto K(t, u)$. Assumption (iii) yields the continuity of Y and the bi-continuity of $K(\cdot, \cdot)$ which in turn guarantees the Skorohod flatness condition (c2).

(3) The condition $L(T) \le \xi(t),\ t \in [0, T]$, will be explained in connection with the proof of Proposition 6.

3. Existence and uniqueness of solutions to reflected BSVIEs

In this section we derive existence, uniqueness and a comparison result for continuous reflected BSVIEs. We have

Theorem 4. *Suppose the assumptions (A1), (A2) and (A3) are satisfied. Then the reflected BSVIE (1) associated with (f, ξ, L) admits a unique solution (Y, Z, K) which satisfies (a), (b) and (c). Moreover, we have $\mathbb{P}$-a.s. the representation, for every $t \in [0, T]$,*

$$Y(t) = \operatorname*{ess\,sup}_{\tau \ge t} \mathbb{E}\left[\int_t^{\tau} f(t, s, Y(s), Z(t, s)) ds + L(\tau)\mathbb{1}_{\{\tau < T\}} + \xi(t)\mathbb{1}_{\{\tau = T\}} \Big| \mathcal{F}_t\right], \tag{5}$$

where the essential supremum is taken over $\mathbb{F}$-stopping times τ taking values in $[0, T]$.

Remark 5. We note that the representation (5) does not imply that Y is a supermartingale, as it can easily be checked.

Inspired by the approach suggested in [1,10] and [3] to solve the ordinary BSVIE by identifying an accompanying true martingale, we derive a unique solution to the reflected BSVIE (1) associated with (f, ξ, L) by applying the notion of Snell envelope along with a contraction argument to an accompanying supermartingale $\widetilde{Y}(t, \cdot)$, parametrized by t, defined below from which we obtain Y by setting $Y(t) := \widetilde{Y}(t, t)$. To this end, we first consider the case where the driver f does not depend on (Y, Z). Then, we consider the general case where f depends on (Y, Z).

3.1. Driver independent of Y and Z

Consider the following reflected BSVIE

$$Y(t) = \xi(t) + \int_t^T f(t, s) ds + \int_t^T K(t, ds) - \int_t^T Z(t, s) dW(s), \tag{6}$$

where the driver f does not depend on (Y, Z). We have

Proposition 6. *Under Assumptions (A1), (A2) and (A3), there exists a unique solution (Y, Z, K) to the reflected BSVIE (6) associated with (f, ξ, L) which satisfies the properties (a), (b) and (c). Moreover, it admits the representation (5).*

Proof. The proof is based on existence and uniqueness of the process $(\widetilde{Y}, Z, K)$ which satisfies the following reflected BSDE, parametrized by t, associated with (f, ξ, L): $\mathbb{P}$-a.s., for every $t \in [0, T]$,

$$\widetilde{Y}(t, u) = \xi(t) + \int_u^T f(t, s) ds + \int_u^T K(t, ds) - \int_u^T Z(t, s) dW(s), \quad u \in [0, T], \tag{7}$$

where

(d) $\widetilde{Y}(t, \cdot) \in \mathcal{S}^2,\ K(t, T) \in L^2(\mathbb{P}),\ Z(t, \cdot) \in \mathcal{H}^2$;
(e) $\widetilde{Y}(t, u) \ge L(u)$ $\mathbb{P}$-a.s. , $0 \le u \le T$;
(f) $K(t, \cdot)$ is continuous and increasing, $K(t, 0) = 0$ and satisfies the following version of the Skorohod flatness condition: for each $0 \le \alpha < \beta \le T$,

$$K(t, \alpha) = K(t, \beta) \quad \text{whenever} \quad \widetilde{Y}(t, u) > L(u) \quad \text{for each } u \in [\alpha, \beta] \quad \mathbb{P}\text{-a.s.} ,$$

which is equivalent to the property

$$\int_t^T (\widetilde{Y}(t, u) - L(u)) K(t, du) = 0, \quad \mathbb{P}\text{-a.s.}$$

A solution (Y, Z, K) to (6) is obtained from $(\widetilde{Y}, Z, K)$ by setting $Y(t) := \widetilde{Y}(t, t)$.

- **Existence of a solution**. For a fixed $t \in [0, T]$, set

$$\Gamma(t, u) := \int_t^u f(t, s) ds + L(u)\mathbb{1}_{\{u < T\}} + \xi(t)\mathbb{1}_{\{u = T\}}, \quad u \in [0, T].$$

In view of (i) and the continuity of the obstacle process L, the map $u \mapsto \Gamma(t, u)$ is continuous on $[0, T)$. Moreover, by (A1) and (A3), it holds that, for each $t \in [0, T]$,

$$\sup_{0 \le t \le T} \mathbb{E}\left[\sup_{0 \le u \le T} |\Gamma(t, u)|^2\right] \le 8 \sup_{0 \le t \le T} \mathbb{E}\left[|\xi(t)|^2 + \left(\int_t^T |f(t, u)| du\right)^2 + \sup_{0 \le s \le T} (L(s))^2\right] < \infty.$$

Consider the process $(\widetilde{Y}(t, u),\ u \in [0, T])$ defined by

$$\widetilde{Y}(t, u) := \operatorname*{ess\,sup}_{\tau \ge u} \mathbb{E}\left[\int_u^{\tau} f(t, s) ds + L(\tau)\mathbb{1}_{\{\tau < T\}} + \xi(t)\mathbb{1}_{\{\tau = T\}} \Big| \mathcal{F}_u\right], \tag{8}$$

where the essential supremum is taken over $\mathbb{F}$-stopping times τ taking values in $[0, T]$. The process $X_t(u) := \widetilde{Y}(t, u) + \int_t^u f(t, s) ds$, $0 \le u \le T$ satisfies

$$X_t(u) = \operatorname*{ess\,sup}_{\tau \ge u} \mathbb{E}\left[\int_t^{\tau} f(t, s) ds + L(\tau)\mathbb{1}_{\{\tau < T\}} + \xi(t)\mathbb{1}_{\{\tau = T\}} \Big| \mathcal{F}_u\right], \quad u \in [0, T],$$

i.e. it is the Snell envelope of the processes $(\Gamma(t, u))_{0 \le u \le T}$ which is the smallest continuous supermartingale, parametrized by t, which dominates the continuous process $\Gamma(t,\cdot)$. Furthermore, by Doob's inequality it holds that

$$\sup_{0\le t\le T}\mathbb{E}\left[\sup_{0\le u\le T}|\widetilde{Y}(t,u)|^2\right]\le C\sup_{0\le t\le T}\mathbb{E}\left[\sup_{0\le u\le T}|\Gamma(t,u)|^2\right]<\infty. \quad (9)$$

In particular, the processes $\Gamma(t,\cdot)$ and $\widetilde{Y}(t,\cdot)$ are uniformly integrable and the processes $\Gamma(\cdot,u)$ and $\widetilde{Y}(\cdot,u))$ are square-integrable. Furthermore, the supermartingale $(\widetilde{Y}(t,u)+\int_t^u f(t,s)ds)_{0\le u\le T}$ is square-integrable.

