Sterner, H. (2021)
Working on graphs in elementary school: a pathway to the generalization of patterns
In: M. Inprasitha, N. Changsri, & N. Boonsena, (Eds.) (ed.), *Proceedings of the 44th conference of the International Group for the Psychology of Mathematics Education, Khon Kaen Thailand* (pp. 113-121). Thailand Society of Mathematics Education

WORKING ON GRAPHS IN ELEMENTARY SCHOOL – A PATHWAY TO THE GENERALIZATION OF PATTERNS

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Pattern generalization is a key element of early algebra. However, it is also an area that causes significant problems for students as well as teachers, as it has proved challenging for elementary school students to understand the meaning of generalization. To address these problems, an intervention was done to introduce the graph and functions in relation to pattern generalizations in Grades 1 and 6. Working on graphs was new for these teachers because, in Sweden, graphs are normally not introduced in school until Grade 7. The results show that the introduction of graphs became a tool to understand and talking about a pattern generalization. As a result, their teaching on linear functions and patterns changed, and the implications of the results on mathematics education in elementary school are discussed in this paper.

INTRODUCTION

Algebra learning includes the ability to express and generalize relationships among quantitates. One way of introducing young students to generalizing and functional relationships is through pattern generalizations (e.g., Blanton et al., 2019; Radford, 2010; Wilkie, 2019). The main problem with generalizing in early grades is that the meaning and activity of generalizing in mathematics has proved challenging to understand (Stylianides & Silver 2009). Therefore, generalizing in early grades warrants further investigation. This paper presents results from a Swedish educational design research study on linear functions and the generalization of patterns in elementary school. The intervention was conducted in close collaboration between three teachers and one researcher (the author). This paper focuses on the three teachers' learning during the nine months of the intervention. More explicitly, the focus is on how these teachers come to implement graphs in their teaching of generalization of patterns and how they reflected upon their changed teaching when using graphs. In the intervention, the teachers initially expressed that they lacked the words to describe and teach the generalization of patterns, and they found it hard to discern when the generalization had been realized (Sterner 2019). This is in line with Blanton's et al. (2019) and Wilkie's (2019) statement that teachers lack awareness about functional thinking, and the authors point to the graph's contributing to functional thinking and pattern generalization.

In this paper, I will present how the introduction of graphs made the teachers aware of the generalization of patterns, and how, as a result of this awareness, their teaching of functions and patterns changed. More explicitly, the following research question will

2020. In NNN (Eds.). Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education, Vol. 1, pp. XX-YY. Khon Kaen, Thailand: PME.

be in focus: How does the introduction of the graph representation make it possible for elementary teachers to discern and teach the generalization of patterns?

LITERATURE REVIEW AND CONCEPTUAL FRAMEWORK

One main concept of algebra involves generalization. Early algebra can be seen as a way to address traditional transition problems from elementary to secondary school. One key aspect of this is working with functions rather than equations, which means addressing the relationship between quantitates and letters as variables rather than unknowns. This invites the possibilities to work with patterns of various sorts, including figural patterns and pattern generalizations.

Dörfler (1991) makes a distinction between theoretical and empirical generalization, where the *empirical generalization* can be seen as a generalization from one situation to another, while *theoretical generalization* is including some form of abstraction. Elements from Dörfler's interpretation of generalization can be found in Radford's concept *algebraic generality*, which is the main theoretical perspective in this paper. Radford (2010) explains algebraic generality in different layers: *factual-*, *contextual-*, and *symbolic generality*. Factual generality could be described as generality articulated on, for example, numbers, words, and gestures related to the task, for example, when a student talks about a pattern and says, "increases by 3". The layers of contextual and symbolic generality are interpreted as; generality expressed in some linguistically way, for example, through symbols or language. In this study, the difference between the layers of contextual and symbolic generality is interpreted in the following way. In the symbolic generality, the generalization is expressed in variable notation. In contrast, the contextual generality creates opportunities to use language and actions to create meaning of the variable notations, to create what Radford calls *knowledge objectification* (Radford, 2003; 2010). Working with the algebraic generality in different layers, Radford (2010) points to the process of noticing something general and making sense of the general. Radford calls this process of sense-making 'knowledge objectification'. However, from a teacher's perspective, questions arise about how generalization can benefit students' learning in early grades and what generalization is in elementary school. Carraher, Martinez, and Schliemann (2008) and Wilkie (2019) emphasize the importance of both the representation and the reasoning behind the conventional notation when teaching generalization in early grades.

