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# Teaching toddlers the meaning of numbers-connecting modes of mathematical representations in book reading 

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#### Abstract

In this article, we direct attention to what becomes critical in teaching activities for toddlers (1-3-year-olds) to learn the meaning of numbers. One activity we thoroughly explore is interactive book reading, based on previous research indicating positive learning outcomes from this type of mathematical activity, as it has shown to simultaneously embrace the child's perspective and encourage interaction and 'number talk.' A specially designed picture book presenting small quantities was developed, and variation theory principles were embedded in both the book design and the teaching acts. Through qualitative analyses, we aim to identify what is critical in the interactive book reading sessions for toddlers to discern essential aspects of numbers, with a specific focus on the conditions for making modes of representations into resources for learning. Preschool teachers frequently read the book to 27 toddlers over the course of a year. Video documentation of their reading sessions was analyzed, and exposed the significance of addressing the child's perspective when choosing what representation to emphasize and in what ways connections within and between representations can be made. Thus, the study contributes knowledge on the teaching of numbers with toddlers, and problematizes as well as extends the potential of interactive book reading as a quality-enhancing educational tool.


Keywords Early childhood education • Learning • Numbers • Modes of representation • Teaching • Toddlers

## 1 Introduction

This article reports from a Swedish educational research project concerning toddlers' (1-3-year-olds) numerical development, focusing on what becomes critical in preschool education for young children to discern basic aspects of numbers. While there is a large body of research on numerical development, few studies focus on how this development can be facilitated or how the teaching of numbers can be made meaningful for toddlers;

[^0]that is, meaningful in both a mathematical and a child-responsive sense. Here, we address both these issues by analyzing the potential for learning of numbers in an interactive book reading activity.

The activity was chosen based on previous research indicating positive learning outcomes from this type of mathematical activity, as it may simultaneously embrace the child's perspective and encourage interaction and talk about numbers' meaning and use. Numbers represented in different modes, such as symbolically by counting words or finger patterns, groups of items found in pictures, and manipulative materials, were especially focused on. This focus is based on Lesh (1981; see also Lesh et al., 1987) that using different modes of representation and becoming aware of connections within and between them is important in order for children to discern necessary aspects of, for example, numbers. The aim, then, is to identify what becomes critical in the interactive book reading sessions for the children to discern essential aspects of numbers. To fulfill this aim, we pose the following research question: What are the conditions for making modes of representations into resources for learning? To answer this question, we looked for qualitative differences regarding how the meaning of numbers (re)presented to the toddlers (thus the representations made into resources for learning). This, to reveal under what teaching conditions, the participating toddlers actively engaged in exploring numbers' meaning. The empirical data used for analysis is video documentations of authentic reading sessions in which preschool teachers read a specially designed book with 27 toddlers, for a period of one year.

### 1.1 Picture book reading in early mathematics education

Picture books are often used as pedagogical artifacts in early childhood education because pictures attract children's interest and picture books have the potential to frame learning content in familiar contexts and through visual representations. An exciting and inspiring narrative may also direct and maintain attention in a shared reading activity. However, it cannot be taken for granted that mathematical meaning is mediated through picture books. For instance, the appearance of varying number sets in pictures alone was found insufficient for preschool children to direct attention to new aspects of numbers (Björklund \& Palmér, 2020). Also, Elia et al. (2010) observed few utterances of a mathematical type made by children during joint reading (teacher-child), even though the book was designed for mathematical learning purposes. Nevertheless, participating in the reading of picture books of high literary quality has shown to engage mathematical thinking among young children (Van den Heuvel-Panhuizen \& Van den Boogaard, 2008), indicating that picture books may be useful as pedagogical tools, but research also suggests a need for social interaction in order to unpack their full potential for mathematics learning. To date, the way preschool teachers act in order to direct attention towards mathematical content and connections has been explored only to a limited extent (for examples of such studies see Björklund, 2014; Ekdahl, 2020; Van den Heuvel-Panhuizen \& Elia, 2013).

### 1.2 Young children's learning of numbers and representations

A large body of research offers quite a good understanding of the trajectory of children developing number knowledge (see Fuson, 1992; Baroody \& Purpura, 2017). Research (e.g., Björklund et al., 2021) also furthers this knowledge by interpreting numerical skills as perceiving necessary aspects of numbers, which enables the child to make use of powerful strategies in numerical problem-solving. One example of a key aspect of early number
knowledge is cardinality; seeing numbers as constituting a composite set (Sarnecka \& Carey, 2008). However, cardinality is not enough for enumerating or handling numbers in arithmetic settings; numbers' ordinality is also an essential aspect, with each number related to other numbers in an orderly fashion (Fuson, 1988). To determine the exact number of items in a set, it is necessary to simultaneously understand numbers' dual meaning of cardinality and ordinality (Fuson, 1992; Davydov \& Andronov, 1981). This duality means that when a child has determined that there is a number of items in a set and one more is added, the next counting word also means 'one added' to the original set. A critical question for early mathematics education, then, is how children come to perceive this dual meaning of numbers, and how to facilitate their learning of such an abstract notion as numbers.

Duval (2006) points out that abstract notions, such as numbers, are only accessible through representations like spoken language, graphical expressions, or images. The mastery of such representations is not inherent but are, according to Van Oers (2010), learnt through communication with others whereby representations connected to children's actions mediate mathematical meaning. Research shows that the quality of verbal number talk is participating in dialogues where numbers' meaning and use can be explored, plays a significant role in children's development of an understanding of numbers (e.g., Björklund \& Palmér, 2021; Levine et al., 2010). However, verbal number talk is only one mode of representation and in many situations, there are at least two representations implicitly or explicitly used, depending on the mathematical activity (Duval, 2006).

