Theory-informed lesson study as practice based research: identifying what is critical for grade 2 and 3 pupils' learning of negative numbers

Estudo de aula informado pela teoria como investigação baseada na prática: identificando o que é crítico para a aprendizagem dos números negativos de alunos dos 2.º e 3.º anos

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Abstract. The aim of the paper is to demonstrate how a theory-informed lesson study can be a form of practice-based research where knowledge is generated within the process of teachers' actions. Learning study shares features with lesson study, such as the iterative design, the teacher driven approach and with attention to student learning, but is guided by a theoretical framework. The dominant theory has been variation theory. In learning study, the focus is the object of learning and what must be learned to make the object of learning one's own. A learning study about learning and teaching negative numbers to young pupils (age 8-9) in a Swedish context is used as an example. Our proposal is in resonance with Morris' and Hiebert's (2011) suggestion that lesson study is a system that can generate instructional products that are sharable and open for improvement by other actors. The 'instructional product' from learning study is a theoretical description of the object of learning, how it is constituted and can be taught. In the learning study reported, three teachers worked in collaboration to identify the critical aspects for realizing the existence of negative numbers. The critical aspects emerged and were successively specified in the process and as a result of a thorough analysis of data on pupils' learning and the lessons.

Keywords: Lesson study; learning study; theory and practice; teacher collaboration; negative numbers; variation theory.

Resumo. O objetivo do artigo é demonstrar como um estudo de aula informado pela teoria pode ser uma forma de investigação baseada na prática onde o conhecimento é gerado dentro do processo das ações dos professores. O estudo de aprendizagem compartilha caraterísticas com o estudo de aula, como o processo iterativo, a abordagem centrada no professor e a atenção à aprendizagem do aluno, mas é guiado por um quadro teórico. A teoria dominante tem sido a teoria da variação. No estudo de aprendizagem, o foco é o objeto de aprendizagem e o que deve ser aprendido para que o objeto de aprendizagem seja apropriado. Um estudo de aprendizagem sobre aprendizagem e ensino de números negativos a jovens alunos (idade 8-9 anos) num contexto sueco é usado como exemplo. A nossa proposta está em ressonância com a sugestão de Morris e Hiebert (2011) de que o estudo de aula é um sistema que pode gerar produtos de ensino que são compartilháveis e abertos para melhoria por outros atores. O 'produto de ensino' do estudo de aprendizagem é uma descrição teórica do objeto de aprendizagem, como é constituído e pode ser ensinado. No estudo de aprendizagem relatado, três professores trabalharam em colaboração para identificar os aspetos críticos para perceber a existência de números negativos. Os aspetos críticos emergiram e foram sucessivamente especificados no processo e como resultado de uma análise de dados minuciosa dos sobre a aprendizagem dos alunos e as aulas.

Palavras-chave: Estudo de aula; estudo de aprendizagem; teoria e prática; colaboração docente; números negativos; teoria da variação.

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Introduction

There are extensive reports on the effectiveness of lesson study for teacher professional development and learning. The collaborative nature of lesson study and elements of inquiry and reflection contribute to establish a culture of learning communities and teacher collaboration (e.g. Chichibu & Kihara 2013; Hunter & Back, 2011; Toshiya & Toshiyuki, 2013) and improved teaching skills. The Japanese version of lesson study received much attention when it was promoted by Stigler and Hiebert (1999), Yoshida (1999) and Lewis (2002) as a form of teacher driven professional development and has been adopted globally since then. The problems with introducing a model rooted in a cultural context into another, with other conditions, have been discussed, however (e.g. Arani, Keisuke & Lassegard, 2010; Huang & Shimizu, 2016). It appears as if the development of lesson Studies in the European context is different from the Asian and US context, for example. In this paper, we present a version of lesson study developed in Hong Kong and Sweden, but applied in other contexts including for example, South Africa, Brunei, the UK, and Austria. We would suggest that lesson study informed by a pedagogical theory can be more than just reflective practice documented and reported. The aim of the paper is to demonstrate how a theory informed lesson study can be a form of practice-based research where knowledge is generated within the process of teachers' actions. A study about a team of teachers' inquiry into young children's learning of negative numbers will be used as an illustration. The character of the knowledge product that can be a result of such practice-based research will be discussed.

Lesson study and teacher research

The effects of lesson study on the improvement of teaching skills, how teachers learn to reflect on their actions, analyze students' thinking in more mathematically detailed ways, on changes in motivation and capacity to improve instruction, and the development of content and pedagogical content knowledge have been reported (Fernandez, 2005; Lewis, Perry & Hurd, 2009; Robinson & Leikin, 2011). Furthermore, it is often pointed out how lesson study can promote the establishment of learning communities of mutual accountability, shared goals for instruction, and a common language for analyzing instruction (e.g. Chichibu & Kihara 2013; Hunter & Back, 2011; Toshiya & Toshiyuki, 2013). To us, with these purposes, lesson study will be restricted to a model for professional development only, not as a system that can generate new and relevant knowledge recognized as a legitimate knowledge source for professionals.

The idea of researching classroom teaching and learning *with* teachers — not *on* teachers — was among the first recognized by Lawrence Stenhouse (1975). This has been taken up by others (e.g. Cochran-Smith & Lytle, 1990; 1999), and it has been argued that teachers must be key stake holders and co-producers of professional knowledge (Kieran, Krainer, & Shaughnessy, 2013*).* However, teacher research sometimes seems to be more concerned with teachers' professional development than with the generation of knowledge (Carlgren, 2012). Although the overall aim might be common — to improve teaching and learning — in many studies involving teachers and researchers there might be a risk that the interests and object of research sometimes diverge. The researchers' interest can be on many layers for instance, to study the process of teacher inquiry *per se*, teacher change, or how to link scientific and professional knowledge. In contrast to many approaches of teacher research, a significant feature of lesson study is the teacher autonomy (Kieran et al., 2013). Thus, lesson study has a potential to be the teacher research that Stenhouse promoted: teachers as *the* key stakeholders in the research process of generating knowledge for professionals.

