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GENERALIZED MITTAG-LEFFLER KERNELS AND GENERALIZED SCALING OPERATORS IN MITTAG-LEFFLER ANALYSIS

WOLFGANG BOCK AND ANG ELYN GUMANOY

Abstract. Generalized scaling operators and generalized Gauss kernels are fundamental concepts in Gaussian analysis with application to path integrals and PDEs via the Feynman-Kac formula. In non-Gaussian analysis, particularly in Mittag-Leffler analysis, i.e., in the case when compared to a Gaussian characteristic function the exponential is replaced by a Mittag-Leffler function, these concepts are unknown. In view of this, we elaborate in this article the generalized scaling and generalized Mittag-Leffler kernels and prove a form of a Wick-type product formula. We give some first examples for generalized scaling.

1. Introduction

Generalized Gauss kernels and generalized scaling operators play a fundamental role in different aspects and applications of Gaussian analysis. Gaussian analysis and white noise analysis became a rapidly developing theory with various applications such as mathematical physics as statistical mechanics, quantum field theory, quantum mechanics and polymer physics as well as in applied mathematics and Stochastic Analysis, Dirichlet forms, Stochastic (Partial) Differential equations and finance, see e.g. the monographs [10, 22, 13]. Various characterization theorems [24, 14, 6] are proven to build up a strong analytical foundation. Almost at the same time, first attempts were made to introduce a non-Gaussian infinite dimensional analysis, by transferring properties of the Gaussian measure to the Poisson measure [11], with the help of a biorthogonal generalized Appell systems [2, 1, 16]. This approach is suitable for many measures, like the Gaussian measure and the Poisson measure [15]. In [4] a similar concept was used to establish the Mittag-Leffler Analysis. The grey noise measure [27, 21] is included as a special case in the class of Mittag-Leffler measures, which offers the possibility to apply the Mittag-Leffler analysis to fractional differential equations, in particular to fractional diffusion equations [26, 27], which carry numerous applications in science, like relaxation type differential equations or viscoelasticity. There is a well-known connection between PDEs and stochastic processes, provided by the Feynman-Kac formula. By investigating a heat equation, where the time derivative is a Caputo derivative of fractional order, Schneider introduced grey Brownian motion (gBm) in [27]. He showed that a solution to the time-fractional heat equation is given in terms of grey Brownian motion like in the Feynman-Kac case. The link between grey Brownian motion and fractional differential equations is also studied by Mura and

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Mainardi in the framework of fractional diffusion equations in [21]. In [5] also a relation between the fractional heat kernel in this setting and the associated process grey Brownian motion was proved.

In this article, we define a set of generalized Mittag-Leffler kernels in Mittag-Leffler analysis which plays the role of generalized Gauss kernels in Gaussian analysis. We show its relation to generalized scaling operators and provide a version of a generalized Wick formula. At the end of the paper, we study the application of a generalized scaling operator to Donsker’s Delta function in Mittag-Leffler Analysis.

2. The Mittag-Leffler Measure

Let \( d \in \mathbb{N} \) and \( L^2_d := L^2(\mathbb{R}) \otimes \mathbb{R}^d \). The space \( L^2_d \) is unitary isomorphic to a direct sum of \( d \) identical copies of \( L^2 := L^2(\mathbb{R}) \), (i.e., the space of real-valued square integrable measurable functions with Lebesgue measure). Any element \( f \in L^2_d \) may be written in the form

\[
f = (f_1 \otimes e_1, \ldots, f_d \otimes e_d),
\]

where \( f_i \in L^2(\mathbb{R}) \), \( i = 1, \ldots, d \) and \( \{e_1, \ldots, e_d\} \) denotes the canonical basis of \( \mathbb{R}^d \). The inner product in \( L^2_d \) is given by

\[
(f, g)_0 = \sum_{k=1}^d (f_k, g_k)_{L^2} = \sum_{k=1}^d \int_{\mathbb{R}} f_k(x) g_k(x) \, dx,
\]

where \( g = (g_1 \otimes e_1, \ldots, g_d \otimes e_d) \), \( f_k \in L^2 \), \( k = 1, \ldots, d \), \( f \) as given in (2.1). The corresponding norm in \( L^2_d \) is given by

\[
|f|_0^2 := \sum_{k=1}^d |f_k|_{L^2}^2 = \sum_{k=1}^d \int_{\mathbb{R}} f_k^2(x) \, dx.
\]

As a densely embedded nuclear Fréchet space in \( L^2_d \) we choose \( S_d := S(\mathbb{R}) \otimes \mathbb{R}^d \), where \( S(\mathbb{R}) \) is the Schwartz test function space. An element \( \varphi \in S_d \) has the form

\[
\varphi = (\varphi_1 \otimes e_1, \ldots, \varphi_d \otimes e_d),
\]

where \( \varphi_i \in S(\mathbb{R}) \), \( i = 1, \ldots, d \). Together with the dual space \( S'_d := S'(\mathbb{R}) \otimes \mathbb{R}^d \) we obtain the basic Gel’fand triple

\[
S_d \subset L^2_d \subset S'_d.
\]

The dual pairing between \( S'_d \) and \( S_d \) is given as an extension of the scalar product in \( L^2_d \) by

\[
(f, \varphi)_0 = \sum_{k=1}^d (f_k, \varphi_k)_{L^2},
\]

where \( f \) and \( \varphi \) as in (2.1) and (2.2), respectively. In \( S'_d \) we choose the Borel \( \sigma \)-algebra \( \mathcal{B} \) generated by the cylinder sets.