Research, as well as the summary of PME contributions of topics of functions and calculus, indicate the importance of algebraic thinking and using a functional approach in early grades (e.g., Blanton et al., 2015; Hitt & González-Martín, 2016). Working with pattern generalizations is one tool to stimulate algebraic thinking and the idea of generalizations in early grades (Blanton et al., 2015; Wilkie, 2019). The graph representation, along with figural pattern generalizations, could be a pathway to functional thinking and algebraic generalization. Researchers point to the importance of reasoning about the relationship between two (or more) varying quantities when teaching functional thinking (e.g., Radford, 2010; Wilkie, 2019). However, research indicates that students have difficulties identifying covariational relationships, which involves describing how two quantities vary in relation to each other (Wilkie 2019). Similar ideas emerge in (Blanton et al., 2019), where the authors point out teachers' lack of awareness about the functional thinking and how to use the graph representation to visualize covariational relationship and proportionality. Blanton et al. (2019) and Wilkie (2019) stress the importance of further research on how to support elementary teachers in functional thinking in early algebra. Research shows how difficulties in proportional reasoning emerge in the early grades, indicating that students do not use the zero point on the x-axis when working with graph representations (Wilkie, 2019). With the above literature review as background, two general goals for teaching were formulated for a long-term intervention study (see Sterner 2019). In this paper, there is a particular focus on the second theme:

1: The students should be given opportunities to identify a pattern, structure the pattern, and generalize the pattern.

2: The students should be given opportunities to work with algebraic reasoning, including functional thinking and proportional relationship, and determining relations between two or more varying quantities.

METHOD

The intervention in this study is designed as an educational design research, and a continuation of a project, including mathematics teachers from grades 1-6 (Sterner 2015). The author and three mathematics teachers collaborated (one from Grade 1 and two from Grade 6) in three recurring design cycles. The selection of the teachers' teaching groups was done naturally since it was in grades 1 and 6 the teachers' work when the intervention took place. The two goals for teaching, the themes, are seen as Design Principles (DPs) (McKenney & Reeves, 2012) and are used as a theoretical guide for the intervention. Hereafter, these themes are referred to as DP1 and DP2. The background and the content of the DPs are described in more detail in Sterner (2019).

The Swedish context

There are goals for algebra in the Swedish curriculum materials (National Agency of Education, 2017) for Grades 3–6, but functions and functional thinking are not introduced until Grade 7. However, most teachers, including the participants in the current project, have little experience with functional thinking because of the lack of emphasis on functions in the elementary school. Three mathematics teachers (Clara, Irma, and Jonna) from different schools in Sweden have all of them more than twenty years of experience from teaching in Grades 1–6. Two of the teachers are semi-specialized mathematics teachers and teach in three subjects, mathematics, science, art, or music in Grades 1–6. The third teacher is a general subjects teacher in Grades 1–3.

Empirical materials in this paper

The intervention took place over a period of nine months, in the meetings, the teachers and the researcher planned and evaluated the teaching. Excerpts from lessons in Grade

6 are also used to illustrate the challenges that occurred in the teachers' discussions. The teachers' individual reflections about these lessons are also included in the analysis. The empirical data analyzed in this paper included 15 hours of video recordings from the meetings mentioned above where the teachers and the researcher planned and evaluated the teaching, 21 hours of lessons, and 5.5 hours of teachers' individual reflections with the researcher directly after teaching.

Knowledge objectification and algebraic generality as an analytical frame

In this study, Radford's knowledge objectification with focus on algebra generality (2003; 2010) is used as a conceptual frame in relation to the empirical data. The algebraic generality (factual-, contextual-, and symbolic generality) is used to explore and exemplify how the graph representation makes it possible for elementary teachers to discern and teach the generalization of patterns. The frame is also used to exemplify how the teachers move within and between the different layers of algebraic generality when using the graph in pattern generalizations. The algebra generality is used as a theoretical frame for the analyses, while the DPs, are seen as a theoretical frame for the intervention and goals for teaching. Through the analysis, the teachers' reflections upon their changed teaching when using graphs were analyzed, i.e., their knowledge objectification.