We agree with Fernandez (2005) who argues that, besides a systematic inquiry based on generated data, there must be some sort of critical lens in the process. We would advocate that a theory is needed to improve the quality of lesson studies and assessing how teaching actions affect student learning, for instance (Elliott, 2012; Nuthall, 2004).

Although the role of a theory in lesson study is mostly unclear or rarely made explicit (Elliott, 2012 p. 114), there are Lesson studies where the significance of theories is reported (e.g. Clivaz, 2015; Martin & Clerc-Gorgy, 2015; Martin & Towers, 2016; Pillay & Adler, 2015). So, for example, Martin and Towers use the Pierie-Kieren theoretical model of the growth of mathematical understanding, and particularly the notion of folding-back, as a conceptual tool in the lesson study for considering the way in which learners work with, utilize, and build on existing mathematical knowledge. These studies demonstrate that a theoretical framework enables teachers to improve the quality of their lesson study by making them critically aware of the tacit beliefs and assumptions that shape practice in classrooms (Runesson, 2015). Due to a difference in the epistemological and ontological assumptions underpinning the theory, however, the lesson studies will have different foci (Runesson, 2016).

The modified version of lesson study we will report on here, the learning study (Cheng, & Lo, 2013; Holmqvist Olander, 2015; Marton & Pang, 2003; Marton & Runesson, 2015; Runesson, 2008), is underpinned by variation theory (Marton & Booth, 1997; Marton, 2015), which will be described in detail below.

Learning study — a theory-informed lesson study

Learning study is basically a version of lesson study developed in a Hong Kong context circa 2000 and further developed and used in other countries (e.g. Sweden[1], Brunei, South Africa, the UK, Austria, Indonesia). Since it is not the lesson that is the focus, but the object of learning, it was named learning study. It shares features with lesson study, such as the collaboration among teachers and the iterative design of planning, implementing, observing and revising of the lesson, but is framed by a theory of learning (predominantly) — variation theory (Marton, 2015).

Just as with lesson study, there are reports of the positive effects of learning study on teachers' professional development (e.g. Kullberg, Runesson et al., 2016; Lo, 2012; Lo, Chik & Pang, 2006). Although, teachers learn from learning study, the aim is beyond teacher professional development, but to produce knowledge that is sharable and can be utilized in other contexts.

> A Learning Study is simply a study of the relationship between learning and the conditions of learning, carried out by a group of teachers, with the double aim of boosting the participating teachers' ability to help their students to learn, on the one hand, and to produce new insights into learning and teaching that can also be shared with teachers who do not participate in the study, on the other hand. (Marton & Runesson, 2015, p. 104)

With this purpose, learning study is in line with Morris and Hiebert's (2011) claim that lesson study is a system that can create shared instructional products that guide classroom teaching (p. 5). The instructional product generated in learning study is a theoretical description of the object of learning; how it is constituted.

The focus in learning study: the object of learning and critical aspects

Pillay and Adler (2015, p. 224) have suggested that learning study is a response to Ponte and Chapman's call (2006) for research design that deals with researching pedagogical settings and teacher activities as concerns the object of learning and how it is constituted. The object of learning plays a central role in learning study. Basically, the object of learning is an answer to the question: "What is to be learned?". This question could be answered in three different ways, however (Marton, 2015). First, it can simply be the topic (e.g. 'negative numbers'). Second, it can be an educational objective (such as 'realizing the existence of negative numbers'). Third, it can be a critical aspect, thus something necessary to learn (e.g. learn that 3–1≠1–3, see below). In learning study, we are primarily dealing with the third meaning of the object of learning. 'What is to be learned' are the critical aspects and these must be found empirically for every group of learners.

The notion of critical aspects, which is central in variation theory and learning study, emanates from phenomenographic studies (Marton & Booth, 1997) where it was found that differences in the experience of the same thing were due to differences in aspects discerned of the phenomenon in question. From this follows that learning is a change in ways of experiencing or seeing the phenomenon and furthermore, that this change comes from discerning new aspects (Marton, 2015; Runesson, 2005). From a variation theory perspective, learning is seen as a change in ways of experiencing and learning failures are explained in a specific way; when learners fail to learn what was intended, they have not (yet) discerned aspects necessary to discern. So 'what is to be learned' are things that the students have not yet learned, but which are necessary for attaining particular educational objectives.

To answer the question: "What is to be learned?", aspects of the concept from the point of view of the discipline are not sufficient. Critical aspects cannot be derived from the subject *only*, but need to be explored and identified in relation to the learners and tested in the classroom (Mårtensson, 2015; Pang & Ki, 2016). Furthermore, critical aspects are not identical to what students have problems with, although these give keys to what the critical aspects might be. Similarly, what has been found to be problematic and reported in research literature is a valuable source for anticipating critical aspects.