Lesh (1981) and Lesh et al. (1987) emphasize five modes of representation that constitute a model of how mathematical concepts can be represented in mathematics education: real-world situations (connecting to events, situations, and objects from the learner's life), pictures (drawings or photographs of physical objects, or diagrams), verbal symbols (the spoken language representing numbers or numerical concepts), written symbols (letters, digits, and other graphical symbols), and manipulatives (objects designed to demonstrate a mathematical concept by the learner manipulating them). These representations may be perceptually similar but mathematically different (such as finger patterns of two and three fingers), or perceptually different but mathematically similar (such as two blocks and a pattern of two fingers). According to Lesh et al., the learning of mathematics is reflected in the ability to make connections between (translations) and within (transformations) these modes of representation. Duval (2006) also classifies representations in different semiotic representation systems, for example, systems of iconic, visual, and oral representations, and similar to Lesh, Duval emphasizes what he names transformation between semiotic representation systems. He distinguishes between two kinds of transformations: treatment, which implies transformations within, and conversion, which implies transformation between semiotic representation systems. Thus, even though the theoretical foundations and wordings of Lesh and Duval slightly differ, they both highlight that the learning of mathematics involves and can be strengthened by the ability to make connections within and between representations. In this perspective, it is critical for learning that children have experiences of numbers represented in different modes of representation. But research in early childhood settings has also convincingly shown that using several modes of representation is not enough, as aspects or connections to be discerned are not automatically realized by children but often have to be pointed out and demonstrated to the child (Björklund \& Palmér, 2020). Ekdahl (2019), for instance, pointed out the diverse opportunities for learning depending on teachers' ways of connecting and bridging aspects of a concept to
be learnt. Thus, the interaction directed at certain content with which the child is engaged plays an important role in what becomes possible for him or her to discern.

### 1.3 Conditions for learning-theoretical views

In the teaching activity we investigate here, the intended object of learning concerns basic aspects of numbers, presented to toddlers in book reading sessions. Modes of representation become an important framework for differentiating the potential that the activity, and particularly the interaction between teachers, peer, and artefacts may have for the toddlers to whom numbers are novel. Considering these young learners and their encounters with representations of numbers of different kinds, we need to point out an important issue: representations are one aspect of numbers that toddlers have to discern in order to understand the idea of numbers (numbers are communicated through modes of representation); also, representations are used to mediate a certain meaning (numerical meaning). In our case, we primarily focus on the latter, as a means to illuminate a certain numerical meaning. Duval (2006) poses a question that becomes especially critical in this query: 'How can they [learners] distinguish the represented object from the semiotic representation used if they cannot get access to the mathematical object apart from the semiotic representations?' (p. 107). Our study is an effort to better understand, based on empirical findings, how this is realized among the youngest learners.

Both Lesh (1981) and Duval (2006) have developed frameworks on representations designed for mathematics education with older students, in which teaching is framed differently than in the social-pedagogical preschool settings (see Karlsson Lohmander \& Pramling Samuelsson, 2015) in which this study is situated. One significant difference is that the choice of tasks and content for learning is pre-determined in formal education to a greater extent, while in preschool learning objectives can vary during one and the same activity. Nevertheless, there are sufficient similarities in mathematics education regardless of educational level for us to adapt and use Lesh's model of modes of representation as an outset for our inquiry (see Fig. 1).


Fig. 1 Modes of representation— adapted from Lesh (1981)

The adapted model starts with the real-world situations. As the learners in our study are 1 to 3 years old, the setting for their learning is their lived experience in the current situation, which needs to be meaningful and relevant (Van Oers, 2010). The real-world situations, then, embrace the entire learning activity and everything that occurs in the context of book reading, and cannot be separated as a distinct mode of representation. In Lesh's original model, written and verbal symbols are two different modes of representation. In early childhood, children also encounter different symbolic representations, but they are not limited to these two modes: finger patterns are also common representations in children's acts with numbers and can be used to represent a number, for example, showing three fingers to represent a quantity of three items (Björklund \& Reis, 2020). Thus, in an early childhood context, different modes characterized by their common symbolic nature could be seen as distinctions within one mutual mode of representation: symbols. Pictures and manipulatives are the same in the adapted model as in Lesh's original. However, considering the early childhood context, fingers may also be used as manipulatives, for example, when keeping track of counted items by showing one finger at a time while counting out loud. The fingers are then representing items that can be counted, which expresses a different meaning than when used in a symbolic sense.

As mentioned above, both Lesh et al., (1987, see also Lesh, 1981) and Duval (2006) emphasize the importance of connections within and between modes of representation to bring the mathematical object to the fore. What becomes critical in teaching is then how such connections can be offered to young children who have not yet got 'access to the mathematical object' (see Duval above). Variation theory of learning (Marton, 2015) does however give suggestions to how an earlier unseen mathematical object can be made discernable: the meaning of a mathematical object (as it appears to the learner) is constituted of what aspects of that object the learner is able to discern. Numbers, for example, constitute of aspects such as cardinality but also ordinality, part-whole relations, and representations (Björklund et al., 2021). To the young child, all of these aspects may not yet be discerned, which leads to the child experiencing the meaning of numbers in a qualitatively different way than adults, who most likely have discerned these aspects of numbers and thus have a more nuanced understanding of, for example, numbers' meaning mediated through a certain representation. Discerning new aspects broadens the meaning of the mathematical object to the learner. This is made possible if so called 'values' (see Marton, 2015, p. 47) within the undiscerned aspect are made explicit against an invariant background. Values within cardinality may be altering numbers appearing in a certain situation. Values within the mode of symbolic representation may be finger patterns, verbal notions, or graphical symbols sharing for example the same number 'three' (as mentioned above). Consequently, in a teaching act, the teacher first ought to have a reasonably clear idea of what meaning she wants the learner to 'see,' thus an intended object of learning. She then brings about certain patterns of sameness and difference among tasks, instances and representations by pointing out those patterns and relations to the learner. There has to be a careful choice of what is kept invariant and what is varied (Runesson, 2005), but also what a certain representation may afford the learner to 'see,' as this is assumed to enable the child to develop their own way of seeing and understanding and to become aware of the (intended) meaning of the mathematical object. For example, if numbers' cardinality is not an aspect that the child has discerned before, it may become discernible if one set of items is contrasted with another set of similar items. The number 2 is then seen as a cardinal value against another cardinal value such as 3 or 5 , but only if the items that the numbers are represented with are kept invariant (same color, shape, and size). In sum, there are certain aspects of numbers that learners have to learn to discern to be able to complete certain
tasks. This means that learning is considered a change in ways of experiencing numbers in terms of discerned aspects, which opens up for the learner to act with and handle numbers in ways that were not possible before. For example, a child who recites the number sequence while randomly pointing at a set of items expresses a different way of experiencing numbers than a child who carefully connects number words to items one-to-one, finishing by saying the last uttered number word once more. Through the lens of variation theory, the first child is likely not discerning the cardinality of number words, only the ordinality since the words are recited in a fixed order and that number words are representing some quantitative aspect of the world, however indistinct. The latter child has, on the other hand, likely discerned the ordinality, representational and cardinality aspects of number words, which is expressed in the way they handle enumeration tasks. This line of reasoning also leads to methodological gains, as changed ways of acting may be interpreted as changed ways of experiencing the meaning of numbers. In taking the above described theoretical principles as the foundation for interpreting how learning may occur, it is significant to pay attention not only to the fact that different representations are used in an act of teaching, but also to which representations are chosen and how they are presented to the learner.