By the well known techniques related the Snell envelope which use the Doob–Meyer decomposition along with the martingale representation theorem (see e.g. [18], Proposition 5.1), there exists a unique adapted increasing continuous process $K(t,\cdot)$ such that $K(t,0)=0$ and $K(t,T)\in L^2(\mathbb{P})$ and an $\mathbb{F}$-adapted process $Z(t,\cdot)\in\mathcal{H}^2$ such that, $\mathbb{P}$-a.s., for every $t\in[0,T]$,

$$\widetilde{Y}(t,u)=\xi(t)+\int_u^T f(t,s)ds+\int_u^T K(t,ds)$$
$$-\int_u^T Z(t,s)dW(s),\ \ u\in[0,T],$$

for which the properties (d), (e) and (f) are satisfied. Moreover, the process
$\int_0^u Z(t,s)dW(s),\ u\in[0,T]$, is a uniformly integrable martingale and

$$\mathbb{E}\left[\sup_{0\le u\le T}|\widetilde{Y}(t,u)|^2+\int_t^T|Z(t,s)|^2ds+K^2(t,T)\right]\le C\mathbb{E}\Big[|\xi(t)|^2 + \int_t^T|f(t,s)|^2ds+\sup_{0\le u\le T}(L(u))^2\Big]. \quad (10)$$

Hence, since $Y(t)=\widetilde{Y}(t,t)$, it satisfies $\mathbb{P}$-a.s. the representation (5) and the following estimate for (Y,Z,K) holds:

$$\mathbb{E}\left[\int_0^T|Y(t)|^2dt+\int_0^T\int_t^T|Z(t,s)|^2dsdt+\int_0^T K^2(t,T)dt\right]$$
$$\le C\mathbb{E}\left[\int_0^T|\xi(t)|^2dt+\int_0^T\int_t^T|f(t,s)|^2ds+T\sup_{0\le u\le T}(L(u))^2\right], \quad (11)$$

which is finite by the assumptions (A1), (i) and (A3). Thus, $(Y,Z,K(\cdot,T))\in\mathcal{H}^2\times\mathbb{L}^2\times\mathcal{H}^2$.

Next, we will show that the maps $(t,u)\mapsto\widetilde{Y}(t,u)$, $\int_t^u f(t,s)ds$, $\int_0^u Z(t,s)dW(s)$ and $K(t,u)$ are continuous.

Before we establish bi-continuity, we derive the following estimates.

Lemma 7. *For any $\beta>1$, there exists a positive constant C_β depending only on β, c_1 and T such that*

$$\mathbb{E}\left[\left(\int_0^T|Z(t,s)-Z(t',s)|^2ds\right)^{\beta/2}\right]\le C_\beta\mathbb{E}\left[|\xi(t)-\xi(t')|^\beta\right]$$
$$+\mathbb{E}\left[\sup_{0\le u\le T}|\widetilde{Y}(t,u)-\widetilde{Y}(t',u)|^\beta\right]$$
$$+|t-t'|^{\frac{\alpha\beta}{2}}\left(\mathbb{E}\left[\sup_{0\le u\le T}|\widetilde{Y}(t,u)-\widetilde{Y}(t',u)|^\beta\right]\right)^{1/2}. \quad (12)$$

Proof. First we note that, given $t,t'\in[0,T]$, denoting by $\Delta\xi:=\xi(t)-\xi(t')$, $\Delta\widetilde{Y}(u):=\widetilde{Y}(t,u)-\widetilde{Y}(t',u)$, $\Delta Z(u):=Z(t,u)-Z(t',u)$, $\Delta f(u):=f(t,u)-f(t',u)$ and $\Delta K(ds):=K(t,ds)-K(t',ds)$ and applying Itô's formula to $|\widetilde{Y}(t,u)-\widetilde{Y}(t',u)|^2$ and the Skorohod flatness condition (f), we have, for any $0\le u\le T$,

$$|\Delta\widetilde{Y}(u)|^2+\int_u^T|\Delta Z(s)|^2ds=|\Delta\xi|^2+2\int_u^T\Delta\widetilde{Y}(s)\Delta f(s)ds$$
$$+2\int_u^T\Delta\widetilde{Y}(s)d\Delta K(ds)-2\int_u^T\Delta\widetilde{Y}(s)\Delta Z(s)dW(s)$$
$$\le|\Delta\xi|^2+2\int_u^T\Delta\widetilde{Y}(s)\Delta f(s)ds-2\int_u^T\Delta\widetilde{Y}(s)\Delta Z(s)dW(s). \quad (13)$$

Taking expectation, we obtain

$$\mathbb{E}\left[|\widetilde{Y}(t,u)-\widetilde{Y}(t',u)|^2+\int_u^T|Z(t,s)-Z(t',s)|^2ds\right]\le\mathbb{E}\left[|\xi(t')-\xi(t)|^2\right]$$
$$+2\mathbb{E}\left[\int_u^T(\widetilde{Y}(t,u)-\widetilde{Y}(t',u))(f(t,s)-f(t',s))ds\right]. \quad (14)$$

Moreover, by Burkholder–Davis–Gundy's inequality, (iii) and Young's inequality it follows that, for any $\beta>1$, there exists a positive constant C_β depending only on β, c_1 and T such that (12) holds. □

• *Continuity of the map $(t,u)\mapsto\widetilde{Y}(t,u)$.* Using (iii) we have, for $0\le t,t',u\le T$,

$$|\widetilde{Y}(t,u)-\widetilde{Y}(t',u)|$$
$$=\Big|\operatorname*{ess\,sup}_{\tau\ge u}\mathbb{E}\left[\int_u^\tau f(t,s)ds+L(\tau)\mathbb{1}_{\{\tau<T\}}+\xi(t)\mathbb{1}_{\{\tau=T\}}\Big|\mathcal{F}_u\right]-$$
$$\operatorname*{ess\,sup}_{\tau\ge u}\mathbb{E}\left[\int_u^\tau f(t',s)ds+L(\tau)\mathbb{1}_{\{\tau<T\}}+\xi(t)\mathbb{1}_{\{\tau=T\}}\Big|\mathcal{F}_u\right]\Big|$$
$$\le\operatorname*{ess\,sup}_{\tau\ge u}\mathbb{E}\left[\int_u^\tau|f(t,s)-f(t',s)|ds+|\xi(t)-\xi(t')|\Big|\mathcal{F}_u\right]$$
$$\le\mathbb{E}\Big[\int_u^T|f(t,s)-f(t',s)|ds$$
$$+\ |\xi(t)-\xi(t')|\Big|\mathcal{F}_u\Big]$$
$$\le Tc_1|t'-t|^\alpha+\mathbb{E}\left[|\xi(t)-\xi(t')|\Big|\mathcal{F}_u\right].$$