Identifying the critical aspects in learning study

Critical aspects are relative to the educational objectives, but they are also relative to the learners: they differ with the learners; "critical aspects are relational in nature in that they are related to the qualitatively different ways of experiencing the same phenomenon manifested by learners" (Pang & Ki, 2016, p. 6). Therefore, it is necessary to study in depth the different ways the students experience or understand the phenomenon. This is usually done by analyzing students' answers on a diagnostic pre- and post-test, either as a paper-and-pen test and/or in an interview. From the analysis, hypothetical critical aspects are identified. Based on these findings, the research lessons are planned to make the assumed critical aspects discernable. In this way, the classroom becomes a "laboratory" (Elliott, 2012; Dewey, 1910/1974) where hypotheses of conditions for learning can be tested. A careful analysis of data of the outcomes on the post-test (did the students respond differently to the tasks in the post-test compared with before they were taught?) and the video-recorded research lesson, give further insights into what is critical for learning and how the content must be handled to promote learning. This becomes the basis for the planning of the second lesson in the cycle, taught by a new teacher, and to new students, and again the observed/recorded lesson and the diagnostic post-test are analyzed. The iteration proceeds until all classes are taught. The critical aspects are found in a transactional process comprising the learners, their learning (what they learn), what is targeted, or using Dewey and Bentley's (1949) description: a transaction of the known, the knowing and the knower. Sometimes unexpected results in a learning study can lead the team to identify aspects critical for student learning, however initially taken-for-granted and thus, beyond teachers' awareness. When the learning study team iteratively test different ways of handling the content *together with* students' learning as a point of reference, the critical aspects emerge and are specified (Mårtensson, 2015).

Variation theory as a design tool

Besides providing theoretical concepts (such as object of learning, critical aspects) that enable the team to have a common focus and common language to talk about teaching and learning, variation theory can also be used for designing the lesson (Kullberg, Mårtensson & Runesson, 2016).

One central idea of variation theory is that discernment of critical aspects is due to seeing differences rather than similarities. Marton (2015) argues that the discernment of a feature requires the experiencing of a difference between (at least) two things or parts of the same thing. To discern a new concept, one needs to experience contrast (variation) between the new concept and another concept, and hence how it differs from the other concept. So, for instance, it is probably easier to get acquainted to specific features of the natural numbers if these are contrasted with negative numbers. Furthermore, to discern an aspect of the concept this must be opened up as a dimension of variation against a stable background. Thus, a necessary condition for discernment is the experience of variation. A similar proposal is made by Watson and Mason (2006); variation can structure sense making by drawing attention to the targeted aspects and that different kinds of variation in exercises afford different learning possibilities (c.f. Goldenberg & Mason, 2008; Rowland, 2008). When comparing two lessons arranged similarly and with the same topic taught, several studies have demonstrated that differences in the pattern of variation seem to have a significant role for student learning (e.g. Kullberg et al., 2013; Lo, 2012; Pang & Lo, 2012).

A learning study about the existence of negative numbers

Next, we will report a learning study on teaching and learning of negative numbers in early grades in a Swedish school s an illustration of how a team of teachers can generate knowledge within a process of their actions.

Background

According to the Swedish national curriculum, negative numbers are not formally taught until grade 7 (13-14 years old), but contextualized at an earlier age within discussions about temperature below and above zero and with the help of the thermometer. However, there might be a limitation if negative numbers are only contextualized as 'degrees below zero'. The number system and the ordering of integers might not be visible when negative numbers are talked about as 'minus degrees' (in Swedish: 'minus-grader'). Every child probably knows that it is colder when the temperature is −10 degrees C compared to a temperature of 3 degrees C. This may be confusing when they must learn that −10 is a less number than 3. The team of teachers conducting the learning study reported here, had this experience, and wanted to explore whether it was possible to introduce negative numbers in earlier grades and not just in the context of temperature. They had noticed that many students believed that 'there are no numbers < 0' and that subtractions like 2 – 4 = were insolvable, or the difference is 0. So, the point of departure for the study was a learning problem they had encountered in their practice. Initially in the study, when the team studied the literature, they learned that gaining understanding of the nature of negative numbers has been problematic for early mathematicians to comprehend (Bishop et al., 2014), as well as for teachers to teach and learners to learn (e.g. Ball, 1993). The difficulties have to do with the meaning of the numerical system and the magnitude and direction of the number, the meaning of arithmetic operations, and the meaning of the minus sign (Altiparmak & Özdoğan, 2010). For Swedish students, the meaning of the minus sign is probably particularly difficult. In Swedish, a number such as –2 (in English: negative two) is pronounced 'minus två' (minus two) and written –2. Thus, there is no linguistic and symbolic difference between the minus as a sign for the operation and as a sign for the number. The team of teachers also learned that teaching of negative numbers should take the point of departure in real life problems or situations known from the children's experience and transformed into mathematical models. For instance, using 'a house' with floors above and below the ground floor, or a bird flying/diving above/below sea level, has been suggested (e.g. Ball, 1993). From this, the aim of the learning study was formulated: *to find the critical aspects for realizing the existence of negative numbers*.

Design of the study

The learning study was conducted in a school in a small municipality in Sweden with four classes in grade 2 and 3 (8 and 9 years old). The students had a similar socio-economic background; mostly middle-class. All of them were native Swedish speakers. No one had a non-Swedish background. For the participating students, the guardians had given their written consent.

The learning study team consisted of three teachers with different experience of teaching mathematics to young children. One teacher was certified to teach grade 1-9 in mathematics, one to teach mathematics in grade 1-3. The third teacher was not certified as a mathematics teacher but had some experience of teaching mathematics. Their teaching experience varied; 19 years, six years and one and a half years. Apart from 'reading the thermometer' none of them had taught negative numbers before. So, this was really a new area for them to explore. Before conducting the learning study, the teachers were introduced to learning study and variation theory in a one-day seminar. One of the teachers (second author) was the team leader and this study was a part of her master program in education (Lövström, 2015). She had been allocated a certain amount of time for the project and took care of practicalities, recorded the lessons and prepared the meetings. She also took notes at the planning/evaluation meetings and had the main responsibility for the analysis made after the last cycle. Being a master-student, she had the opportunity to discuss the study with her supervisor (first author), but the analysis during and after the learning study (thus, what is reported on here) was made by the team and the supervisor never met the team.