In addition to the theoretical premises above, it is not sufficient to simply direct attention to representations to discern a certain meaning. The teacher and child also have to coordinate their perspectives on that which they are attending to, as the same phenomenon being attended to may be perceived very differently. In other words, they need to share attention and establish intersubjectivity through mutual verbal and non-verbal engagement and response to the other's initiatives (Rommetveit, 1992; Trevarthen \& Delafield-Butt, 2016). From this follows that the teacher needs to be responsive to the learner's perspective and what aspects of numbers the child has or has not yet discerned, which may become visible in communicative interaction. Thus, teaching involves first establishing sufficient intersubjectivity, and then coordinating perspectives that allow the participants to continue a joint activity in which new features can be introduced (Pramling et al., 2019).

## 2 The study

This study is part of a collaborative research project with three preschool teachers at three Swedish preschools (with 27 participating toddlers in total) and two researchers. During two years, teachers and researchers held biweekly meetings to design, evaluate and further develop mathematics - teaching activities suitable for toddlers. As part of this project, a picture book was designed to be read to the toddlers. The teachers conducted all the activities with the toddlers, of which video documentations were used both as an outset for developing practice in collaboration and for scientific analyses.

Based on variation theory principles (Marton, 2015), we conjectured that numbers' cardinality, among other aspects, would be possible to discern if contrasting cardinalities were (re)presented simultaneously in an invariant (book) environment, and generalized when keeping the number invariant and varying other aspects irrelevant to cardinality. These patterns of variation were the core of the picture book's design (see example in Fig. 2). As merely the appearance of varying number sets in pictures has been found to be insufficient for most children to discern new aspects of numbers (Björklund \& Palmér, 2020), the book reading sessions included a puppet (a dog resembling the character in the book) to open up for play-oriented interaction, along with physical building blocks similar to those in the pictures. This addition of items to the shared reading is also grounded in the theoretical


Fig. 2 Left picture: one block on the left spread makes a contrast to the two blocks on the right spread. The text (in Swedish) says ‘Look! A block!' and 'Two blocks!'. (In Swedish, the word for 'a' and 'one' is the same.) Right picture: Three blocks stacked vertically are contrasted to three blocks spread out on the floor. The text says 'Now the tower is tall' and 'The tower falls down!'
framing by Lesh (1981) that children's learning of mathematics is reflected in the ability to make connections between and within several modes of representation.

### 2.1 Empirical data

A total of 407 min of video-documented reading sessions were analyzed ( 75 observations, mean length 5 min ). Observations were made over a 12 -month period, at the start of which the participating children were 12-27 months old. The picture book was frequently read to the toddlers by their preschool teacher, both individually and in groups. The participating children's legal guardians had given written consent for the children to take part in the activities and to be filmed for research purposes. The project, of which this particular study is a part, has been approved by the Swedish Ethical Review Authority (reg. no. 2019-01037).

To prepare the data for analysis, we selected cases of documented interaction in which there were at least three turns of action; for example, the teacher makes a statement, the child responds in words, gestures, or actions, and the teacher responds to the child's act, or vice versa. This selection process is based on the conjecture that detailed analyses of interaction between teacher and child allow us to inquire and illuminate the nature of the learning processes observed in the teaching practice. A total of 216 instances of interaction directed at some aspect of numbers were found and transcribed verbatim (both verbal and gestural utterances).

### 2.2 Analysis

To answer the research question, we conducted analyses in several steps. First, we identified what modes of representation were coming through in the book reading (symbolic representations, manipulatives, and pictorial representations). Then, we identified distinctions within each of the three modes of representation:

S Symbolic representations
SF Finger patterns: fingers shown simultaneously to illustrate the quantity of a set
SV Verbal counting words

SW Written numerals
M Manipulatives
MB Blocks: physical blocks
MF Fingers: used as countables, one finger for each block or counting word
P Pictorial representations
PS Isolated numerical content in picture, e.g. blocks grouped together
PV Varying numerical content in picture, e.g. two sets of blocks in two simultaneously visible pictures or in a single picture

The frequency of representations appearing in the empirical data resulted in an overview of how different modes of representation were appearing isolated or in combination with others. To gain further knowledge of the interaction constituted in the reading sessions, we looked for qualitative differences regarding how the meaning of numbers was (re)presented to the toddlers. Three different ways of (re)presenting numbers' meaning appeared: a) instructive interaction, b) confirming interaction, and c) challenging interaction.