By Doob's maximal inequality and (iii), since $\beta>1/\alpha\ge2$,

$$\mathbb{E}\left[\sup_{0\le u\le T}\left(\mathbb{E}\left[|\xi(t)-\xi(t')|\Big|\mathcal{F}_u\right]\right)^\beta\right]\le\left(\frac{\beta}{\beta-1}\right)^\beta\mathbb{E}[|\xi(t)-\xi(t')|^\beta]$$
$$\le c_2\left(\frac{\beta}{\beta-1}\right)^\beta|t-t'|^{\alpha\beta}.$$

Hence,

$$\mathbb{E}\left[\sup_{0\le u\le T}|\widetilde{Y}(t,u)-\widetilde{Y}(t',u)|^\beta\right]\le C|t'-t|^{\alpha\beta}, \quad (15)$$

where $C:=2^\beta\left(Tc_1+c_2\left(\frac{\beta}{\beta-1}\right)^\beta\right)$.

In view of Kolmogorov's continuity theorem (see [19], Theorem I.2.1.), $(t,u)\mapsto\widetilde{Y}(t,u)$ is continuous (admits a bicontinuous modification). In particular, $t\mapsto Y(t):=\widetilde{Y}(t,t)$ is continuous.

• *Continuity of the map $(t,u)\mapsto g(t,u):=\int_t^u f(t,s)ds$.* For any $0\le t<t'\le T$, we have

$$\mathbb{E}\left[\sup_{0\le u\le T}|g(t',u)-g(t,u)|^\beta\right]$$
$$\le C\mathbb{E}\left[\Big|\int_t^{t'}f(t,s)ds\Big|^\beta+\left(\int_0^T|f(t',s)-f(t,s)|ds\right)^\beta\right].$$

By Hölder's inequality and (iii), we have

$$\mathbb{E}\left[\Big|\int_t^{t'}f(t,s)ds\Big|^\beta\right]\le|t'-t|^{\beta/2}\mathbb{E}\left[\left(\int_0^T|f(t,s)|^2ds\right)^{\beta/2}\right]$$
$$\le|t'-t|^{\beta/2}\sup_{0\le t\le T}\mathbb{E}\left[\left(\int_0^T|f(t,s)|^2ds\right)^{\beta/2}\right]$$
$$\le C|t'-t|^{\beta/2},$$

where the constant C is due (4). Moreover,

$$\mathbb{E}\left[\left(\int_0^T|f(t',s)-f(t,s)|ds\right)^\beta\right]\le(c_1T)^\beta|t'-t|^{\alpha\beta}.$$

Therefore, since $\alpha \in (0, 1/2]$, we have

$$\mathbb{E}\left[\sup_{0\le u\le T}|g(t',u)-g(t,u)|^{\beta}\right] \le C(1+|t'-t|^{\beta(\frac{1}{2}-\alpha)})|t'-t|^{\alpha\beta} \le C|t'-t|^{\alpha\beta}. \quad (16)$$

Since $\alpha\beta > 1$, by Kolmogorov's continuity theorem $t \mapsto g(t,u) := \int_t^u f(t,s)ds$ is continuous (admits a bicontinuous modification).

• *Continuity of the map* $(t,u) \mapsto \int_0^u Z(t,s)dW(s)$. By Burkholder–Davis–Gundy's inequality and (12), we have

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le u\le T}|\int_0^u\left(Z(t,s)-Z(t',s)\right)dW(s)|^{\beta}\right] &\le (\tfrac{\beta}{\beta-1})^{\beta}\mathbb{E}\left[\left(\int_0^T|Z(t,s)-Z(t',s)|^2ds\right)^{\beta/2}\right]\\ &\le (\tfrac{\beta}{\beta-1})^{\beta}C_{\beta}\mathbb{E}\left[|\xi(t)-\xi(t')|^{\beta}+\sup_{0\le u\le T}|\widetilde{Y}(t,u)-\widetilde{Y}(t',u)|^{\beta}\right]\\ &+(\tfrac{\beta}{\beta-1})^{\beta}C_{\beta}|t-t'|^{\frac{\alpha\beta}{2}}\left(\mathbb{E}\left[\sup_{0\le u\le T}|\widetilde{Y}(t,u)-\widetilde{Y}(t',u)|^{\beta}\right]\right)^{1/2}.\end{aligned}$$

Therefore, by (iii) and (15), it holds that

$$\mathbb{E}\left[\sup_{0\le u\le T}|\int_0^u\left(Z(t,s)-Z(t',s)\right)dW(s)|^{\beta}\right] \le \widehat{C}_{\beta}|t-t'|^{\alpha\beta}.$$

Hence, in view of Kolmogorov's continuity theorem, $(t,u) \mapsto \int_0^u Z(t,s)dW(s)$ is continuous (admits a bicontinuous modification).

• *Continuity of the map* $(t,u) \mapsto K(t,u)$. This follows from the fact that

$$K(t,u) = \widetilde{Y}(t,u)-\widetilde{Y}(t,0)-\int_0^u f(t,s)ds+\int_0^u Z(t,s)dW(s), \quad (t,u)\in[0,T]^2,$$

and the bi-continuity of each of the terms on the r.h.s.

With $Y(t) := \widetilde{Y}(t,t)$, we obtain a solution$(Y,Z,K)$ to (6) which satisfies (a), (b) and (c1). It remains to check the Skorohod flatness condition (c2). Indeed, in view of the property (f), for each fixed $t \in [0,T]$, it holds that, for each $0 \le \alpha < \beta \le T$,

$$K(t,\alpha) = K(t,\beta) \quad \text{whenever } \widetilde{Y}(t,u) > L(u) \text{ for each } u\in[\alpha,\beta] \quad \mathbb{P}\text{-a.s. },$$

Thanks to the bi-continuity of $\widetilde{Y}(\cdot,\cdot)$ and $K(\cdot,\cdot)$, by sending t to u it holds that for each $0 \le \alpha < \beta \le T$,

$$K(u,\alpha) = K(u,\beta) \quad \text{whenever } Y(u)=\widetilde{Y}(u,u) > L(u) \text{ for each } u\in[\alpha,\beta] \quad \mathbb{P}\text{-a.s.}$$

By virtue of the bi-continuity of these mappings, Eq. (7) holds for every $t \in [0,T]$, $\mathbb{P}$-a.s.