The iterative process comprised four cycles of planning, analyzing the data and revising the lessons. Each class was taught one lesson (approx. 50 mins.). Two of the teachers taught one lesson/class each (L1 and 2). Class/L3 and 4 were taught by the third teacher. In total, the team had 10 meetings (app. 2 hours/meeting).

The focus in the process was to identify the critical aspects, thus, what the students must learn to realize the existence of negative numbers (i.e. the object of learning). Initially the team designed a pre-test (See appendix), which was given to all students. Based on the results from this, the team anticipated the critical aspects and planned the first lesson. After the lesson, the test was given again. The results were analyzed together with the recorded lesson. This lead to a revision of the presumed critical aspects and to changes in how the content was taught in the following cycles. The empirical data (pre- and post-test data see appendix 1[2] and Table 1 and 2, and four video-recorded lessons) were analyzed by the team after each cycle throughout the process. This process is described in section I-IV below.

After the last cycle, the data were re-analyzed, mainly by the team leader (second author). This post-analysis was much more in depth and resulted in a specification of the assumed critical aspects (section V below).

In the following we will report on the cyclic process; the lessons, students' learning and the reflections and conclusions the teacher team drew from systematically analyzing results on the pre- and post-tests, recordings and transcripts of the recorded lessons. We will describe how they successively identified the critical aspects and refined the lessons to make the critical aspects discernable during the four cycles and finally, specified the identified critical aspects after cycle 4.

Cycle 1

Based on a discussion of own experience and from readings of the literature the team constructed a pre-test (See appendix.) From the analysis of the results on the pre-tests three critical aspects (CA) were identified:

CA 1. To discern the value of numbers in the numerical range –10 to 10.

CA 2. To discern the direction of subtraction on the number line.

CA 3. a) To discern the sign for negative numbers and the sign that indicates subtraction. b) To discern that negative numbers always have a visible sign.

These made the basis for the planning of the first lesson.

To make CA 1 discernable, pairs of negative and positive numbers on a number line were compared in the lesson. However, only one pair of numbers was compared; –2 and 2 (Figure 1).

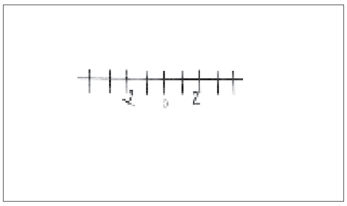

Figure 1. Comparison of –2 and 2 by means of a horizontal number line on the board

The teacher asked about similarities and differences between the two numbers. In the discussion, the focus became the minus sign in front of the negative two, which was contrasted to the missing minus sign for the positive 2. (CA 1). Similarly, a model of a house with floors above and below the ground was used to demonstrate that 2 and –2 is not the same; that floors below the ground floor are indicated by a minus sign and that degrees below zero are indicated in the same way. In this way, the different notations (–2 and 2) were compared, not the values, however. Thus, CA 1 was not made possible to learn in L1.

To make CA 2 visible, operations were made with the help of the number line, a model of and a house with floors above and below the ground and a thermometer. So, for example, the teacher pointed to +3 on the thermometer and said: "We are at plus three degrees. Then, it gets four degrees colder. Where do we end up at?" The students were able to indicate the point –1 on the thermometer. However, when they were asked to represent the change of temperature by a number sentence, they were confused. Suggestions like 3–4=0 and 3–4=1 came up.

When the operations 3+2=and 1–3= were conducted on the number line (CA 2), the starting/end point, the direction and the number of steps taken on the number line were noticed: "Where do we start on the number line? Which way should we go?" The teacher indicated the direction with gestures; to the right, to the left and asked; "Shall we go in this or that direction? How many steps? Where do we end up?"

After L1 the results on the post-test were analyzed and discussed by the teachers (See Table 1). The team noticed that task 1b, identifying the highest value among integers, did not seem to be a problem (17 of 19 answered correctly). On task 1d, however, finding the number with the highest value among the negative numbers: –1,–9,–6,–2, -5, only four more students could give the correct number after being taught (see Table

1). When analyzing L1, they became aware that *value* of the numbers (e.g. −2 and 2) was not actually made explicit to the class. When e.g. −2 and 2 was compared (See above) the emphasis on the minus sign in −2 highlighted the notation only. This made the team consider whether the range of numbers *per se* was the critical thing. They asked for instance: Would it be necessary to make *the relation* between different values of numbers visible, rather than the range of numbers? Would emphasizing the relation between different numbers open up for different possibilities to discern the values of integers? This discussion lead into a re-formulation of CA 1. It was also noticed that only taking one pair of numbers was probably not sufficient. More examples were needed.

Table 1. Number of correct answers in class 1-4 respectively on task 1b and 1d (choose the highest number) on the pre- and post-test

Lesson/class	**Task 1b** (O, -4, -3,-10)		**Task 1d** (-1,-9,-6,-2,-5)	
	Pre-test	*Post-test*	*Pre-test*	*Post-test*
L1 (N=19)	13	17	0	4
L2 (N=16)	15	15	7	7
L3 (N=15)	12	13	4	6
L4 (N=14)	13	14	3	10
N=64	53	59	14	27

On task 3 (Table 2), the team noticed improvement on task 3d 4−6= and 3g −2−2=. After L1, 9 respectively 7 (of 19) students could solve the subtractions. However, they noticed that the number of correct answers to item 3c had decreased. It seemed as if some students had been confused by meeting operations with negative difference. They could no longer solve familiar operations like 6−4=. This led into a discussion about the importance of noticing that commutativity is not valid for subtraction.