Define the operator \( M^\alpha_2 \) on \( S(\mathbb{R}) \) by

\[
M^\alpha_2 \varphi := \begin{cases} 
K_2 D_-^{-(\alpha-1)/2} \varphi, & \alpha \in (0, 1), \\
\varphi, & \alpha = 1, \\
K_2 I^{(\alpha-1)/2} \varphi, & \alpha \in (1, 2),
\end{cases}
\]
where the normalization constant $K_{\frac{\alpha}{2}} := \sqrt{\alpha \sin(\frac{\pi \alpha}{2})} \Gamma(\alpha)$ and $D^r_\alpha, I^r_\alpha$ denote the left-side fractional derivative and fractional integral of order $r$ in the sense of Riemann-Liouville, respectively:

\[
(D^r_\alpha f)(x) = \frac{-1}{\Gamma(1-r)} \frac{d}{dx} \int_x^\infty f(t) (t-x)^{-r} \, dt,
\]

\[
(I^r_\alpha f)(x) = \frac{1}{\Gamma(r)} \int_x^\infty f(t) (t-x)^{r-1} \, dt, \quad x \in \mathbb{R}.
\]

We refer to [25] or [12] for the details on these operators. It is possible to obtain a larger domain of the operator $M_{\alpha, \beta}$ in order to include the indicator function $\mathbb{1}_{[0,t]}$ such that $M_{\alpha, \beta} \mathbb{1}_{[0,t]} \in L^2$, for the details we refer to Appendix A in [5]. We have the following

**Proposition 2.1** (Corollary 3.5 in [5]). For all $t, s \geq 0$ and all $0 < \alpha < 2$ it holds that

\[
(M_{\alpha, \beta} \mathbb{1}_{[0,t]}, M_{\alpha, \beta} \mathbb{1}_{[0,s]})_{L^2} = \frac{1}{2} (t^\alpha + s^\alpha - |t-s|^\alpha).
\]  

(2.3)

Note that this coincides with the covariance of the fractional Brownian motion with Hurst parameter $H = \frac{\alpha}{2}$.

In order to construct ggBm we will use the Mittag-Leffler function which is introduced by G. Mittag-Leffler in a series of papers [18, 19, 20].

**Definition 2.1** (Mittag-Leffler function). For $\beta > 0$ the Mittag-Leffler function $E_\beta$ is defined as an entire function by the following series representation

\[
E_\beta(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)}, \quad z \in \mathbb{C}.
\]  

(2.4)

Here $\Gamma$ denotes the well-known Gamma function which is an extension of the factorial to complex numbers such that $\Gamma(n + 1) = n!$ for $n \in \mathbb{N}$.

Note that for $\beta = 1$, the Mittag-Leffler function coincides with the classical exponential function, i.e., $E_1(z) = e^z$ for all $z \in \mathbb{C}$. We also consider the so-called the $M$-Wright function $M_\beta$ for $0 < \beta \leq 1$ (in one variable) where its series expansion is given by

\[
M_\beta(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \Gamma(-\beta n + 1 - \beta)}.
\]

For the choice $\beta = \frac{1}{2}$ the corresponding $M$-Wright function reduces to the Gaussian density

\[
M_{\frac{1}{2}}(z) = \frac{1}{\sqrt{\pi}} \exp \left(-\frac{z^2}{4}\right).
\]  

(2.5)

The Mittag-Leffler function $E_\beta$ and the $M$-Wright are related through the Laplace transform

\[
\int_0^\infty e^{-s\tau} M_\beta(\tau) \, d\tau = E_\beta(-s).
\]  

(2.6)

The Mittag-Leffler measures $\mu_\beta, 0 < \beta \leq 1$ is a family of probability measures on $S_d'$ whose characteristic functions are given by Mittag-Leffler functions, see Definition 2.1. Using the Bochner-Minlos theorem, see [3] or [9], we obtain the following definition.

**Definition 2.2** (cf. [4]). For any $\beta \in (0, 1]$ the Mittag-Leffler measure is defined as the unique probability measure $\mu_\beta$ on $S_d'$ by fixing its characteristic functional

\[
\int_{S_d'} e^{i(w, \varphi)} \, d\mu_\beta(w) = E_\beta \left(-\frac{1}{2} |\varphi|^2\right), \quad \varphi \in S_d.
\]  

(2.7)
Remark 2.1.  
1. The measure \( \mu_\beta \) is also called grey noise (reference) measure, cf. \([4]\) and \([5]\).  
2. The range \( 0 < \beta \leq 1 \) ensures the complete monotonicity of \( E_\beta(-x) \), see Pollard \([23]\), i.e., \((-1)^nE_\beta^{(n)}(-x) \geq 0 \) for all \( x \geq 0 \) and \( n \in \mathbb{N}_0 := \{0, 1, 2, \ldots \} \). In other words, it is sufficient to show that  
\[
S_d \ni \varphi \mapsto E_\beta \left( -\frac{1}{2} |\varphi|_0^2 \right) \in \mathbb{R}
\]
is a characteristic function in \( S_d \).