Next, we identified occurrences of learning opportunities. The indication for learning opportunities was determined to be instances in which a child responded to a presented mode of representation in a different way than they had initially. In line with variation theory, this means that we looked for instances in which the child expressed extending their way of understanding the learning object, by discerning aspects they had not discerned before and was now able to 'see' the learning object in a qualitatively different way, thus handling a task in a more advanced or flexible way than before. To find these instances of learning, we looked at both the representations provided in the observed interaction (S, M, and P and subcategories) and the characteristics of the interaction. In this combined analysis, certain patterns of how the meaning of numbers was made discernible for the children appeared.

## 3 Results

### 3.1 Frequency of modes of representation

First, we can conclude that all modes of representation outlined by Lesh (1981), with adaptations to this specific setting, are included in the book reading sessions either as a single mode or in combination with other modes. The frequency with which the modes appear in the reading differs, however (see Fig. 3):

Fig. 3 The frequency of observed cases in which attention to numbers' meaning is directed through some mode of representation or a combination of several modes


As book reading is a communicative activity, it is not surprising that symbolic representations are present, solely or in combination with other modes, in most $(98 \%)$ of the observations (in verbal, gestural, or written symbols). Furthermore, it is very rare for only one mode of representation to be used in the reading sessions, while symbolic representations along with manipulatives or pictorial representations are observed in about $17 \%$ of the observations, respectively. Mostly, all three modes of representation are present in the interactive reading ( $59 \%$ of the observations).

### 3.2 Categories of interaction

Among the 216 instances of interaction, three categories were identified: (a) instructive, (b) confirming, and (c) challenging (see Fig. 4). Approximately half of the instances fall into the instructive and confirming interaction categories ( $48 \%$ ), and about half into the category of challenging interaction (52\%):

### 3.2.1 Instructive interaction

Instructive interaction is characterized by the teacher taking the initiative to offer the child different modes of representation to be explored simultaneously but not pointing out connections in meaning between them. With only a few exceptions, these are verbal symbols (counting words) in combination with isolated numerical content in a picture. For example, the teacher directs attention to the act of counting by asking 'How many are there, can you count?' and the child responds by pointing to either physical blocks or blocks in a picture. The teacher then repeats the child's counting act or corrects the child if the counting procedure was incorrect. Representations are present in this kind of interaction, but lack connections made between the modes of representation.

### 3.2.2 Confirming interaction

Confirming interaction is characterized by the child taking initiatives, for instance, counting items (physical or pictorial), and the teacher affirming the child's counting act. There are many instances of confirming interaction, and instructive interaction that turns into confirming interaction. However, both these kinds of interaction mainly acknowledge what the child already knows and the skills they have already mastered. According to our definition of learning as a changed way of experiencing the meaning of numbers (grounded in variation theory), teaching in early childhood should afford children opportunities to broaden their experiences and to see different aspects of numbers that will deepen their way of understanding what numbers mean and can be used for. In this category, similar to the instructive interaction category, only a few connections are made within one and the same mode of representation and the modes that are present often lack connection. This

Fig. 4 Model of the analysis of interaction



Picture 1 The child counts blocks on the picture, the teacher counts out loud and raising fingers
we can see in the following example, which illustrates how instructive interaction turns to confirming interaction (original Swedish utterances in italic, see also Picture 1):

Teacher: ‘How many blocks does Dutten have?' Hur många klossar har Dutten? [3 purple blocks in the picture].
Aina: 'One, two, three.' En, två, tre (points at each block in the picture).
Teacher: 'Yes, one, two, three.' Ja, en, två, tre (unfolds one finger for each counting word).

Pictorial representations (the picture with three purple blocks), symbolic representations (counting out loud), and manipulatives (using fingers to illustrate added units in the counting act) are all present in the interaction. The child is the one making a connection between the picture's set of items and verbal counting words, but no connections are made by the teacher between the different modes of representation; thus, it ends as a confirming interaction that does not extend the child's way of understanding regarding numbers. In fact, it is not confirmed that the child even discerns the cardinality of numbers used in her counting act, as the teacher merely imitates and thereby verifies the counting sequence.

### 3.2.3 Challenging interaction

Challenging interaction occurs when the teacher extends the child's experiences by staying within one and the same mode of representation but offering different values (see section about variation theory) within this mode to be contrasted. In these observations the teacher may, for instance, offer verbal symbols along with finger patterns (both symbolic representations), illustrating the cardinality of numbers: the number is kept invariant but the representations vary within the same mode of representation. Similarly, we can see the teacher using manipulatives (blocks) and mapping fingers one-to-one as a means to emphasize similarities and differences in number in using countables within the same mode of representation. Connections are also made between modes of representation in this category, but the emphasis on variation within one and the same mode of representation is distinct ( $78 \%$ of the observations in this category, compared to only $19 \%$ in the instructive and confirming interaction combined). From a theoretical standpoint (in our case, the variation theory of learning), this way of presenting new aspects to be discerned by the children should afford them the best opportunities to develop their knowledge of numbers, as necessary aspects are brought to the fore through patterns of variation and invariance. Observations being
characterized as challenging interaction means that the interaction is informed by an intention to visualize the meaning of numbers through contrasting cardinal numbers, by adding some new representation or connecting representations that the child had not discerned before.

### 3.3 Facilitative conditions for learning

Differences found in how modes of representation are used in the interaction reveal that instances characterized as challenging interaction seem to offer the children better opportunities to learn the meaning of numbers. However, this conclusion is based on a theoretical construct and had to be empirically evaluated, thus leading to a closer analysis of those instances ( $n=113$ ) in which the teacher offers (theoretically) the toddlers the best possible opportunities for learning, to reveal what makes a difference in the children's opportunities to learn about numbers' meaning.