• **Uniqueness of the solution**. Let (Y',Z',K') be another solution to (1) associated with (f,ξ,L) satisfying (a), (b) and (c). Define $\widehat{Y} := Y-Y', \widehat{Z} := Z-Z'$ and $\widehat{K} := K-K'$, and correspondingly $\widehat{\widetilde{Y}} := \widetilde{Y}-\widetilde{Y}'$. Applying Itô's formula to $|\widehat{\widetilde{Y}}|^2$ and taking expectation we obtain, for each fixed $t \in [0,T]$,

$$\mathbb{E}\left[|\widehat{\widetilde{Y}}(t,u)|^2+\int_u^T|\widehat{Z}(t,s)|^2ds\right] = 2\mathbb{E}\left[\int_u^T\widehat{\widetilde{Y}}(t,s)\widehat{K}(t,ds)\right], \quad u\in[0,T].$$

But, by the flatness condition (f), we have $\int_u^T\widehat{\widetilde{Y}}(t,s)\widehat{K}(t,ds) \le 0$ $\mathbb{P}$-a.s. which implies that $\mathbb{E}\left[|\widehat{\widetilde{Y}}(t,u)|^2\right] = 0$ and $E\left[\int_u^T|\widehat{Z}(t,s)|^2ds\right] = 0$ for all $u \in [0,T]$. Therefore, $\widetilde{Y}(t,\cdot) = \widetilde{Y}'(t,\cdot)$ and $Z(t,\cdot) = Z'(t,\cdot)$ and then $K(t,\cdot) = K'(t,\cdot)$ $\mathbb{P}$-a.s. Hence $Y = Y'$ $\mathbb{P}$-a.s. , by the continuity of the map $(t,u) \mapsto \widetilde{Y}(t,u)$. □

3.2. *The general case. Proof of Theorem 4*

Let $\mathcal{E} := \mathcal{H}^2 \times \mathbb{L}^2$ be the Hilbert space endowed with $\|(Y,Z)\|^2_{\mathcal{E}} = \|Y\|^2_{\mathcal{H}^2}+\|Z\|^2_{\mathbb{L}^2}$.

Consider the map Φ from $\mathcal{E}$ to itself for which $(Y,Z) = \Phi(U,V)$, where

$$\int_0^t K(t,ds) := Y(t)-Y(0)-\int_0^t f(t,s,U(s),V(t,s))ds + \int_0^t Z(t,s)dW(s), \quad t\in[0,T]. \quad (17)$$

Proposition 8. *Assume (A1), (A2) and (A3). Then there exists $\delta > 0$ depending only on the Lipschitz constant of f such that Φ is a contraction mapping on the space $\mathcal{E}([T-\delta,T])$.*

Proof. Let $X := (U,V)$ and $X' := (U',V')$ be two elements of $\mathcal{E}$ and define $\overline{X} = (\overline{U},\overline{V}) = X-X'$, $\overline{Y} = Y-Y'$, $\overline{Z} = Z-Z'$, $\overline{K} = K-K'$ and for $s,t \in [0,T]$, $\bar{f}(t,s) := f(t,s,U(s),V(t,s)) - f(t,s,U'(s),V'(t,s))$. We have

$$\int_0^t \overline{K}(t,ds) := \overline{Y}(t)-\overline{Y}(0)-\int_0^t\bar{f}(t,s)ds+\int_0^t\overline{Z}(t,s)dW(s), \quad 0\le t\le T.$$

From Proposition 6, we have $\mathbb{P}$-a.s.

$$Y(t) = \operatorname*{ess\,sup}_{\tau\ge t}\mathbb{E}\left[\int_t^{\tau}f(t,s,U(s),V(t,s))ds+L(\tau)\mathbb{1}_{\{\tau<T\}}+\xi(t)\mathbb{1}_{\{\tau=T\}}\Big|\mathcal{F}_t\right],$$
$$Y'(t) = \operatorname*{ess\,sup}_{\tau\ge t}\mathbb{E}\left[\int_t^{\tau}f(t,s,U'(s),V'(t,s))ds+L(\tau)\mathbb{1}_{\{\tau<T\}}+\xi(t)\mathbb{1}_{\{\tau=T\}}\Big|\mathcal{F}_t\right].$$

Thus,

$$|\overline{Y}(t)| \le \operatorname*{ess\,sup}_{\tau\ge t}\mathbb{E}\left[\int_t^{\tau}|\bar{f}(t,s)|ds\Big|\mathcal{F}_t\right] \le \mathbb{E}\left[\int_t^T|\bar{f}(t,s)|ds\Big|\mathcal{F}_t\right].$$

Noting that by the Cauchy–Schwarz inequality and (ii) we have

$$\begin{aligned}\mathbb{E}\left[\int_{T-\delta}^T\left(\int_t^T|\bar{f}(t,s)|ds\right)^2dt\right] &\le \delta\mathbb{E}\left[\int_{T-\delta}^T\int_t^T|\bar{f}(t,s)|^2dsdt\right]\\ &\le 4c_f^2\delta\mathbb{E}\left[\int_{T-\delta}^T\int_t^T|\overline{U}(s)|^2dsdt+\int_{T-\delta}^T\int_t^T|\overline{V}(t,s)|^2dsdt\right]\\ &\le 4c_f^2(\delta^2+\delta)\|(\overline{U},\overline{V})\|^2_{\mathcal{E}([T-\delta,T])}.\end{aligned} \quad (18)$$

Therefore,

$$\begin{aligned}\mathbb{E}\left[\int_{T-\delta}^T|\overline{Y}(t)|^2dt\right] &\le \mathbb{E}\left[\int_{T-\delta}^T\left(\mathbb{E}\left[\int_t^T|\bar{f}(t,s)|ds\Big|\mathcal{F}_t\right]\right)^2dt\right]\\ &\le \mathbb{E}\left[\int_{T-\delta}^T\left(\int_t^T|\bar{f}(t,s)|ds\right)^2dt\right]\\ &\le 4c_f^2(\delta^2+\delta)\|(\overline{U},\overline{V})\|^2_{\mathcal{E}([T-\delta,T])}.\end{aligned}$$