By \( L^2(\mu_\beta) := L^2(S'_d, \mathcal{B}, \mu_\beta) \) we denote the complex Hilbert space of square integrable measurable functions defined on \( S'_d \) with scalar product  
\[
\langle F, G \rangle_{L^2(\mu_\beta)} := \int_{S'_d} F(w)\overline{G}(w) \, d\mu_\beta(w), \quad F, G \in L^2(\mu_\beta).
\]
The corresponding norm is denoted by \( \| \cdot \|_{L^2(\mu_\beta)} \). It follows from \((2.7)\) that all moments of \( \mu_\beta \) exists and we have

Lemma 2.1. For any \( \varphi \in S_d \) and \( n \in \mathbb{N}_0 \) we have  
\[
\int_{S'_d} \langle w, \varphi \rangle_0^{2n+1} \, d\mu_\beta(w) = 0,
\]
\[
\int_{S'_d} \langle w, \varphi \rangle_0^{2n} \, d\mu_\beta(w) = \frac{(2n)!}{2^n\Gamma(n+1)} |\varphi|_0^{2n}.
\]
In particular, \( \| \cdot \|_{L^2(\mu_\beta)}^2 = \frac{1}{\Gamma(\beta+1)} |\varphi|_0^2 \) and by polarization for any \( \varphi, \psi \in S_d \) we obtain  
\[
\int_{S'_d} \langle w, \varphi \rangle_0 \langle w, \psi \rangle_0 \, d\mu_\beta(w) = \frac{1}{\Gamma(\beta+1)} \langle \varphi, \psi \rangle_0.
\]

3. Generalized grey Brownian motion in dimension d

For any test function \( \varphi \in S_d \) we define the random variable  
\[
X^{\beta}(\varphi) : S'_d \to \mathbb{R}^d, \; w \mapsto X^{\beta}(\varphi, w) := (\langle w_1, \varphi_1 \rangle, \ldots, \langle w_d, \varphi_d \rangle).
\]
As a consequence of Lemma 2.1 and the characteristic function of \( \mu_\beta \) which is given in \((2.7)\), the random variable \( X^{\beta}(\varphi) \) possess the following properties:

Proposition 3.1. Let \( \varphi, \psi \in S_d, \; k \in \mathbb{R}^d \) be given. Then  
1. The characteristic function of \( X^{\beta}(\varphi) \) is given by  
\[
\mathbb{E}(e^{i(k, X^{\beta}(\varphi))}) = E_\beta \left( -\frac{1}{2} \sum_{j=1}^d k_j^2 |\varphi_j|_{L^2}^2 \right).
\]  
2. The characteristic function of the random variable \( X^{\beta}(\varphi) - X^{\beta}(\psi) \) is  
\[
\mathbb{E}(e^{i(k, X^{\beta}(\varphi) - X^{\beta}(\psi))}) = E_\beta \left( -\frac{1}{2} \sum_{i=1}^d k_i^2 |\varphi_i - \psi_i|_{L^2}^2 \right).
\]  
3. The expectation of the \( X^{\beta}(\varphi) \) is zero and  
\[
\|X^{\beta}(\varphi)\|_{L^2(\mu_\beta)}^2 = \frac{1}{\Gamma(\beta+1)} |\varphi|_0^2.
\]
(4) The moments of \( X^\beta(\varphi) \) are given by
\[
\int_{S_d} \left| X^\beta(\varphi, w) \right|^{2n+1} d\mu_\beta(w) = 0,
\]
and
\[
\int_{S_d} \left| X^\beta(\varphi, w) \right|^{2n} d\mu_\beta(w) = \frac{(2n)!}{2^n\Gamma(\beta n + 1)} |\varphi|^{2n}.
\]

**Remark 3.1.**

(1) The property (3.10) of \( X^\beta(\varphi) \) gives the possibility to extend the definition of \( X^\beta \) to any element in \( L^2_\beta \), in fact, if \( f \in L^2_\beta \), then there exists a sequence \( (\varphi_k)_{k=1}^\infty \subset S_d \) such that \( \varphi_k \to f, k \to \infty \) in the norm of \( L^2_\beta \). Hence, the sequence \( (X^\beta(\varphi_k))_{k=1}^\infty \subset L^2(\mu_\beta) \) forms a Cauchy sequence which converges to an element denoted by \( X^\beta(f) \in L^2(\mu_\beta) \).

(2) For \( \beta = 1 \) property (3.10) yields the Itô isometry.

We define \( \mathbb{I}_{[0,t]} \in L^2_\beta, t \geq 0 \), by
\[
\mathbb{I}_{[0,t]} := (\mathbb{I}_{[0,t]} \times e_1, \ldots, \mathbb{I}_{[0,t]} \times e_d)
\]
and consider the process \( X^\beta(\mathbb{I}_{[0,t]}) \in L^2(\mu_\beta) \) such that the following definition makes sense.