RESULTS

In the results, transcripts and figures are used to visualize the teachers' process of knowledge objectification of pattern generalizations, and the selection of transcripts will show crucial moments in this process.

The graph opened up for different representations

The introduction of the graph revealed that the general formula needs to be visualized in different representations. In the initial process of the intervention, the teachers strongly opposed using the graph as a representation for pattern generalizations. As mentioned, working with graphs as a representation of pattern generalizations was a new challenge for these teachers (Sterner 2019). However, the complexity of understanding, expressing, and making justifications of a general formula becomes visible in the teaching when the teachers challenge their students to explain what the variable notation symbolizes in an equation for example $y = 3x + 5$. This equation represents a pattern generalization that the teachers called *the canoes* (see Fig. 1).

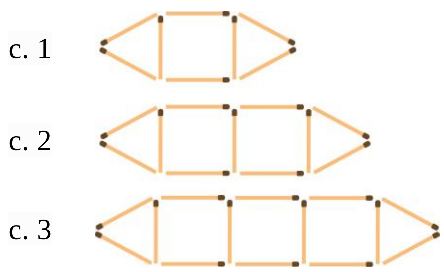

Figure 1. Image of a pattern called 'the canoes.'

This task was a crucial task in the intervention to explore the slope – the change in input and the corresponding change in output. In discussions with each other, the teachers became aware that the general formula itself, do not explains neither the teachers' nor the students' understanding of generalizations. The teachers realized that they had neither the language nor tools to talk about the structure of a general formula. Therefore, I introduced the graph representation as a tool to make a justification for pattern generalizations, in line with Wilkie (2019).

The graph visualized the structure in the general formula

By introducing the graph, the importance of visualizing the structure in a general formula emerge. The teachers used various examples in their teaching, illustrating linear functions and pattern generalizations. One task exemplifying pattern generalization was 'the canoes' (Fig. 1). Another task illustrating direct proportionality was a pattern concerning a number of dogs and their corresponding number of tails, ears, and legs. The graph made it possible to visualize the slope (m), and the y-intercept (c), ($y = mx + c$). The teachers talk about the 'start-value' when c has the value of zero. The teacher asked the students to work with the figural patterns in various representations, for example, using matches and tables, using the coordinates from the table to make a graph representation, and finding a general formula (see Fig. 2).

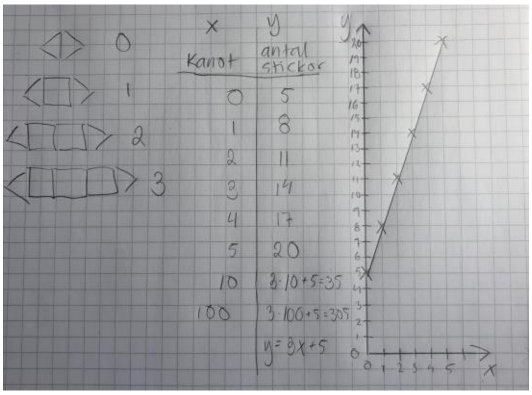

Figure 2. A student's solution of the pattern of 'the canoes.'

The following transcript illustrates a conversation in the whole class discussion when the teacher (Irma) asked two of the students to describe what they had realized when using the graph and the equation to represent the pattern generalization. This conversation illustrates how Irma, in the meeting with her students, comes to realize the potential of using the graph. The transcript indicates that the students (Anna and Kim) used the graph to understand the rate of change and used the graph to understand what happens when the independent value is 0.

Irma: Kim and Anna, can you tell us what you found when you compared the pattern of matches with the table, the graph, and the general formula?

Kim Yes…we tried to draw a straight line through the origin, but it didn't work…It didn't end up as a straight line…That made us understand the meaning of the number of 5 in the general formula ($y = 3x + 5$).

Irma: Alright, go on.

Anna: We just realized the meaning of figure 0 in the table. We hadn't thought about it earlier, but now, when we look at the graph, and we didn't manage to get a straight line through the origin…we realized…and saw the 1 step in the right (x-axis) and 3 steps up (y-axis).