On the other hand, by Itô's formula applied to $|\overline{\widetilde{Y}}(t,u)|^2$, taking expectation and using Young's inequality, we obtain

$$\mathbb{E}\left[\int_u^T|\overline{Z}(t,s)|^2ds\right] \le 2E\left[\int_u^T\overline{\widetilde{Y}}(t,s)\bar{f}(t,s)ds\right].$$

In view of (8), we have for any $u \in [0,T]$,

$$|\overline{\widetilde{Y}}(t,u)| \le \operatorname*{ess\,sup}_{\tau\ge u}\mathbb{E}\left[\int_u^{\tau}|\bar{f}(t,s)|ds\Big|\mathcal{F}_u\right] \le \mathbb{E}\left[\int_u^T|\bar{f}(t,s)|ds\Big|\mathcal{F}_u\right].$$

Therefore,

$$\begin{aligned}\mathbb{E}\left[\int_u^T|\overline{Z}(t,s)|^2ds\right] &\le 2E\left[\int_u^T\overline{\widetilde{Y}}(t,s)\bar{f}(t,s)ds\right]\\ &\le 2E\left[\int_u^T\bar{f}(t,s)\mathbb{E}\left[\int_s^T|\bar{f}(t,r)|dr\Big|\mathcal{F}_s\right]ds\right]\\ &\le 2E\left[\int_u^T|\bar{f}(t,s)|\mathbb{E}\left[\int_u^T|\bar{f}(t,r)|dr\Big|\mathcal{F}_s\right]ds\right]\\ &= 2E\left[\left(\int_u^T|\bar{f}(t,r)|dr\right)^2\right].\end{aligned}$$

In particular, in view of (18),

$$\mathbb{E}\left[\int_{T-\delta}^T\int_t^T|\bar{Z}(t,s)|^2dsdt\right]\le 2E\left[\int_{T-\delta}^T\left(\int_t^T|\bar{f}(t,r)|dr\right)^2dt\right]$$
$$\le 8c_f^2(\delta^2+\delta)\|(\overline{U},\overline{V})\|^2_{\mathcal{E}([T-\delta,T])}.$$

Summing up, we have

$$\|(\overline{Y},\overline{Z})\|^2_{\mathcal{E}([T-\delta,T])}\le 8c_f(\delta^2+\delta)\|(\overline{U},\overline{V})\|^2_{\mathcal{E}([T-\delta,T])}.$$

Now, choosing $\delta>0$ such that $\delta^2+\delta<1/8c_f$, yields that Φ is a contraction mapping on $\mathcal{E}([T-\delta,T])$ and thus admits a unique fixed point which yields the unique solution of the reflected BSVIE (1) associated with (f,ξ,L) over $[T-\delta,T]$. □

We are now ready to give a proof of Theorem 4.

Proof. By repeatedly applying the fixed point argument of Proposition 8 over adjacent time intervals of fixed length δ, satisfying $\delta^2+\delta<1/8c_f$, and pasting the solutions that we describe below, we finally obtain existence of a unique solution in $\mathcal{H}^2\times\mathbb{L}^2\times\mathcal{K}^2$ to the reflected BSVIE (1) over the whole time interval $[0,T]$.

Since we are dealing with Volterra-type integrals where the driver f depends on t, we paste two adjacent solutions as follows. Fix such a δ and let (Y^0,Z^0,K^0) be the unique solution of (1) on $\mathcal{E}([T-\delta,T])$. Consider the accompanying reflected BSDE $(\widetilde{Y}^0(t,\cdot),\widetilde{Z}^0(t,\cdot),\widetilde{K}^0(t,\cdot))$, with lower obstacle L, parametrized by $t\in[0,T]$, defined by

$$\begin{aligned}\widetilde{Y}^0(t,u)&=\xi(t)+\int_u^T f(t,s,Y^0(s),\widetilde{Z}^0(t,s))ds\\&\quad+\int_u^T\widetilde{K}^0(t,ds)-\int_u^T\widetilde{Z}^0(t,s)dW(s),\\&\quad u\in[T-\delta,T],\end{aligned}$$

where $\widetilde{K}^0(t,T-\delta)=0$. In particular, $\widetilde{Y}^0(t,T-\delta)\in L^2(\mathcal{F}_{T-\delta})$. Moreover, by Proposition 8, we have $\mathbb{P}$-a.s.

$$(\widetilde{Y}^0(t,t),\widetilde{Z}^0(t,s),\widetilde{K}^0(t,s))=(Y^0(t),Z^0(t,s),K^0(t,s)),$$
$$(t,s)\in[T-\delta,T]^2,\ t\le s.$$

On the time interval $[T-2\delta,T-\delta]$, we consider the BSVIE

$$\begin{aligned}Y^1(t)&=\widetilde{Y}^0(t,T-\delta)+\int_t^{T-\delta}f(t,s,Y^1(s),Z^1(t,s))ds\\&\quad+\int_t^{T-\delta}K^1(t,ds)-\int_t^{T-\delta}Z^1(t,s)dW(s),\\&\quad t\in[T-2\delta,T-\delta].\end{aligned}\tag{19}$$

Since the length of the time interval $[T-2\delta,T-\delta]$ is δ, we may apply Proposition 8 to obtain a unique solution $(Y^1,Z^1,K^1)\in\mathcal{H}^2([T-2\delta,T-\delta])\times\mathbb{L}^2([T-2\delta,T-\delta])\times\mathcal{K}^2([T-2\delta,T-\delta])$ to the reflected BSVIE (19).

Now, with

$$Y(t):=\begin{cases}Y^0(t), & t\in[T-\delta,T],\\ Y^1(t), & t\in[T-2\delta,T-\delta],\end{cases}$$

and

$$(Z(t,s),K(t,s))$$
$$:=\begin{cases}(Z^0(t,s),K^0(t,s)), & (t,s)\in[T-\delta,T]^2,\ t\le s,\\ (\widetilde{Z}^0(t,s),\widetilde{K}^0(t,s)), & t\in[T-2\delta,T-\delta]\times[T-\delta,T],\\ (Z^1(t,s),K^1(t,s)), & (t,s)\in[T-2\delta,T-\delta]^2,\ t\le s,\end{cases}$$

the process (Y,Z,K) solves the reflected BSVIE (1) over the time interval $[T-2\delta,T]$.

By repeating the same reasoning on each time interval $[T-(m+1)\delta,T-m\delta]$, $m=1,2,\dots,n$ (where n is arbitrary) with a similar dynamics but terminal condition $\widetilde{Y}^{m-1}(t,T-m\delta)$ at time $T-m\delta$, for $t\in[T-(m+1)\delta,T-m\delta]$, we construct recursively, for $m=1,2,\dots,n$, a solution $(Y^m,Z^m,K^m)\in\mathcal{H}^2\times\mathbb{L}^2\times\mathcal{K}^2$ on each time interval $[T-(m+1)\delta,T-m\delta]$. Pasting these processes, we obtain a unique solution (Y,Z,K) of the reflected BSVIE (1) on the full time interval $[0,T]$. □

A closer look at the way the estimate (10) is derived in e.g. [18] or [20], we obtain the following proposition as a direct consequence of the estimate leading to it.