**Definition 3.1.** For any \( 0 < \alpha < 2 \) we define the process
\[
S_d \ni w \mapsto B^{\beta,\alpha}(t, w) := \langle (w, (M^2 \mathbb{I}_{[0,t]} \times e_1), \ldots, (w, (M^2 \mathbb{I}_{[0,t]} \times e_d))
\]
\[
= \langle (\mathbb{I}_{[0,t]} \times M^2 \mathbb{I}_{[0,t]}), \ldots, (\mathbb{I}_{[0,t]} \times M^2 \mathbb{I}_{[0,t]}), \rangle \rangle, t > 0 \quad (3.11)
\]
as an element in \( L^2(\mu_\beta) \). This process is called a version of \( d \)-dimensional generalized grey Brownian motion (ggBm). Its characteristic function has the form
\[
\mathbb{E}(e^{i(k, B^{\beta,\alpha}(t))}) = E_\beta \left( -\frac{|k|^2}{2} t^\alpha \right), k \in \mathbb{R}^d. \quad (3.12)
\]

**Remark 3.2.**

(1) By Remark 3.1 the \( d \)-dimensional ggBm exist as a \( L^2(\mu_\beta) \)-limit and hence the map \( S_d \ni w \mapsto \langle w, \mathbb{I}_{[0,t]} \rangle \) yields a version of ggBm, \( \mu_\beta - a.s. \), but not in the pathwise sense.

(2) For a fixed \( 0 < \alpha < 2 \) one can show by the Kolmogorov-Centsov continuity theorem that the paths of the process are \( \mu_\beta - a.s. \) continuous.

**Proposition 3.2.** For any \( 0 < \alpha < 2 \), the process \( B^{\beta,\alpha} := \{ B^{\beta,\alpha}(t), t \geq 0 \} \), is \( \frac{\alpha}{2} \) self-similar with stationary increments.

**Remark 3.3.** The family \( \{ B^{\beta,\alpha}(t), t \geq 0, \beta \in (0,1], \alpha \in (0,2) \} \) forms a class of \( \frac{\alpha}{2} \) self-similar process with stationary increments (\( \frac{\alpha}{2} \)-sssi) which includes:

(1) For \( \beta = \alpha = 1 \), the process \( \{ B^{1,1}(t), t \geq 0 \} \) is a standard \( d \)-dimensional Brownian motion.

(2) For \( \beta = 1 \) and \( 0 < \alpha < 2 \), \( \{ B^{1,\alpha}(t), t \geq 0 \} \) is a \( d \)-dimensional fractional Brownian motion with Hurst parameter \( \frac{\alpha}{2} \).

(3) For \( \alpha = 1 \), \( \{ B^{1,1}(t), t \geq 0 \} \) is \( \frac{1}{2} \) self-similar non Gaussian process with
\[
\mathbb{E}(e^{i(k, B^{1,1}(t))}) = E_\beta \left( -\frac{|k|^2}{2} t^\alpha \right), k \in \mathbb{R}^d. \quad (3.13)
\]

(4) For \( 0 < \alpha = \beta < 1 \), the process \( \{ B^\beta(t) := B^{\beta,\beta}(t), t \geq 0 \} \) is \( \frac{\beta}{2} \) self-similar and is called \( d \)-dimensional grey Brownian motion (gBm for short). Its characteristic function is given by
\[
\mathbb{E}(e^{i(k, B^\beta(t))}) = E_\beta \left( -\frac{|k|^2}{2} t^\beta \right), k \in \mathbb{R}^d. \quad (3.14)
\]
For $d = 1$, gBm was introduced by W. Schneider in [26, 27].

4. DISTRIBUTIONS AND CHARACTERIZATION THEOREMS

With the help of the Appell systems, a test function and a distribution space in non-Gaussian analysis can be constructed, the details of this construction can be found in [16], [4], [5] and references therein. In between the many choices of triples which can be constructed, we choose the Kondratiev triple

$$(S_d)_{\mu,\beta}^1 \subset (H_p)_{\mu,\beta}^1 \subset L^2(\mu,\beta) \subset (H_{-p})_{\mu,\beta}^{-1} \subset (S_d)_{\mu,\beta}^{-1}.$$ 

The space $(H_p)_{\mu,\beta}^1$ is defined as the completion of the $\mathcal{P}(S_d')$ (the space of smooth polynomials on $S_d'$) w.r.t. the norm $\| \cdot \|_{p,q,\mu,\beta}$ given by

$$\|\varphi\|_{p,q,\mu,\beta}^2 := \sum_{n=0}^{\infty} (\beta e)^{nq} |\varphi^{(n)}|^2_p, \quad p, q \in \mathbb{N}_0, \quad \varphi \in \mathcal{P}(S_d').$$

The dual space $(H_{-p})_{\mu,\beta}^{-1}$ is a subset of $\mathcal{P}^*(S_d')$ such that if $\Phi \in (H_{-p})_{\mu,\beta}^{-1}$, then

$$\|\Phi\|_{p-q,\mu,\beta}^2 := \sum_{n=0}^{\infty} 2^{-nq} |\Phi^{(n)}|^2_p < \infty, \quad p, q \in \mathbb{N}_0.$$ 