Table 2. Number of correct answers in class 1-4 respectively on task 3c, 3d and 3g on the pre- and post-test

Lesson/class	**Task 3c)** 6-4=		**Task 3d)** 4-6=		**Task 3g)** -2-2=	
	Pre-test	*Post-test*	*Pre-test*	*Post-test*	*Pre-test*	*Post-test*
L1 (N=19)	18	15	0	9	0	7
L2 (N=16)	15	14	4	11	3	9
L3 (N=15)	15	10	0	3	0	0
L4 N=14)	12	14	1	10	0	11
N=64	60	53	5	33	3	27

When it comes to the different signs for subtraction and negative numbers (CA 3), they seemed to be mixed up during L1. The team noticed that the model of the house was insufficient for representing the difference between, on the one hand, the minus sign for a number (e.g. –2) and on the other hand, as a sign for operation. This made them change CA 3. They assumed that emphasizing numbers both as places and distances on the number line was needed.

These findings made the team reconsider the presumed critical aspects and they were therefore revised as follows:

> CA 1. To discern the value of negative numbers in relation to other integers (before cycle 1: in the numerical range –10–10).
>
> CA 2. To discern that subtractions are not governed by the commutative law (new).
>
> CA 3. To discern numbers both as places and as distances on the number line (new).

Cycle 2-4

The revised critical aspects remained unchanged in the following cycles (2–4), whereas the treatment of them in the lesson was changed. For instance, the critical aspect (1) "to discern the value of negative numbers in relation to other integers" was planned to be made possible to discern by choosing specific pairs of numbers and by using the number line as a representation for comparison. Furthermore, the team decided to take more examples, so a task was planned where the students had to compare the following three pairs of numbers: –2 and 2 (the same as in cycle 1), –3 and 2, –4 and –2. These pairs of numbers were deliberately chosen because it was anticipated that they would challenge the students' understanding of the numerical values of negative numbers. For instance, in the pair –3 and 2, '3' has a higher magnitude than '2', but –3 has less value. In the same way, comparing the numbers –4 and –2, is an example that they thought could challenge a generalization from natural numbers (i.e. 4>2).

However, the planning was not followed completely. In L2 only one task with the planned examples (–2 and –2) to compare was given, and in L3, this task was not present at all. It was planned to be taught at the end of the lessons, but the changes made from cycle 1 took up more time than expected. So, the planned examples were never done in these lessons. It was not until L4 the comparison between two negative numbers was given as a task to the students. For instance, the teacher asked: "Which number has the highest value; negative two or positive two?" In this lesson (although not planned) another example was conducted (–3 and 2) instead of –4 and –2.

In the post-lesson seminars after L2 and 3, the team concluded that leaving the planned examples out probably affected the post-test results. This conjecture was confirmed; on task 1d after L2, there was no improvement at all (7 correct answers before and 7 after the lesson), and only a slight improvement after L3 (See Table 1). After L4, however, the team could observe a rather significant increase in correct answers; from 3 correct answers in the pre-test to 10 in the post-test. Thus, almost ¾ of the class could find the highest value of negative numbers after the lesson where all the planned examples were taught.

Specification of critical aspects

After the last cycle, it was found that the results on the post-test in class 4 on some of the tasks were significantly much better compared to the previous lessons (See Table 1 and 2). To verify the assumed critical aspects, an in-depth post-analysis of the transcribed video-recordings was done mainly by the team leader (second author). Again, the focus was if and how the test results reflected how the content was treated and what was made possible to learn in the lesson. This analysis resulted in a specification of the critical aspects.

After L4, the students performed much better on task 1d (Table 1) in comparison with their counterparts. Almost ¾ of the class (10/14) could find the number with the highest value among negative numbers, whereas in L2 and L3 only about half of the class answered correctly. There was a difference in the answers given also. In the post-test after L1 and L2, −9 was the most common choice. This error was interpreted as a generalization from the natural numbers.

As mentioned above, only in L1, L2 and L4, pairs of numbers were compared, and for the former lessons only one pair (–2 and 2). The immediate reflection was that the lower scores on the post-test after L1 and L2 could be explained by the students not having the opportunity to experience a comparison of negative numbers, since only one example was used. However, a comparison of the numbers –2 and 2 could have the potential to illustrate number value, depending on how the comparison is made. In L1 the comparison mainly concerned the notation (i.e. if there is a minus sign in front of the number or not). In L2 the numbers were compared in relation to zero, which can be seen from the following, when similarities and differences between the two numbers were discussed with the help of a number line:

> [8] T: If we look at the zero. How many steps should we take to get to that two [negative 2]? How much do we move? How many steps?
>
> [9] Bjorn: Two.
>
> [10] T: If we want to get to that two [2], how many steps do we have to take from zero?
>
> [11] Benjamin: Two as well.
>
> (Excerpt A, L2)

Excerpt A shows that the focus was mainly on the distance to zero; “How many steps away from zero must we take?” and thus, that both numbers have the same distance to zero. The values of the numbers were not in focus, however.

Furthermore, the analysis showed the team that, besides more examples taken in L4, there were other differences between L1, L2, L3 on the one hand and L4 on the other that might have been of importance for students' opportunities to learn. Firstly, it was noticed that using a number line with an arrow indicating infinity and 'the more to the right the higher the value', which happened in L4 only, seemed to be helpful when comparing integers.