Within the category challenging interaction, differences are observed in the responses by the children that indicated that not all the interactions are facilitative for developing their knowledge of numbers. Within the challenging interaction, which theoretically would have the best potential for learning, about half ${ }^{1}$ of the observations do not lead to any response or further interaction from the toddlers, while the other half of the observations does result in some kind of changes in their ways of responding regarding numbers' meaning. This final step in the analysis identifies when the interaction becomes developable. One feature of the (challenging) interaction stands out: The teacher takes the child's directed attention as a starting point. We will illustrate this key finding through contrasting empirical examples in the following.

Regarding what makes a challenging interaction developable or not is best seen when contrasts are made between when a child's attention is and is not adhered. For example, in a simple act of enumeration there might appear very different ways of experiencing the meaning of numbers, which interrupts the interaction and possible connections between or within representations to be discerned:

Teacher: 'There are two blocks: one, two.' Det är två klossar: en, två (points at two blocks in the picture)
Alma: ‘Three.' Tre.
Teacher: 'Are there three? One, two, three.' Är det tre? En, två, tre (points at a block on the previous spread, then at two blocks on the current spread) 'Yes, there are three together.' Ja, de är tre tillsammans.

The counting task is often treacherous when it comes to toddlers, as in the example above (see Picture 2). The teacher counts two blocks in the picture and the child continues, saying 'three.' The teacher interprets the child's utterance as an expression of cardinal-ity-that the child sees three blocks in the pictures-and seems to try to find a reason for such an understanding, which is found by adding a single block seen on the parallel spread

[^1]

Picture 2 Pictures with one and with two blocks are similarly present in the book reading
to the two counted by the teacher. However, the child does not respond in any way to the teacher's acts of connecting two sets found in the pictures into one whole set of three. We suggest that this is because the child responds to the teacher's first counting act with the next word in the counting sequence; an ordinal meaning of the number words but not necessarily a cardinal meaning. Thus, the child and the teacher do not share perspectives, and the connections made by the teacher between pictorial representations that include several sets (PS and PV) and symbolic representations in number words (SV) are not discerned by the child. Thus, it seems to be critical to share attention not only to what is present in the pictures but also to a shared meaning of numbers. In the following example, we can see the teacher trying to extend the child's understanding of numbers by adding and connecting between and within modes of representation, but as the child's attention is directed at another aspect of the task at hand, she does not respond to (or discern) what the teacher intends to highlight:

The teacher gives the child two wooden blocks and one red one in the same shape as three purple ones in the picture.
Teacher: 'Do you have equally many?' Har ni lika många?
Assra places the blocks beneath the picture, pushes the red one away, points at the space between the wooden blocks and is given another wooden block. Assra looks satisfied.
Teacher: 'Look here, Assra, one, two, three. Three blocks.' Titta här Assra, en, två, tre. Tre klossar (points at the wooden blocks, then shows the finger pattern 'three') 'If I take that away' Om jag tar bort den (takes the middle wooden block away) 'And add this one' Och lägger till den (adds the red block in the middle, points and counts) 'One, two, three, there are still three blocks.' En, två, tre, det är fortfarande tre klossar.
Assra starts building a tower with the blocks.
The child is apparently engaged in a sorting activity in which it is essential to make groups of similar blocks (see Picture 3). The teacher introduces the aspect of abstraction; visual features such as color are irrelevant for enumerating a set of objects. She does this by counting the set of blocks that the child created, determining that there are three in all, and then switching one of the blocks for a similarly shaped but differently colored one and repeating the enumeration. According to theoretical principles for best learning opportunities, counting by using symbolic verbal representations (SV) in combination with symbolic finger patterns (SF), and connected to sets of three blocks (MB) that are altered concerning irrelevant features, would allow the child to discern the numerical aspect of the sets.


Picture 3 Blocks of mixed colors are not accepted by the child. The teacher counts and shows finger patterns to make a numerical connection between a group of similar blocks and mixed-colour blocks

However, the child's attention is directed at classification in her creating groups with similar features, to which a numerical aspect seems irrelevant (taking the child's perspective). Using counting words and symbolic finger patterns does not help the child discern numerical meaning in this case. A more successful way of directing attention to the numerical aspect seems to be to take starting point in the child's attention (to classification, in the example above), and through altering the number of blocks between sets with similar features, numerical relations may be discerned. In the following example, this is shown as the teacher introduces new aspects that make sense in accordance with the child's intention, yet keeps irrelevant features to a minimum to allow for the necessary aspect to be discerned. The aspect of number relations is introduced to the child in ways that add to her intention, with a successful outcome and potential for learning:

There are three blocks in the picture. Otilia places two red blocks on the picture's blocks.
Otilia: ‘I need one more, in between.' Jag behöver en till, emellan (Otilia gets another block, which she places between the two)
Teacher: 'How many do you have now?' Hur många har du nu då?
Otilia shows her thumb, index, and middle fingers at the same time.
Teacher: 'How many is that?' Hur många är det?
Otilia: 'Three' Tre (holds her fingers against the table, keeping the same finger pattern)
Teacher: 'Is this also three?' Är det här också tre? (shows finger pattern 'four')
Otilia shakes her head.
Teacher: ‘How many should I take away to make three?' Hur många måste jag ta väck för att det ska bli tre?
Otilia: (struggles to make a finger pattern, holding down her little finger with her other hand) ‘There!’ Så!
Teacher: ‘If I do this, is it three then?' Om jag gör så då, är det tre då? (holds down thumb and little finger)
Otilia: 'Have to take away one more' Måste ta bort en till (shakes her head and folds down all fingers)
Teacher: 'Is this three now?' Är det tre nu? (showing index and middle fingers)
Otilia nods and smiles.
Teacher: 'I think you're joking with me.' Nu tror jag du skojar med mig (holds her two fingers against two of the three blocks in the picture) 'There weren't enough fin-


Picture 4 Contrasting finger patterns are connected to a set of three blocks
gers! Look, now there are three.' Fingrarna räckte ju inte till! Titta, nu var det tre (unfolds her ring finger)
Otilia: (holds her index and middle fingers on two of the blocks; the blocks move slightly) 'Whoops!' Oj, hoppsan! (reorganizes the blocks)
Teacher: 'Do you have equally many now?' Har ni lika många nu?
Otilia nods.
Teacher: 'Dutten has three and Otilia has three.' Dutten har tre och Otilia har tre (moves the blocks to the table)

The child is engaged in mapping physical blocks (MB) to the picture's blocks (PS) one-to-one. She also connects to a finger pattern when asked how many there are (SF) (see Picture 4). The teacher picks up on the child's finger pattern use and challenges her knowledge of cardinality by showing a different finger pattern and relating it to how a pattern of three could be created (varying the numbers within the same mode of representation, SF). When the child seems unsure about the cardinality of the finger patterns, the teacher reconnects with the mapping activity the child had initiated, mapping one finger to one block and leaving one without a finger pair, to which the teacher interchanges the finger patterns 'two' and 'three.'