The dual pairing between $(S_d')_{\mu,\beta}^{-1}$ and $(S_d)_{\mu,\beta}^1$, denoted by $\langle \cdot, \cdot \rangle_{\mu,\beta}$ is a bilinear extension of scalar product in $L^2(\mu,\beta)$. For any $\varphi \in (S_d)_{\mu,\beta}^1$ and $\Phi \in (S_d')_{\mu,\beta}^{-1}$ we have

$$\langle \Phi, \varphi \rangle_{\mu,\beta} = \sum_{n=0}^{\infty} n! \Phi^{(n)}(\varphi) \varphi^{(n)}.$$ 

The set of $\mu,\beta$-exponentials

$$\left\{ e_{\mu,\beta}(\cdot, \varphi) := \frac{e^{(\cdot, \varphi)}}{E(e^{(\cdot, \varphi)}),} \quad \varphi \in S_{d,\mathbb{C}}, \quad |\varphi|_p < 2^{-q} \right\}$$

forms a total set in $(H_p)_{\mu,\beta}^1$ and for any $\varphi \in S_{d,\mathbb{C}}$ such that $|\varphi|_p < 2^{-q}$ we have $\|e_{\mu,\beta}(\cdot, \varphi)\|_{p,q,\mu,\beta} < \infty$.

Let us introduce an integral transform, the $S_{\mu,\beta}$-transform, which is used to characterize the spaces $(S_d)_{\mu,\beta}^1$ and $(S_d)_{\mu,\beta}^{-1}$. For any $\Phi \in (S_d)_{\mu,\beta}^{-1}$ and $\varphi \in U \subset S_{d,\mathbb{C}}$, where $U$ is a suitable neighborhood of zero, we define

$$S_{\mu,\beta}(\Phi)(\varphi) := \frac{\langle \Phi, e^{(\cdot, \varphi)} \rangle_{\mu,\beta}}{E(e^{(\cdot, \varphi)})} = \frac{1}{E_{\beta\left(\frac{1}{2}(\cdot, \varphi)\right)}} \langle \Phi, e^{(\cdot, \varphi)} \rangle_{\mu,\beta}.$$ 

The characterization theorem for the space $(S_d)_{\mu,\beta}^{-1}$ via the $S_{\mu,\beta}$-transform is done using the spaces of holomorphic functions on $S_{d,\mathbb{C}}$. We denote by $\text{Hol}_0(S_{d,\mathbb{C}})$ the space of holomorphic functions at zero where we identify two functions which coincides in a neighborhood of zero. The space $\text{Hol}_0(S_{d,\mathbb{C}})$ is given as the inductive limit of a family of normed spaces, see [16] for the details and the proof of the following characterization theorem.

**Theorem 4.1** (cf. [16, Theorem 8.34]). The $S_{\mu,\beta}$-transform is a topological isomorphism from $(S_d)_{\mu,\beta}^{-1}$ to $\text{Hol}_0(S_{d,\mathbb{C}})$.

As a corollary from the characterization theorem the following integration result can be deduced.

**Theorem 4.2.** Let $(T, B, \nu)$ be a measure space and $\Phi_t \in (S_d)_{\mu,\beta}^{-1}$ for all $t \in T$. Let $U \subset S_{d,\mathbb{C}}$ be an appropriate neighbourhood of zero and $0 < C < \infty$, such that
(1) $S_{\mu_\beta}^\varphi(t) : T \to \mathbb{C}$ is measurable for all $\xi \in \mathcal{U}$.
(2) $\int_T |S_{\mu_\beta}^\varphi(t)| d\nu(t) \leq C$ for all $\xi \in \mathcal{U}$.

Then, there exists $\Phi \in (S_d)^{-1}$ such that for all $\xi \in \mathcal{U}$

$$S_{\mu_\beta}^\varphi(t) = \int_T S_{\mu_\beta}^\varphi(t) d\nu(t).$$

We denote $\Psi$ by $\int_T \Phi_i d\nu(t)$ and call it the weak integral of $\Phi$.

In the following we will use the $T_{\mu_\beta}$-transform which is defined as follows.

**Lemma 4.1.** Let $\Phi \in (S_d)^{-1}$ and $p, q \in \mathbb{N}$ such that $\Phi \in (H_-)^{-1, \mu_\beta}$. Then, the $T_{\mu_\beta}$-transform given by

$$T_{\mu_\beta}^\varphi = \int \exp(i\langle \cdot, \varphi \rangle)$$

is well-defined for $\varphi \in U_{p,q}$ and we have

$$T_{\mu_\beta}^\varphi = E_{\beta} \left( -\frac{1}{2} \langle \varphi, \varphi \rangle \right) S_{\mu_\beta}^\varphi.$$

In particular, $T_{\mu_\beta}^\varphi \in \text{Hol}(S_d; \mathbb{C})$ if and only if $S_{\mu_\beta} \subset \text{Hol}(S_d; \mathbb{C})$. Moreover, Theorem 4.2 also holds if the $S_{\mu_\beta}$-transform is replaced by the $T_{\mu_\beta}$-transform.