Secondly, the interpretation was that the character of the pair of numbers in the examples taken seemed to be of importance. It seems as if these pairs must have certain features to make the difference between magnitude and value visible. The example –2 and 2 might not be the most powerful one to use. Even if the students can choose the right number in the pair, can we be sure that they know the values of integers? Does this example (a comparison of integers i.e. positive and negative numbers) open for the possibility to discern values of negative numbers? A closer look at Erica's response in the following excerpt indicates that this might not be the case.

> [1] T: Listen up! I've been thinking about something. Which of these numbers is worth most? Is it minus two or is it plus two? Negative two or plus two? What do you say, Erica?
>
> [2] Erika: Positive two.
>
> [3] T: You think so? Why do you think that?
>
> [4] Erika: Er … that's minus and that's plus. That one is a lot … plus is higher than minus.
>
> (Excerpt B, L4)

Erica says: "plus is higher than minus". The team asked whether it could be the case that Erica means that positive numbers are bigger than negative? Indeed, this is correct, but it is not sufficient to know when comparing two negative numbers; –3 and –2, for example. Here neither "plus is higher", or previous experience of values and magnitude of numbers is of help. The following excerpt was taken as an indication of that.

The teacher asks the students to compare –3 (negative 3) and –2 (negative 2).

> [1] T: Which has the highest value?
>
> [2] Erik: minus three-
>
> [3] T: So, you think that minus three has the highest value, do you? How are you thinking, Erik?
>
> [4] Elin: Because it's one of those numbers.
>
> (Excerpt C, L4)

One cannot know for sure what is behind Eric's response, but Elin, who replied to the teacher's question said: "Because it's one of those numbers". It seems as if she neglected the minus sign and focused on what she knew about the positive numbers; that 3>2.

Thus, she focused on the magnitude. This lead to the conclusion that the critical aspect, as it was formulated in cycle 2–4 – *to discern the value of negative numbers in relation to other integers* – could be verified, but needed to be specified, however. It was concluded that what numbers that are compared is of importance, and thus, that some examples seemed to be more powerful than others. In this case, examples with negative numbers only and with numbers that challenge the idea of magnitude (e.g. 4>2, but –4<–3) would probably be more powerful. This insight enabled CA 1 to be specified into *to differentiate negative numbers*.

The results on the post-test showed improvement on subtraction 4–6= in all cycles except cycle 3 (Table 2). Only three students could correctly solve task 3g –2–2= before the lesson (cycle 2). After being taught, about ⅓ of class/L1 1, ½ of class 2/L2 and ¾ of class 4/L4 scored correctly, whereas none in class 3. Thus, there were improvements in all classes, except one, but mostly in class 4. Therefore, a closer look at L3 in relation to the others, and specifically L4, was needed.

To discern that the commutative law is not valid for subtraction, a well-planned pattern of a+b=b+a and a-b≠b-a was implemented in the L3 and L4. Taking a closer look at L3 and L4, the team could see that the subtractions were contextualized differently and different representations were used. I L3 1–3= was contextualized in terms of "what is missing?", and fingers were used for calculating the difference. The following excerpt shows how the suggested alternatives 1–3=-2 and 1–3=0 were discussed in L3 and how the teacher told the students to check the calculation with their fingers:

(The teacher and the students use their fingers. The teacher holds up one finger.)

> [7] T: We can start by taking away one. (takes away one finger)
>
> [8] Students: And then two.
>
> [9] T: … are missing. Could it be so?
>
> *The students think about this and say somewhat hesitantly "yes".*
>
> [10] T: What do you think Carola?
>
> [11] Carola: No
>
> [12] Charlie: Three is more than one, so it must be zero. Because three is more than if you take one. If you take three away from one, then it is zero because then it's nothing.
>
> (Excerpt D, L3)

The teacher held up one finger, took it away and asked what was missing. However, the students cannot see the missing two fingers, and probably not experience them as negative numbers, which Charlie's comment indicates. He said: "If you take three away from one, then it is zero because then it's nothing", which is exactly what the teacher is showing. A closer look at the post-test showed that 9/15 (in class 3) students answered –2–2=0. Thus, instead of a negative number, 0 was given as the difference.

It was concluded that it was not made possible to discern negative numbers in this situation and subsequently not the difference between $a-b=$ and $b-a=$, thus the assumed critical aspect. However, in what ways was this made possible in the other lessons where the scores were the highest?

Since those classes that had met the number line in the lesson, scored better (L1, L2 and L4) on the post-test, it was concluded that the representation (number line) was better for representing negative numbers than talking about quantities and how much is missing. However, the class 4 was better than the others. Why? the team asked.

A closer look at L3 and L4 showed that only in L4 a comparison between addition and subtraction was made. Thus, the team found that only in this lesson it was made possible to experience that $a+b=b+a$ but $a-b\neq b+a$. The deeper analysis of the transcripts showed another thing; the lessons were different in respect to how the functions of the minuend and subtrahend were taught. In L4 3+1=4 and 1+3=4 were compared with 3–1=2 and 1–3=–2 by representing the operations on the number line and the different operations were visible on the board simultaneously. The operations were made by talking about the starting point on the number line, the direction, the number of steps taken and the end-point. For instance, with the subtraction 1–3=, the teacher started by asking: "Where should we start?".

> [6] Elof: Three
>
> [7] T: Do we start on three in this question?
>
> [8] Elof: No, one.
>
> [9] T: We actually start on one. Do you get that?
>
> [10] Students: Mm
>
> [11] T: We start on one. I'll put a star here so we know where we start. There we are. What happens now?
>
> (Excerpt E, L4)

Elof's answer "three" was interpreted by the team as if he was not clear about the function of the minuend and the subtrahend. This was noticed in the lesson by the teacher [7] and Elof then corrected himself [1]. The teacher repeated that the starting point was 1 and marked this on the number line [11]. After having asked what would happen, she continued and asked for the direction of the operation and the number of steps they must take on the number line (the direction and the number of steps were marked on the number line):

> [1] T: Should we go forwards or backwards?
>
> *The teacher gesticulates. The students are engaged.*
>
> [2] T: What do you think Emil?