Proposition 9. *If instead of (A1) and (A2), we assume that*

$$\mathbb{E}[\sup_{0\le t\le T}|\xi(t)|^2]<\infty\quad\text{and}\quad\mathbb{E}\left[\sup_{0\le t\le T}\left(\int_t^T|f(t,s,0,0)|ds\right)^2\right]<\infty,$$

then

$$\|Y\|^2_{\mathcal{S}^2}:=\mathbb{E}\left[\sup_{0\le t\le T}|Y(t)|^2\right]<\infty.$$

3.3. A comparison result for reflected BSVIEs

In this section we derive a comparison theorem, similar to that of [18], Theorem 4.1 for standard reflected BSDEs, which extends [8], Theorem 3.4., for non-reflected BSVIEs. We follow the method of the proof in [8] and first derive a comparison when the driver f does not depend of y, extending [8], Proposition 3.3, since in this case the proof is based on the comparison principle for reflected BSDEs, and then proceed to the proof of the general case using an approximation scheme. Some of the imposed conditions on the coefficients can be relaxed at the expense of heavy technical details that we omit to make the content easy to follow.

Proposition 10. *Let (f,ξ,L) and (f',ξ',L') be two sets of processes which satisfy the assumptions (A1), (A2) and (A3) and suppose further that*

(i) $\xi(t)\le\xi'(t)$, *$\mathbb{P}$-a.s. ,* $0\le t\le T$,
(ii) $f(t,s,z)\le f'(t,s,z)$, *for all* $(t,z)\in[0,s]\times\mathbb{R}$, *a.s., a.e.* $s\in[0,T]$,
(iii) $L(s)\le L'(s)$, $0\le s\le T$, *$\mathbb{P}$-a.s.*

Let (Y,Z,K) be a solution of the reflected BSVIE associated with (f,ξ,L) and (Y',Z',K') be a solution of the reflected BSVIE associated with (f',ξ',L'). Then

$$Y(t)\le Y'(t),\quad 0\le t\le T,\quad \mathbb{P}\text{-a.s.}$$

Proof. For each fixed $t\in[0,T]$, let $\widetilde{Y}(t,\cdot)$ and $\widetilde{Y}'(t,\cdot)$ be the standard BSDEs accompanying Y and Y' respectively, constructed in a similar way to the one described in the proof of Proposition 6: For each fixed $t\in[0,T]$,

$$\begin{aligned}\widetilde{Y}(t,u)&=\xi(t)+\int_u^T f(t,s,Z(t,s))ds\\&\quad+\int_u^T K(t,ds)-\int_u^T Z(t,s)dW(s),\quad u\in[0,T].\end{aligned}$$

A similar form for $\widetilde{Y}'$ holds. We may apply the comparison theorem ([18], Theorem 4.1) for standard reflected BSDEs to obtain that, for each fixed $t\in[0,T]$,

$$\widetilde{Y}(t,u)\le\widetilde{Y}'(t,u),\quad 0\le u\le T,\quad \text{a.s.}$$

Since, the maps $(t,u)\mapsto\widetilde{Y}(t,u),\widetilde{Y}'(t,u)$ are continuous, it follows that

$$Y(t):=\widetilde{Y}(t,t)\le\widetilde{Y}'(t,t)=:Y'(t),\quad 0\le t\le T,\quad \text{a.s.}\quad\square$$

Theorem 11. *Let (f,ξ,L) and (f',ξ',L') satisfy the assumptions (A1), (A2) and (A3) and that either the map $y\mapsto f(t,s,y,z)$ or*

$y \mapsto f'(t,s,y,z)$ is nondecreasing. Assume further that

(H1) $\xi(t) \le \xi'(t)$, $\mathbb{P}$-a.s. , $0 \le t \le T$,
(H2) $f(t,s,y,z) \le f'(t,s,y,z)$, for all $(t,z) \in [0,s] \times \mathbb{R}$, a.s., a.e. $s \in [0,T]$,
(H3) $L(s) \le L'(s)$, $0 \le s \le T$, $\mathbb{P}$-a.s.

Let (Y,Z,K) be a solution of the reflected BSVIE associated with (f,ξ,L) and (Y',Z',K') be a solution of the reflected BSVIE associated with (f',ξ',L'). Then

$$Y(t) \le Y'(t), \quad 0 \le t \le T, \quad \mathbb{P}\text{-a.s.}$$

Proof. The proof follows the same steps of the proof of Theorem 3.2 in [8]. We sketch it and leave some technical details related to Peng's monotone convergence theorem [21] which are by now standard in the literature related to reflected BSDEs.

Assume the map $y \mapsto f(t,s,y,z)$ is nondecreasing. Set $Y_0(\cdot) = Y'(\cdot)$ and for $n \ge 1$, consider, the following sequence of reflected BSVIEs with the same lower obstacle L

$$Y_n(t) = \xi(t) + \int_t^T f(t,s,Y_{n-1}(s),Z_n(t,s))ds + \int_t^T K_n(t,ds) - \int_t^T Z_n(t,s)dW(s).$$

In view of the assumption (H2) and the monotonicity of f in y, we may apply Proposition 10, to obtain that, for every $n \ge 1$,

$$Y'(t) = Y_0(t) \ge Y_1(t) \ge \cdots \ge Y_n(t) \ge Y_{n+1}(t), \quad t \in [0,T], \text{ a.s.}$$

If we show that $(Y_n, Z_n, K_n)_{n \ge 1}$ converges to (Y,Z,K) w.r.t. the norm defined, for a given $\theta > 0$, by

$$\|(Y,Z,K)\|_\theta^2 := \mathbb{E}\left[\int_0^T e^{\theta t}|Y(t)|^2 dt + \int_0^T e^{\theta t}\int_t^T |Z(t,s)|^2 dsdt + \int_0^T e^{\theta t}|K(t,T)|^2 dt\right]$$

which is equivalent with the norm defined on the space $\mathcal{H}^2 \times \mathbb{L}^2 \times \mathcal{H}^2$, then the limit process (Y^*, Z^*, K^*) satisfies the reflected BSVIE

$$Y^*(t) = \xi(t) + \int_t^T f(t,s,Y^*(s),Z^*(t,s))ds + \int_t^T K^*(t,ds) - \int_t^T Z^*(t,s)dW(s)$$

along with the Skorohod flatness condition (obtained by taking a.s. converging subsequences) w.r.t. the same lower obstacle L. By uniqueness of the solution, we have $Y^*(\cdot) = Y(\cdot)$ which in turn entails the comparison principle.