Interaction that is characterized as developing have one specific feature in common: The teacher's initiatives take the toddler's directed attention as their starting point. In the following example, the child points to a 'tower' of two blocks in the picture and initially says 'on the same spot'. This indicates his directed attention, which the teacher aims to extend:

Teacher: 'Dutten wants to build higher, he adds another block.' Nu vill Dutten bygga högre, då lägger han till en kloss (points at two blocks in a tower in the picture)
Gustav: ‘On the same spot.' På samma ställe.
Teacher: 'That's right. Two blocks there and one there. How many blocks are there together?' Precis. Två klossar där och en kloss där. Hur många klossar är det tillsammans? (circles her index finger around the two blocks and a single block behind Dutten)
Gustav: ‘Two.' Två (points at the tower) 'One.' En (points at the single block) 'Two' Två (points at the tower) 'One, two there' En, två där (points at each block in the tower).
Teacher: 'How many were there, let's count.' Hur många var det, ska vi räkna? (points at the single block first, then the two in the tower) 'One...' En.
Gustav: ‘Two, three!’ Två, tre!
Teacher: ‘Three blocks, that's right. Two there.' Tre stycken ja. Två stycken där (points with two fingers at the two blocks in the tower) 'And one there' Och en där (pointing with one finger at the single block) 'Three blocks together.' Tre stycken tillsammans (circles all three blocks)

Gustav seems to regard the two blocks as one set, labeled 'two', but also addresses an ordinal meaning when pointing to and labeling the blocks 'one, two'. The picture allows the child to discern the blocks as one set with both cardinal and ordinal meaning (PS). Initially, he does not appear to consider the third block on the side as part of a set, as he makes no attempt to count it. The teacher points to all three blocks and even circles around all three of them to extend the child's awareness and include the single block in the set of two that he had initially focused on 'two there and one there.' The pictures are used here to bring forward the set of 'one' and 'two,' as contrasting cardinal numbers (PV). The teacher then picks up the child's counting of the tower, 'one, two,' and acknowledges that the child seems to know the cardinality of the set of two blocks. She then offers a variation in the counting act, starting to count from the single block as 'one.' The child then appears to perceive the blocks in a different way, as one composite set whose units are possible to enumerate, as he continues the counting act, 'two, three,' with a content smile on his face. Offering the child the opportunity to discern variation in how to compose a set of blocks within the same mode of representation (pictorial), as well as in connection with the same verbal number words (SV) and finger patterns (SF) labeling different units of blocks (still constituting the same composite set), provides the child ways to discern aspects that are necessary for developing his knowledge of numbers (see Picture 5).

Another example of developing interaction in which the teacher takes the child's directed attention as a starting point can be seen below, where the child has her own way of constituting sets of items:

Teacher: 'How many do I need to get?’ Hur många behöver jag hämta? (3 blocks in the picture, 2 blocks placed on the picture)
Assra: 'One' En. (Assra gets one more block, puts it between the two on the picture) 'Mommy, daddy, baby.' Mamma, pappa, bebi.
Teacher: 'How many is that?' Hur många är det?
Assra: (points at the block in the middle and recites rapidly) 'One-two-three-fourfive.' En-tvà-tre-fyra-fem.
Teacher: 'Is it one, two, three, four, five?' Är det en, två, tre, fyra, fem? (unfolding one finger for each counting word, puts her hand on the table, keeping the finger pattern)


Picture 5 Connecting two and one block to a set of three blocks by counting out loud, circling around the composite set of blocks and mapping finger patterns to the pictorial blocks


Picture 6 Connecting manipulatives with symbolic representations and the child's labeling of objects

> Assra: ‘Baby, daddy, mommy, mommy, daddy.' Bebi, pappa, mamma, mamma, pappa (points at each one of the teacher's fingers)
> Teacher: (puts the blocks at the ends of her little, ring, and middle fingers) 'One, two, three. Look, it wasn't five, it was one, two, three.' En, två, tre. Titta, det var inte fem, det var en, två, tre.
> Assra: (switches places of two blocks) 'Baby, daddy.' Bebi, pappa.
> Teacher: (folds thumb and index finger) 'If I take away two.' Om jag tar väck två.
> Assra: 'Take away.' Ta bort (points at teacher's ring finger)
> Teacher: ‘But now it isn’t three. One, two.' Men nu är det ju inte tre. En, två (points at little and middle fingers, unfolds ring finger) 'One, two, three.' En, två, tre (points at unfolded fingers) 'One, two, three.' En, två, tre (points at the blocks at each unfolded finger).
> Assra: ‘Baby, daddy, mommy.' Bebi, pappa, mamma (points at the blocks one at a time)

It is common that toddlers name sets of items in accordance with other familiar sets, such as family members. The challenge is to extend the child's naming of a group to become numerical in meaning. Above, the child responds to 'how many are there' by rapidly reciting the number sequence without differentiating the words. The teacher repeats the number sequence, however with a slight pause between the words and raising one finger for each word said, thus connecting symbolic representations (SV) with manipulatives (MF) but finishing with a finger pattern (SF). She then relates to the initial question of how many blocks there are by comparing 'three' with 'five' through both symbolic representations (SV, SF) and manipulatives (MB, MF) (see Picture 6). It is noteworthy that even though the child labels the single units with family names while the teacher uses counting words, they establish a shared attention to the numerical aspect by connecting and comparing three blocks (MB) and five fingers (MF), albeit labeled differently by child and teacher, and to how to make the sets equally large. The learning outcome of this developing interaction is thereby not the meaning of counting words but the numerical relation between compared sets.