[3] Emil: Backwards

[4] T: You say backwards … the minus sign means that we should go backwards. How many steps should we go backwards? How many steps? Elinore?

[5] Elinore: Three.

(Excerpt F, L4)

The team came to the conclusion that, since the specific examples taken; 3+1=4,1+3=4,3–1=2 and 1–3=-2, and the four operations illustrated on the number line were visible at the same time on the board in L4, the difference between the minuend (starting point) and the subtrahend (end-point) was made much more explicit compared to the other lessons. This difference, they thought, could shed light on the better performance on the post-test after L4. It seems like it is necessary to learn about *the functions* of the minuend and subtrahend in order not to conclude that $a-b \neq b-a$, thus that commutativity is not valid for subtraction and to realize that the difference in subtraction can be negative. This lead to a specification of critical aspect 2. During cycle 2–4 it was expressed as "to discern that subtractions are not governed the commutative law". After the post-analysis, however, it was specified into "*to differentiate the function of the minuend versus the function of the subtrahend.*

CA 3 was initially formulated: a) To discern the sign for negative numbers and the sign that indicates subtraction and b) To discern that negative numbers always have a visible sign. This was mostly due to reading the literature (e.g. Gallardo, 1995; Lamb et al., 2012; Vlassis, 2004). However, this difference was mixed up in L1. Therefore, it was decided to revise CA 3 before L2. When planning, the team was inspired by Lakoff and Núñez's (2000) metaphors 'places' and 'distance', thus, numbers as places on the number line and operation as 'direction' and 'distance'. When watching the recording, and reading the transcripts of L2 it was found that, although the starting/end point, the direction and the difference was made clear, the difference between the minus sign as indicating operation on the one hand and the sign for negative numbers on the other, was not made explicit, which the following illustrates. The teacher asked the students to compare the results of the operations in the two examples –2–4= and 2–4=. It was concluded they were different. The teacher asked for a justification. Bea replied:

[1] Bea: You start at the number we ended on in the first [example] That which was the answer to the last question one [the result of 2–4=–2].

[2] T: Mmm. Bessie, did you have a thought there?

[3] Bessie: Yeah. You start at negative two and jump backwards so it will be minus three, minus four, minus five, minus six.

(Excerpt G, L2)

So, although planned, this comparison was not done in L2. In L4, however, the examples 3–1=;1–3= were compared with the examples –2–3=;-2–4=. In the whole class discussion, an interesting disagreement about the meaning of the minus sign appeared. Emil thought 2 and 3 in –2–3= were both negative numbers. He said:

> [1] Emil: I think it's like this. The two is a negative two that's on the number line there. Yes, and then we have a negative three there, too.
>
> [2] T: Hm.
>
> [3] Emma: No. It's not a….
>
> [4] Emil: Yes. It's a negative three.
>
> [5] Emma: You just said so.
>
> [6] Emil: It's minus there, it's a negative two and a negative three.
>
> [7] T: Mm. Oh, listen. Now we're on to something interesting, really. There's a difference between this sign [points to –2] and that sign [–3], even if they look similar. An important thing.

Emil changed his mind and said:

> [8] L: It could be a minus two and a positive three.
>
> [9] T: Yes. Minus two minus three. That's very good. So, you think it's a negative two?
>
> [10] Emil: Yeah and a positive three.
>
> [The teacher writes a + in front of 3.]
>
> (Excerpt H, L4)

A few minutes later the following happened. The student Emelie was confused. She announced that she did not know how to interpret '4' in –2–4=. She said: "is it a negative or a positive four?" The student Emrik explained:

> [1] Emrik: It's an ordinary one. It's got just *one* minus in between.
>
> [2] T: … This is a *number*. [*Circles* –2 *and* 4]
>
> [3] Emelie: Aha! Yes!
>
> [4] T: This is a number and here we have a sign. Now, let's see, what do we do? Tell me what to do, Esther.
>
> *The teacher asked Esther to continue and describe the operation on the number line. She said that we start on* –2 *and go four steps backwards [to the left] and end up on* –6.

[13] T: Yeah. That's brilliant. And what's the answer then?

[14] Ss: –6

[15] T: That's right!

[16] Ss: That was not difficult!

(Excerpt I, L4)

Esther's explanation together with the students' reaction "That was not difficult", was interpreted as an indication that the difference in meaning of the minus sign had been made possible to discern. Furthermore, the team realized that seeing this distinction made a difference to students' understanding; it was not difficult anymore. The conclusion was that, to *differentiate the minus sign for negative numbers vs. the sign for subtraction* was a critical aspect. This was identified already before the first cycle, but was changed before cycle 2-4. The post-analysis seemed to verify what was anticipated initially in the learning study.

From discerning to differentiating

The emergence and specification of the assumed critical aspects is illustrated in Figure 2. As described above, the critical aspects were successively specified by the teacher team in the process; from being something to discern (during cycle 1-4), to being something that should be differentiated from another (after cycle 4).

To specify the critical aspects in terms of "differentiation" rather than discernment, Lövström (2015) argues, gives a more detailed description that can inform teachers how to handle the content in the lesson: "This means for example that the first critical aspect, formulated 'to differentiate', at the same time says something about each of the values of the negative numbers and how the numbers relate to each other" (ibid., p. 96). Furthermore, she states that this formulation describes teaching actions in more detail, namely what two things that must be compared: "To differentiate something from something else highlights at the same time: a specific subject matter, students' experiences of the subject content and how this content can be handled in teaching to get students to learn what is planned" (ibid., p. 96).