For details and proofs we refer to [4]. A direct application of the integral theorem is the existence of a generalization of the well-known Donsker’s Delta.

**Theorem 4.3.** [5] Let $0 \neq \eta \in L_2^2$ and $a \in \mathbb{R}$ arbitrary. Then

$$\delta_a(\langle \cdot, \eta \rangle) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp(is(\langle \cdot, \eta \rangle - a)) ds$$

exists in $(S_d)^{-1}$ as a weak integral in the sense of Theorem 4.2 and it is called Donsker’s Delta in $a \in \mathbb{R}$. Its $T_{\mu_\beta}$ transform is given by

$$T_{\mu_\beta}^\delta_a(\langle \cdot, \eta \rangle)(\xi) = \frac{1}{\sqrt{2\pi} \langle \eta, \eta \rangle} \int_0^\infty M_{\beta}(r) r^{-\frac{1}{2}} \exp(-\frac{1}{2} \langle \xi, \xi \rangle - \frac{1}{2} \langle \eta, \eta \rangle) + ia \frac{\langle \xi, \eta \rangle}{\langle \eta, \eta \rangle} - \frac{a^2}{2r\langle \eta, \eta \rangle} dr.$$

5. OPERATORS IN MITTAG-LEFFLER ANALYSIS

5.1. Generalized Mittag-Leffler Kernels. Let $(e_n)_{n \in \mathbb{N}} \subset S_d$ be an orthonormal basis of $L_2^2$. Let $P_{e_n}$ be the bounded linear operator on $L_2^2$ defined by

$$P_{e_n} f = (f, e_n)e_n, \quad \text{with } f \in L_2^2.$$

The projection to the one-dimensional subspace spanned by $e_n$ is a continuous mapping on $S_d$ and can be extended to a continuous mapping on $S_d'$, see e.g. [7].

**Definition 5.1.** Let $K$ be a normal compact operator on $L_2^2$ with eigenvalues $\lambda_n$ and corresponding eigenvectors $(e_n)_{n \in \mathbb{N}} \subset L_2^2$, then we define the quadratic form

$$\langle w, Kw \rangle := \sum_{n=1}^\infty \lambda_n \langle e_n, w \rangle^2.$$

We now define a special class of Mittag-Leffler distributions which are defined by their $T_{\mu_\beta}$-transform, similarly to generalized Gauss kernels in white noise analysis e.g. [7], [10]. Let $B$ be the set of all continuous bilinear mappings $\mathcal{B} : S_d \times S_d \to \mathbb{C}$. Then the functions

$$S_d \ni \xi \mapsto E_{\beta} \left( -\frac{1}{2} \mathcal{B}(\xi, \xi) \right) \in \mathbb{C}$$

are the generalized Mittag-Leffler kernels.
Then we define
\[ \Phi_\mathfrak{B} := T_{\mu_\beta}^{-1} E_\beta \left( -\frac{1}{2} \mathfrak{B}(\cdot,\cdot) \right) \]
are elements of \((S_d)_{\mu_\beta}^{-1}\).

**Definition 5.2.** The set of generalized Mittag-Leffler kernels is defined by
\[ \text{GMLK} := \{ \Phi_\mathfrak{B}, \mathfrak{B} \in \mathcal{B} \}. \]

In case the continuous bilinear form is given via the dual pairing and an operator
\[ B : S_{d,C} \rightarrow S_{d,C}' \] we write \( \Phi_B = T^{-1} E_\beta \left( -\frac{1}{2} (\cdot, B \cdot) \right) \).

**Remark 5.1.** For positive definite operator \( B \), one can also consider \( \Phi_B \) as a generalized version of a Radon-Nikodym derivative. The corresponding measure is then \( \mu_\beta \left( \sqrt{B}^{-1} \right) \), if \( B \) is self-adjoint w.r.t \((\cdot,\cdot)\). In the case the root of the operator does not exists, as in the case of non-positive definite operators \( B \), one can still consider \( \Phi_B \) as a kind of change of measure, although, the "covariance matrix" might not be positive definite.

**Example 5.1.** Let \( B \) be the orthogonal projection on the complement of the subspace spanned by \( \eta \in L^2(\mathbb{R}) \), with \((\eta, \eta) = 1\). Then
\[ T_{\mu_\beta}(\Phi_B)(\xi) = E_\beta \left( -\frac{1}{2} \langle \xi, B \xi \rangle \right) \]
\[ = E_\beta \left( -\frac{1}{2} \langle \xi, \xi - \langle \xi, \eta \rangle \eta \rangle \right) \]
\[ = E_\beta \left( -\frac{1}{2} \langle \xi - \langle \xi, \eta \rangle \eta, \xi - \langle \xi, \eta \rangle \eta \rangle \right) \]
As the \( T_{\mu_\beta} \)-transform of \( \Phi_B \) coincides with the one from the well-known Donsker’s delta function \( \delta_0(\langle \cdot, \eta \rangle) \) for when \( \beta = 0 \), the distributions are the same. Compare also [10]. In the case \( \beta \neq 0 \) this is not the case see e.g. [5].