Irma: Yes, the graph visualized the start-value in the coordinates (0,5) (Fig. 2) and indicates how we can explain the general formula using the values in the table and the graph for the pattern generalization. [When Irma speaks, she makes it as clear as possible by using various representations]. We have the start-value (0,5) by subtracting the rate of change, 3, from the first entry, 8, in the coordinate (1, 8).

In the whole-class discussion, they talked about what several students have comprehended during the lesson. The teacher then returned to the discussions from the previous day, which is about linear functions and direct proportionality representing dogs and their corresponding tails, ears, and legs. The conversation goes on in the whole-class discussion, and they talk about the relationship between quantitates (x and y). The students talk about what happened when they have "one more seat" in the canoes, and other students talk about what happened when they increase one figure, and a third student looked at the graph and said: "every time we go 1 step on the x-axis, we go 3 steps on the y-axis." The graph helped the teachers to talk about the independent and the dependent value. The graph becomes a representation to go from a specific situation to the general. In the reflection after teaching and in the refining phase, the teachers talked about how they now have realized the importance of understanding the relationship between quantities. The graph was an excellent way to visualize this relation.

Jonna: I've always had difficulties explaining the relationship between x and y, and I didn't know what words I could use to explaining the relationship correctly and simply for the students.

Clara: The graph became a tool that I often came back to, making it possible to visualize the values in the table and visualize and talk about the general formula's content…The graph helped me to talk about what I did not have words for – the relationship between x and y.

The quotation below shows how the teachers use the graph representation as an input to talk about a generalization, through other already known representations.

Clara: Can you believe, I've never connected the graph with the table or the pattern of matches before…I never realized the importance of the relationship between x and y. I don't think I've fully understood what proportional relationship is – sometimes, I feel like the text of the (Swedish) curriculum materials is a bit abstract. I would love for all teachers to be part of something like this.

The teachers described how the graph had been an asset for both their own and the students' understanding of pattern generalization. In the discussion, the teachers stated that they had not previously understood the value of paying attention to what they called 'new small details', for example, what they called the *starting-value* or the relationship between quantitates in a generalized formula or a pattern generalization. The results indicate that the graph representation helped the teacher to talk about the functional relationship as well as the proportional relationship of pattern generalizations.

CONCLUSION AND IMPLICATIONS

The results show that the graph representation became a way of understanding and talking about the structure of a general formula in a pattern generalization. However, the graph representation is not enough. The teachers' discussions show that they had to elaborate at multiple representations, to justify and understand what the teachers called 'new small details'. The small details include, for example, the relationship between quantitates and the slope.

The teachers' discussions changed from interpreting a pattern generalization equal with a general formula – *that*, and nothing else. The teachers' initial interpreting of generalization would be described by Radford (2010) as symbolic generalization. At the end of the intervention process, the teachers interpret and justify the pattern generalization in multiple representations. The graph provided an entrance to justify the pattern generalization. The graph was also used as a tool to understand the structure of a generalization. This is what Radford (2003; 2010) called using different layers of generality to understand the algebraic generality. The contextual generalization (Radford, 2010) became visible when the teachers used the graph with already known representations to find the words and the language to talk about the relationships between variables and the slope. That falls in line with what Blanton et al. (2019) and Carraher et al. (2008) indicate from their study, including students and their use of different representations to support the understanding of variable notations. The graph thus becomes a tool to express the symbolic generality in natural language and creates opportunities to make knowledge objectification for pattern generalization. Working with the graph helped the teachers to understand and talk about proportional relationships and functional thinking, which in line with (Blanton et al., 2015; Blanton et al., 2019; Wilkie, 2019). The graph made it possible to visualize how the generalizations apply not only to a specific situation but also to all cases. This demonstrates to the teachers the importance of both empirical and theoretical generalization, which Dörfler (1990) addresses.

In addition to the results answering the research questions of this study, it is worth considering the methodology used. DPs are used as a theoretical frame for the interventions' content and as goals for the teaching. In design research, the DPs are normally changed, and new conceptual ideas are developed during the process (McKenny & Reeves, 2012). However, in this study, the content of the DPs does not change. Instead, the teachers' *understanding* of the DPs changes, thanks to using the graphs and working with different layers of algebraic generality.

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