CA 1. To discern the value of numbers in the numerical range -10 to 10. CA 2. To discern the direction of subtraction on the number line. CA 3. a) To discern: a) the sign for negative numbers and sign that indicate subtraction, b) that negative numbers always have a visible sign.	CA 1. To discern the value of negative numbers in relation to other integers. CA 2. To discern that subtractions are not governed by the commutative law. CA 3. To discern numbers both as places and as distances on the number line.	CA 1. To differentiate the value of two negative numbers. CA 2. To differentiate the function of the minuend versus the function of the subtrahend in a subtraction. CA 3. To differentiate the minus sign for negative numbers versus the minus sign for subtraction.

Figure 2. The emergence and specification of presumed critical aspect during the cyclic process

Theory-informed lesson study as practice-based research?

In this paper, we have shown how a type of theory-informed lesson study can be a mechanism for generating deeper knowledge about the object of learning in terms of what the students need to learn to develop a specific capability, skill or understanding and how this can be handled in teaching. What is specific about a variation theory-informed lesson study is not curriculum development or implementation of reforms *in general,* but about finding key elements in the object of learning; something that some students have not made their own earlier and must therefore learn to grasp. In the study used as an illustration, the critical aspects emerged and were successively revised and specified (Figure 2). The process started with conjectures about what the critical aspects might be. By systematically inquiring what was made possible to learn in the lesson and what students had learned the teachers got deeper insights into the critical aspects. The results from this case study support previous findings (e.g. Holmqvist, 2011); using variation theory helped the teachers to focus on the object of learning and by deeply relating student learning to teaching, identify what was needed to learn; the critical aspects.

Learning study is focused on constructing knowledge concerning the objects of learning. The outcome or result of a lesson study informed by variation theory is a theoretical description of the character of the object of learning in terms of its critical aspects and how the content can be handled to make them possible to discern. This kind of result is in resonance with Morris and Hiebert's (2011) suggestion; that lesson study can create "shared instructional products that guide classroom teaching" (p. 5). We would suggest that the instructional product generated in the learning study reported here can be useful for other teachers. It is not a 'lesson plan' in a traditional sense, still it can serve as a guide for other teachers about what is needed to learn and how powerful tasks and sets of examples can be designed to afford the critical aspects to be discerned.

In learning study, specific objects of learning are theorized and knowledge about the necessary conditions for making them one's own is generated. Such results are indeed generated in a specific context but are descriptions of features of the object of learning – not individuals – and therefore likely transferable to new contexts. For instance, Runesson and Gustafsson (2012) demonstrated that Swedish teachers could use documented Hong Kong research lessons and the identified critical aspects gained as a resource and adapt these to other conditions in the Swedish context thus, that results from learning study can be developed and cumulative. Whether the critical aspects of the object of learning found in this study are critical for learning about the existence of negative numbers in other contexts, if there are others not found here, and if they are transferable to new contexts must be empirically studied, however.

However, one could ask: Is lesson and learning study practice-based research? Lewis (2015) sees parallels with the Japanese version of lesson study and the PDSA- cycle (Plan-do-study-act) in improvement science (See Langley et al., 2009) successfully applied in other sectors than education. Elliott (2012) argues that a theory-informed lesson study "provides a strong basis for the development of a practitioner-based science of teaching" (p. 108). Based on an extensive evaluation of the implementation of learning study in Hong Kong (Elliott & Yu, 2008; 2013), he argues that a theory-informed lesson study like learning study, takes Action research back to its origin," the systematic and cumulative production of pedagogical knowledge in actionable form by teachers" (Elliott, 2012, p. 123) as proposed by Stenhouse. Thus, Elliott argues, and in accordance with our proposal; a theory-informed lesson study could be more than just professional development.

The study reported was driven by teachers' problems. It involved their tacit knowledge and professional experience and the knowledge was generated within the process of teachers' actions. Therefore, we would suggest that the problem-driven feature of the learning study, together with the teachers as the key stakeholders, make it a practice-based research with potential to overlap the research – practice gap. It aims at solving problems in practice by deepening the understanding of the research object in terms of specification and differentiation.

The knowledge product from the study reported here can be seen as 'examples' with a specific learning goal in focus. It is tentative, changeable and thus, open to improvement (Morris & Hiebert, 2011), but detailed enough to guide classroom instruction and possible to be used by other actors.

Notes

1 "The Learning study model was introduced in Sweden in 2003 through a research project financed by the Swedish Research Council and involved close co-operation between the research team at the University and several schools. This was followed by two other research projects, a research school for 24 teachers up to an MPhil degree level, all of them financed by the Swedish Research Council. The Learning study was introduced more in the research context in Sweden than in the context of school development, as was the case in Hong Kong" (Marton & Runesson, 2015, p.104).

2 Originally the tests had 9 tasks. However, 1 and 3 were the tasks that the teacher team paid most attention to and thought gave the most valuable information.

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Appendix 1

Pre- and post-test (selection of 3 out of 9 tasks)

What number has the highest value? (Circle)

a.

5	2	10	6	3

b.

0	-4	7	-3	-10

c.

8	5	8	0	5

d.

9	1	6	2	5

Write the smallest number you know.

____

Calculate

a. 3–2= ______ b. 2–3=______ c. 6–4=______ d. 4–6= _______

e. 2+2=______ f. 2–2=______ g. –2–2=______ h. 0–3= _______