To show convergence, let $\widetilde{Y}_n(t,\cdot)$ be the solution to the standard reflected BSDEs, parametrized by $t \in [0,T]$, accompanying Y_n (i.e. $\widetilde{Y}_n(t,t) = Y_n(t)$), constructed in a similar way to the one described in the proof of Proposition 6, defined, for each fixed $t \in [0,T]$, by

$$\widetilde{Y}_n(t,u) = \xi(t) + \int_u^T f(t,s,Y_{n-1}(s),Z_n(t,s))ds + \int_u^T K_n(t,ds) - \int_u^T Z_n(t,s)dW(s), \quad u \in [0,T],$$

For $n,m \ge 1$, set $\delta\widetilde{Y}(t,s) := \widetilde{Y}_n(t,s) - \widetilde{Y}_m(t,s)$, $\delta Z(t,u) := Z_n(t,u) - Z_m(t,u)$, $\delta f(t,s) := f(t,s,Y_{n-1}(s),Z_n(t,s)) - f(t,s,Y_{m-1}(s),Z_m(t,s))$ and $\delta K(t,u) = K_n(t,u) - K_m(t,u)$.

For any $\theta > 0$, we may apply Itô's formula to $e^{\theta u}|\delta\widetilde{Y}(t,u)|^2$ along with the Skorohod flatness condition to obtain

$$\begin{aligned}
&\mathbb{E}\left[e^{\theta u}|\delta\widetilde{Y}(t,u)|^2 + \int_u^T e^{\theta s}|\delta Z(t,s)|^2 ds\right] \\
&\quad \le 2\mathbb{E}\left[\int_u^T e^{\theta s}\delta\widetilde{Y}(t,s)\delta f(t,s)ds\right] - \theta\mathbb{E}\left[\int_u^T e^{\theta s}|\delta\widetilde{Y}(t,s)|^2 ds\right] \\
&\quad \le -\theta\mathbb{E}\left[\int_u^T e^{\theta s}|\delta\widetilde{Y}(t,s)|^2 ds\right] \\
&\qquad + \mathbb{E}\left[\int_u^T e^{\theta s}\left(\theta|\delta\widetilde{Y}(t,s)|^2 + \tfrac{1}{\theta}|\delta f(t,s)|^2\right)ds\right] \\
&\quad \le \tfrac{2c_f^2}{\theta}\mathbb{E}\left[\int_u^T e^{\theta s}\left(|Y_{n-1}(s) - Y_{m-1}(s)|^2 + |\delta Z(t,s)|^2\right)ds\right].
\end{aligned}$$

Therefore, with $Y_n(t) = \widetilde{Y}_n(t,t)$, we have

$$\begin{aligned}
&E\left[\int_0^T e^{\theta t}|Y_n(t) - Y_m(t)|^2 dt\right] \\
&\qquad + (1 - \tfrac{2c_f^2}{\theta})\mathbb{E}\left[\int_0^T \int_t^T e^{\theta s}|Z_n(t,s) - Z_m(t,s)|^2 dsdt\right] \\
&\quad \le \tfrac{2Tc_f^2}{\theta}\mathbb{E}\left[\int_0^T e^{\theta s}|Y_{n-1}(s) - Y_{m-1}(s)|^2 ds\right].
\end{aligned}$$

Moreover, in view of the estimate in Proposition 3.6 of [18], there exists a constant C independent of n and m such that

$$\begin{aligned}
\mathbb{E}\left[|\delta K(t,T)|^2\right] &\le C\mathbb{E}\left[\int_t^T |\delta f(t,s)|^2 ds\right] \\
&\le 2Cc_f^2\mathbb{E}\left[\int_t^T \left(|Y_{n-1}(s) - Y_{m-1}(s)|^2 + |\delta Z(t,s)|^2\right)ds\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{\theta t}|\delta K(t,T)|^2 dt\right] \\
&\le 2Cc_f^2\mathbb{E}\left[\int_0^T e^{\theta t}\int_t^T |Y_{n-1}(s) - Y_{m-1}(s)|^2 dsdt\right] \\
&\qquad + 2Cc_f^2\mathbb{E}\left[\int_0^T e^{\theta t}\int_t^T |\delta Z(t,s)|^2 dsdt\right] \\
&\le \tfrac{2Cc_f^2}{\theta}\mathbb{E}\left[\int_0^T e^{\theta s}|Y_{n-1}(s) - Y_{m-1}(s)|^2 ds\right] \\
&\qquad + 2Cc_f^2\mathbb{E}\left[\int_0^T e^{\theta t}\int_t^T |\delta Z(t,s)|^2 dsdt\right],
\end{aligned}$$

since

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{\theta t}\int_t^T |Y_{n-1}(s) - Y_{m-1}(s)|^2 dsdt\right] \\
&\quad = \mathbb{E}\left[\int_0^T |Y_{n-1}(s) - Y_{m-1}(s)|^2 ds \int_0^s e^{\theta t}dt\right] \\
&\quad \le \tfrac{1}{\theta}\mathbb{E}\left[\int_0^T e^{\theta s}|Y_{n-1}(s) - Y_{m-1}(s)|^2 ds\right].
\end{aligned}$$

Thus, by choosing $\theta > 2c_f^2(1+2T)$, it follows that $(Y_n, Z_n, K_n)_n$ is a Cauchy sequence w.r.t. the norm $\|\cdot\|_\theta$. □

4. Application to time-inconsistent optimal stopping problems

Let $(X(t), t \in [0,T])$ be an $\mathbb{F}$-adapted process. In many financial applications X may model the price of a commodity. Suppose that (f, L, ξ) satisfies the assumptions (A1), (A2) and (A3) of Section 2. We propose to solve the following optimal stopping problem associated with (f, ξ, L):

$$\sup_{\tau \ge t} J(t,\tau), \tag{20}$$

for

$$J(t,\tau) := \mathbb{E}\left[\int_t^\tau f(t,s,X(s))ds + L(\tau)\mathbb{1}_{\{\tau<T\}} + \xi(t)\mathbb{1}_{\{\tau=T\}}\right], \tag{21}$$

where the supremum is taken over $\mathbb{F}$-stopping times τ taking values in $[t,T]$. More precisely, we would like to find an $\mathbb{F}$-stopping time τ_t^*, indexed by t, such that

$$\tau_t^* = \arg\max_{\tau \ge t} J(t,\tau).$$

Let $(Y(t), t \in [0,T])$ defined by

$$Y(t) := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}\left[\int_t^\tau f(t,s,X(s))ds + L(\tau)\mathbb{1}_{\{\tau<T\}} + \xi(t)\mathbb{1}_{\{\tau=T\}} \,\Big|\, \mathcal{F}_t\right] \tag{22}$$

be the value-function associated to the optimal stopping problem (20).