5.2. **Generalized Scaling Operators.** Pointwise multiplication with a generalized Mittag-Leffler kernel can be considered as a measure transformation. In a view to the previous sections, we want to generalize the notion of scaling to bounded operators. More precisely, we investigate for which kind of linear mappings \( B \in L(S_d', S_d) \) there exists some operator \( \sigma_B : (S_d)_{\mu_\beta}^1 \rightarrow (S_d)_{\mu_\beta}^1 \) such that
\[ \Phi(B B^*) \cdot \varphi := \sigma_B^1 B \cdot \varphi. \]
where \( B^* \) is the dual operator of \( B \) with respect to \( \langle \cdot, \cdot \rangle \). Furthermore we state a generalization of the Wick formula to Mittag-Leffler kernels. We start with the definition of \( \sigma_B \).

**Proposition 5.1.** Let \( B \in L(S_d', S_d) \) and \( \varphi \in (S_d)^1 \) given by its continuous version. Then we define
\[ \sigma_B \varphi(w) = \varphi(Bw), \]
for \( w \in S_d' \).

This can be proved directly by an explicit calculation on the the set of exponentials, a density argument and a verifying of pointwise convergence, compare [22, Proposition 4.6.7, p. 104], for the Gaussian case.

Next we show the continuity of the generalized scaling operator.
Proposition 5.2. Let \( B : S'_d \rightarrow S'_d \) be a bounded operator. For \( \varphi, \psi \in (S_d)^1 \) the following equation holds

\[
\sigma_B(\varphi \psi) = (\sigma_B \varphi)(\sigma_B \psi).
\]

Since we consider a continuous mapping from \((S)^1\) into itself one can define the dual scaling operator with respect to \( \langle \cdot, \cdot \rangle \), \( \sigma_B^1 : (S_d)^{-1} \rightarrow (S_d)^{-1} \) by

\[
\Bigl\langle \bigl\langle \sigma_B^1 \Phi, \psi \bigr\rangle \Bigr\rangle = \Bigl\langle \bigl\langle \Phi, \sigma_B \psi \bigr\rangle \Bigr\rangle.
\]

The Wick formula in White Noise Analysis was stated in [8] for Donsker’s delta function. A similar result is true for Mittag-Leffler-kernels

Proposition 5.3 (Generalized Wick formula). Let \( \Psi, \Xi \in (S_d)^{-1} \), \( \varphi, \psi \in (S_d)^1 \) and \( B \in L(S'_d, S'_d) \). We define

\[
\Psi \circ \Xi := S_{\mu_\beta}^{-1}(S_{\mu_\beta}(\Psi))S_{\mu_\beta}(\Xi).
\]

Then we have

(i) \( S_{\mu_\beta}(\Gamma_B \Psi)(\xi) = S_{\mu_\beta}(\Psi)(B^* \xi), \quad \xi \in S_d. \)

In particular we have

\[
\sigma_B^1 \mathbb{1} = \Phi_{BB^*}.
\]

(ii) \( \Phi_{BB^*} \cdot \varphi = \sigma_B^1(\sigma_B \varphi). \)

(iii) \( \Phi_{BB^*} \cdot \varphi = \Phi_{BB^*} \circ (\Gamma_B \circ \sigma_B(\varphi)). \)

Proof. Proof of (i): Let \( \Psi \in (S_d)^{-1} \) and \( \xi \in S_d \) then we have

\[
S(\sigma_B^1 \Psi)(\xi) = \frac{\langle \sigma_B^1 \Psi, \exp((\cdot, \xi)) \rangle}{E_\beta(\frac{1}{2}(\xi, \xi))} = \frac{\langle \Psi, \sigma_B \exp((\cdot, \xi)) \rangle}{E_\beta(\frac{1}{2}(\xi, \xi))} = \frac{\langle \Psi, \exp((\cdot, B^* \xi)) \rangle}{E_\beta(\frac{1}{2}(\xi, \xi))} = \frac{\langle \Psi, \exp((\cdot, B^* \xi)) \rangle}{E_\beta(\frac{1}{2}(B^* \xi, B^* \xi))} \frac{E_\beta(\frac{1}{2}(B^* \xi, B^* \xi))}{E_\beta(\frac{1}{2}(\xi, \xi))} = S_{\mu_\beta}(\Gamma_B \Psi)(\xi) \cdot S_{\mu_\beta}(\Phi_{BB^*})(\xi)
\]

Proof of (ii): First we have \( \sigma_B^1 \mathbb{1} = \Phi_{BB^*} \circ \Gamma_B \mathbb{1} = \Phi_{BB^*} \).

Thus for all \( \varphi, \psi \in (S_d)^1 \)

\[
\langle (\Phi_{BB^*} \varphi, \psi) \rangle = \langle \sigma_B^1 \mathbb{1}, \varphi \cdot \psi \rangle = \langle \mathbb{1}, (\sigma_B \varphi)(\sigma_B \psi) \rangle = \langle (\sigma_B \varphi), (\sigma_B \psi) \rangle = \langle \sigma_B^1(\sigma_B \varphi), \psi \rangle.
\]

Proof of (iii): Immediate from (i) and (ii).