## 4 Discussion

From the current data, we can only use indications for learning that are observable in the children's verbal or gestural actions. We thus cannot be sure that children who do not respond to the teacher's invitation and offered representations do not learn. A longitudinal study of individual children would bring some clarity to this matter, but such an analysis is
outside the scope of the present study. The main finding, though, is a description of what conditions become critical for making learning possible, not only in a theoretical sense but based on empirical findings and particularly how modes of representation are made into resources for number learning in book reading sessions with toddlers.

In the inquiry concerning what modes of representation emerged in the book reading, we found that each mode included several values. In support of Lesh (1981), for the modes of representation to become facilitators for discerning essential aspects of numbers, each mode has to be presented to the children in ways that highlight the intended specific meaning (here, usually the cardinality of numbers). The mere fact that the representations are there does not mean that the children experience the intended meaning. Our analysis reveals that variation within one mode of representation, described as facilitative conditions for learning above, is more likely to bring to the fore a certain meaning of numbers to the child than is variation between different modes of representation alone (see also Lesh, 1981). This is shown by the few instances of connections made within one and the same mode of representation, and the modes that are present often lack connection, found in the instructive and confirming interaction categories in comparison to the challenging and developable interaction that mainly include several values within one mode of representation and connections made within and between present representations. Conjectures of how novel meaning is discerned in accordance with variation theory (Marton, 2015) also supports this conclusion because that what is to be discerned should be varied to allow a certain meaning to be experienced before it can be generalized, particularly if other features are kept invariant. This was shown to be critical for making a distinction between instructive and confirming interaction and the preferred challenging interaction.

To generalize meaning, some connections between modes of representation are nevertheless necessary (see Duval, 2006), as long as the learning object is kept invariant and is thus possible to recognize as constituting the same meaning (see Marton, 2015). This was a challenge we had to address in studying toddlers' meaning-making, as their understanding of numbers is novel and one representation is not necessarily connected to a certain way of experiencing numbers. The meaning of numbers' cardinality represented by physical blocks or fingers does not appear on its own. Here, we show how variation theory principles, such as varying what is to be discerned against an invariant background (e.g., comparing sets of similar blocks), help direct the child's attention to the intended meaning, within one mode of representation. A key point in the teaching activity thereby turned out to be the introduction of meaning first within modes of representation before connecting between different modes, representing the same mathematical object. As these objects were new to the children, they may very well have discerned aspects other than mathematical ones. To overcome this pedagogical challenge, we empirically found it is necessary to start in what the child directed his or her attention to; what the child discerned as foregrounded by a certain representation. When a teacher introduces a new meaning that adds a broader view of the same object to the child's earlier experienced meaning, it becomes possible to also induce new meaning for the child. In other words, a child can get access to the mathematical object differentiated from the representation itself when it adds meaning to what the child already experiences (cf. Duval, 2006).

Many studies in early childhood education are limited to concluding that activities conducted in preschool should provide good learning opportunities (e.g., Björklund, 2014; Van Oers, 2010). This can be based on the theoretical assumptions regarding what the necessary conditions for learning mathematics are, and serves as a foundation for developing preschool practice. When evidence of learning outcomes is provided, or even indications of children's learning outcomes from participating in such activities, it is often measured in
terms of developed skills (e.g., Li et al., 2020; Mulligan et al., 2020). Rarer are investigations of what makes a difference for children's learning (or not learning) in terms of what an activity or interaction is affording the child to discern, even though this should be a critical issue in early childhood education research but also for justifying theory in empirical contexts (see Clements et al., 2020; Kullberg et al., 2020). Our study thereby contributes to the field of early childhood mathematics education and research by providing empirical findings concerning why learning may occur (or not) that are also grounded in theoretical frameworks.

One of the greatest challenges in early childhood mathematics, and particularly when number concepts are novel, is to facilitate an exploration of numbers' meaning in ways that support exactly what is critical for the child to discern; that is, to build on what the child already knows about numbers and extend his or her way of understanding the meaning of numbers. In fact, finding a suitable 'level' and challenge is quite difficult; at least when, as we do in our study, one is looking for children's responses as a sign of their attentiveness to what the teacher tries to teach that goes beyond confirming what the child already knows.

The results of our analysis have implications for pedagogical practice and the teaching of numbers to young children. We have empirically shown that teaching in accordance with theoretical principles is indeed important for challenging children to expand their ways of experiencing numbers, but the teacher's adherence to the child's directed attention is critical. A shared and sustained attention to a learning object is nevertheless necessary to establish. This is true both for picking up on the child's directed attention and bringing in new aspects through contrasting features within a mode of representation, and for inviting the child to experience a specific content chosen by the teacher. The latter calls for careful design of the teaching in a context that is relevant and engaging (what Lesh would call Real-World Situations), but also for demarcating the mathematical content that is possible to discern.