In financial applications, the functional J represents the yield of an investment in a commodity with price process X, where f is the utility rate per unit time, L is the utility function at the stopping time τ and ξ is the utility at the final time T.

The dependence on t in Eq. (21) implies that the optimal stopping problem is in general time-inconsistent which is similar to time-inconsistency in optimal stochastic control (see e.g. [17,22]).

In time-consistent optimal stopping problems (i.e. when f and ξ do not depend of t), the process Y would be the value function of the stopping problem (20). In particular $Y(0) = \sup_{\tau\geq 0} J(0,\tau)$. This is not the case here. However, we have

$$\sup_{\tau\geq t} J(t,\tau) \leq \mathbb{E}[Y(t)].$$

Now, if we can find an $\mathbb{F}$-stopping time τ_t^* such that

$$Y(t) = \mathbb{E}\left[\int_t^{\tau_t^*} f(t,s,X(s))ds + L(\tau_t^*)\mathbb{1}_{\{\tau_t^*<T\}} + \xi(t)\mathbb{1}_{\{\tau_t^*=T\}} \,\Big|\, \mathcal{F}_t\right]$$

then τ_t^* is optimal for $J(t,\cdot)$ since

$$J(t,\tau_t^*) = \mathbb{E}[Y(t)] \leq \sup_{\tau\geq t} J(t,\tau) \leq \mathbb{E}[Y(t)]. \tag{23}$$

In view of Proposition 6, there exists a unique process $(Z,K) \in \mathbb{L}^2 \times \mathcal{H}^2$ such that

$$Y(t) = \xi(t) + \int_t^T f(t,s,X(s))ds + \int_t^T K(t,ds) - \int_t^T Z(t,s)dW(s),$$

and (Y,Z,K) satisfies the properties (a), (b) and (c) of . Moreover, Y is constructed so that, for every $t \in [0,T]$, $Y(t) = \widetilde{Y}(t,t)$, where $\widetilde{Y}(t,\cdot)$ is the accompanying reflected BSDE associated with $(f(t,s,X(s)), L(s), \xi(t))$, parametrized by t, which satisfies

$$\widetilde{Y}(t,u) = \xi(t) + \int_u^T f(t,s,X(s))ds + \int_u^T K(t,ds) - \int_u^T Z(t,s)dW(s), \quad u \in [t,T],$$

and for which $(\widetilde{Y}(t,\cdot), Z(t,\cdot), K(t,\cdot))$ satisfies the conditions (d), (e) and (f) displayed in the proof of Proposition 6.

Proposition 12. *Suppose the assumptions (A1), (A2) and (A3) are satisfied. For each $t \in [0,T]$, denote by τ_t^* the stopping time*

$$\tau_t^* = \inf\{t \leq u \leq T;\ \widetilde{Y}(t,u) = L(u)\}$$

with the convention that $\tau_t^* = T$ if $\widetilde{Y}(t,u) > L(u),\ t \leq u \leq T$.

Then τ_t^* is optimal in the sense that

$$Y(t) = \mathbb{E}\left[\int_t^{\tau_t^*} f(t,s,X(s))ds + L(\tau_t^*)\mathbb{1}_{\{\tau_t^*<T\}} + \xi(t)\mathbb{1}_{\{\tau_t^*=T\}} \,\Big|\, \mathcal{F}_t\right]. \tag{24}$$

Moreover, τ_t^* is an optimal strategy for $J(t,\cdot)$ i.e.

$$\tau_t^* = \arg\max_{\tau\geq t} J(t,\tau).$$

Proof. We have

$$\begin{aligned} Y(t) &= \xi(t) + \textstyle\int_t^T f(t,s,X(s))ds + \int_t^T K(t,ds) - \int_t^T Z(t,s)dW(s) \\ &= \widetilde{Y}(t,\tau_t^*) + \textstyle\int_t^{\tau_t^*} f(t,s,X(s))ds + \int_t^{\tau_t^*} K(t,ds) - \int_t^{\tau_t^*} Z(t,s)dW(s). \end{aligned}$$

Now, the Skorohod flatness condition (f) along with the continuity of the map $u \mapsto K(t,u)$ imply that

$$\int_t^{\tau_t^*} K(t,ds) = 0.$$

Thus, taking conditional expectation we finally obtain

$$Y(t) = \mathbb{E}\left[\int_t^{\tau_t^*} f(t,s,X(s))ds + L(\tau_t^*)\mathbb{1}_{\{\tau_t^*<T\}} + \xi(t)\mathbb{1}_{\{\tau_t^*=T\}} \,\Big|\, \mathcal{F}_t\right],$$

since, by continuity of the map $u \mapsto \widetilde{Y}(t,u)$, we have $\widetilde{Y}(t,\tau_t^*) = L(\tau_t^*)\mathbb{1}_{\{\tau_t^*<T\}} + \xi(t)\mathbb{1}_{\{\tau_t^*=T\}}$. In view of (23), it follows that $\tau_t^* = \arg\max_{\tau\geq t} J(t,\tau)$. □

Remark 13. The choice of stopping time τ_t^* as the first hitting time of the accompanying Snell envelope $\widetilde{Y}(t,\cdot)$ of the obstacle L instead of the value function Y, as it is the case for standard reflected BSDEs, is simply due to fact that for Volterra type equations we have

$$Y(t) \neq Y(u) + \int_u^T f(t,s,Y(s),Z(t,s))ds + \int_u^T K(t,ds) - \int_u^T Z(t,s)dW(s), \quad u \geq t.$$

Remark 14. The same approach can be applied to find an optimal solution to the following optimal stopping problem related to time-inconsistent recursive utility functions

$$J(t,\tau) := \mathbb{E}\left[\int_t^\tau f(t,s,X(s),Y(s),Z(t,s))ds + L(\tau)\mathbb{1}_{\{\tau<T\}} + \xi(t)\mathbb{1}_{\{\tau=T\}}\right], \tag{25}$$

provided that (f,ξ,L) satisfies (A1), (A2) and (A3).

CRediT authorship contribution statement

Nacira Agram: Visualization, Investigation, Writing - reviewing & editing. **Boualem Djehiche:** Supervision, Writing - reviewing & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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