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## References

Baroody, A., \& Purpura, D. (2017). Early number and operations: Whole numbers. In J. Cai (Ed.), Compendium for research in mathematics education (pp. 308-354). National Council of Teachers of Mathematics.
Björklund, C. (2014). Powerful teaching in preschool - A study of goal-oriented activities for conceptual learning. International Journal of Early Years Education, 22(4), 380-394. https://doi.org/10.1080/ 09669760.2014.988603

Björklund, C., Marton, F., \& Kullberg, A. (2021). What is to be learnt? Critical aspects of elementary arithmetic skills. Educational Studies in Mathematics. https://doi.org/10.1007/s10649-021-10045-0

Björklund, C., \& Palmér, H. (2020). Preschoolers reasoning about numbers in picture books. Mathematical Thinking and Learning, 22(3), 195-213. https://doi.org/10.1080/10986065.2020.1741334
Björklund, C., \& Palmér, H., et al. (2021). Preschool teachers' ways of promoting mathematical learning in picture book reading. In T. Koinzer (Ed.), Early childhood teachers' professional competence in mathematics: International perspectives. Springer.
Björklund, C., \& Reis, M. (2020). Ways of using fingers in preschoolers' numerical reasoning. In M. Carlsen, I. Erfjord, \& P. S. Hundeland (Eds.), Mathematics education in the early years Results from the POEM4 Conference, 2018 (pp. 93-107). Springer.
Clements, D. H., Sarama, J., Baroody, A. J., \& Joswick, C. (2020). Efficacy of a learning trajectory approach compared to a teach-to-target approach for addition and subtraction. ZDM-Mathematics Education. https://doi.org/10.1007/s11858-019-01122-z
Davydov, V. V., \& Andronov, V. P. (1981). The psychological conditions of the origination of ideal actions. Project paper 81-2. Wisconsin Research and Development Center for Individualized Schooling. The University of Wisconsin.
Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. Educational Studies in Mathematics, 61(1), 103-131.
Ekdahl, A.-L. (2019). Teaching for the learning of additive part-whole relations: The power of variation and connections. (Diss.). Jönköping University.
Ekdahl, A.-L. (2020). Different learning possibilities from the same activity-Swedish preschool teachers' enactment of a number relation activity. Scandinavian Journal of Educational Research. https://doi. org/10.1080/00313831.2020.1739131
Elia, I., van den Heuvel-Panhuizen, M., \& Georgiou, A. (2010). The role of pictures in picture books on children's cognitive engagement with mathematics. European Early Childhood Education Research Journal, 18(3), 275-297. https://doi.org/10.1080/1350293X.2010.500054
Fuson, K. (1988). Some further clarifications of numerical terminology using results from young children. Behavioral and Brain Sciences, 11(4), 583-585.
Fuson, K. (1992). Research on whole number addition and subtraction. In D. A. Grouws (Ed.), Handbook of research on mathematics teaching and learning (pp. 243-275). Macmillan.
KarlssonLohmander, M., \& Pramling Samuelsson, I. (2015). Play and learning in early childhood education in Sweden. Psychology in Russia State of the Art., 8(2), 18-26. https://doi.org/10.11621/pir.2015.0202
Kullberg, A., Björklund, C., Brkovic, I., \& Runesson Kempe, U. (2020). Effects of learning addition and subtraction in preschool by making the first ten numbers and their relations visible with finger patterns. Educational Studies in Mathematics, 103(2), 157-172. https://doi.org/10.1007/s10649-019-09927-1
Lesh, R. (1981). Applied mathematical problem solving. Educational Studies in Mathematics, 12(2), 235-264.
Lesh, R., Post, T. R., \& Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), Problems of representation in the teaching and learning of mathematics. Lawrence Erlbaum Associates.
Levine, S. C., Suriyakham, L. W., Rowe, M. L., Huttenlocher, J., \& Gunderson, E. A. (2010). What counts in the development of young children's number knowledge? Developmental Psychology, 46(5), 1309-1319.
Li, L., Zhou, X., Gao, X., \& Tu, D. (2020). The development and influencing factors of kindergarteners’ mathematics problem solving based on cognitive diagnosis assessment. ZDM-Mathematics Education. https://doi.org/10.1007/s1185 8-020-01153-x
Marton, F. (2015). Necessary conditions of learning. Routledge.
Mulligan, J., Oslington, G., \& English, L. (2020). Supporting early mathematical development through a 'pattern and structure' intervention program. ZDM-Mathematics Education, 4(52), 663-676. https:// doi.org/10.1007/s11858-020-01147-9
Pramling, N., Wallerstedt, C., Lagerlöf, P., Björklund, C., Kultti, A., Palmér, H., Magnusson, M., Thulin, S., Jonsson, A., \& Pramling Samuelsson, I. (2019). Play-responsive teaching in early childhood education. Springer. Open Access. https://doi.org/10.1007/2F978-3-030-15958-0
Rommetveit, R. (1992). Outlines of a dialogically based social-cognitive approach to human cognition and communication. In A. Wold (Ed.), The dialogical alternative: Towards a theory of language and mind (pp. 19-45). Scandinavian Press.
Runesson, U. (2005). Beyond discourse and interaction. Variation: A critical aspect for teaching and learning mathematics. Cambridge Journal of Education, 35(1), 69-87.
Sarnecka, B., \& Carey, S. (2008). How counting represents number: What children must learn and when they learn it. Cognition, 108, 662-674.

Trevarthen, C., \& Delafield-Butt, J. (2016). Intersubjectivity in the imagination and feelings of the infant: Implications for education in the early years. In E. J. White \& C. Dalli (Eds.), Under-three year olds in policy and practice (pp. 17-39). Springer.
Van den Heuvel-Panhuizen, M., \& van den Boogaard, S. (2008). Picture books as an impetus for kindergartners' mathematical thinking. Mathematical Thinking and Learning, 10(4), 341-373.
Van den Heuvel-Panhuizen, M., \& Elia, I. (2013). The role of picture books in young children's mathematics learning. In L. D. English \& J. T. Mulligan (Eds.), Reconceptualizing early mathematics learning (pp. 227-251). Springer.
Van Oers, B. (2010). Emergent mathematical thinking in the context of play. Educational Studies in Mathematics, 74(1), 23-37.

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[^1]:    ${ }^{1}$ It is not possible to give an exact number of observed instances, as some of the observations are difficult to make exclusive qualitative interpretations of, mostly due to very subtle, non-verbal responses by the children.