Remark 5.2. The scaling operator can be considered as a linear measure transform. Let \( \varphi \in (S_d)^1 \) and \( B \) a real bounded operator on \( S'_d \). Then we have

\[
\int_{S'_d} \sigma_B \varphi(w) d\mu(w) = \int_{S'_d} \varphi(Bw) d\mu(w) = \int_{S'_d} \varphi(w) d\mu(B^{-1}w).
\]
Moreover we have
\[ \int_{S_d'} \exp(i\langle w, \xi \rangle) \, d\mu(B^{-1}w) = E_\beta(-\frac{1}{2}\langle B^*\xi, B^*\xi \rangle), \]
which is a characteristic function of a probability measure by the Theorem of Bochner-Minlos-Sazonov. Furthermore
\[ \int_{S_d'} \exp(i\langle \xi, w \rangle) \, d\mu(B^{-1}w) = T_{\mu_B}(\sigma_B^\dagger \mathbb{1})(\xi), \]
such that \( \Phi_{BB^*} \) is represented by the positive measure \( \mu \circ B^{-1} \).

6. Examples for Generalized Scaling

**Example 6.1** (Fractional Scaling). Let \( M_\alpha \) denote the operator defined in Section 2. In this example we use generalized scaling operators to scale the underlying noise and hence the processes up to some certain maximal Hölder continuity. Let \( p \) be a polynomial on \( \mathbb{R} \). Let \( \varphi \in S(\mathbb{R}) \). Then
\[ \sigma_{M_\alpha^{\dagger}} (p(\langle \cdot, \varphi \rangle_0)) = p(\langle \cdot, M_\alpha \varphi \rangle_0) = p(\langle \cdot, M_\alpha \varphi \rangle_0), \]
which is again a polynomial in \((S)^1_{\mu_\beta}\), since \( M_\alpha \) leaves \( S(\mathbb{R}) \) invariant, see e.g. [17]. Consider now the random variable
\[ p(\langle \cdot, \mathbb{1}_{[0,t]} \rangle_0) \in (L^2). \]
Then formally as before
\[ \sigma_{M_\alpha^{\dagger}} (p(\langle \cdot, \mathbb{1}_{[0,t]} \rangle_0)) = p(\langle \cdot, M_\alpha \mathbb{1}_{[0,t]} \rangle_0). \]
Indeed this can be made rigorously if we consider a sequence \( \varphi_n \) converging to \( \mathbb{1}_{[0,t]} \) with \( \varphi_n \in S(\mathbb{R}) \) and the use of the convergence theorem for Mittag-Leffler distributions similarly to the findings in [4].

Let us see what would be the representation with the help Proposition 22:
\[ \sigma_{M_\alpha^{\dagger}} \mathbb{1} = \Phi_{M_\alpha \varphi \mathbb{1}}. \]
Furthermore we have that
\[ T_{\mu_\beta}(\Phi_{M_\alpha^{\dagger}})(\xi) = E_\beta \left( -\frac{1}{2}\langle M_\alpha^\dagger \xi, M_\alpha^\dagger \xi \rangle \right), \]
which is the characteristic function of the generalized grey Mittag-Leffler measure.

**Example 6.2** (Scaling with an Orthogonal Operator). Let \( O \in L(S_d', S_d') \) be an orthogonal operator, i.e.
\[ O^* = O^{-1}. \]
Such operators are for example rotations and reflections. Then we have:
\[ \sigma_O^\dagger \mathbb{1} = \Phi_{OO^* \mathbb{1}}. \]
And
\[ T_{\mu_\beta}(\Phi_{OO^* \mathbb{1}})(\xi) = E_\beta \left( -\frac{1}{2} \langle \xi, \xi \rangle \right). \]
This shows that indeed the characteristic function and hence the measure is invariant under orthogonal transformations.
Example 6.3 (Pointwise Products of GMLK with Donsker’s Delta).
Let \( \varphi = \exp(i\lambda(w, \xi)) \), \( \xi \in \mathcal{S}_d \). Let \( B \in L(S'_d, S'_d) \). Then
\[
\sigma_B^\dagger(\sigma_B \varphi) = \sigma_B^\dagger(\sigma_B \exp(i\lambda(w, \xi))) = \Phi_{BB'} \cdot \exp(i\lambda(w, \xi))
\]
As in Theorem 15, we can define Donsker’s Delta as
\[
\frac{1}{2\pi} \int_{\mathbb{R}} \exp(i\lambda(w, \xi))d\lambda.
\]
Since \( \Phi_{BB'} \) is not dependent on \( \lambda \) we can formally write
\[
\int_{\mathbb{R}} \sigma_B^\dagger(\sigma_B \exp(i\lambda(w, \xi)))d\lambda = \int_{\mathbb{R}} \Phi_{BB'} \cdot \exp(i\lambda(w, \xi))d\lambda = \Phi_{BB'} \cdot \delta(\langle w, \xi \rangle).
\]
On the other hand using Theorem 15 (iii) we have with the same arguments and using Fubini with \( \text{fin}S_d \)
\[
S_{\mu_\beta}(\Phi_{BB'} \cdot \delta(\langle w, \xi \rangle))(f) = \int_{\mathbb{R}} S_{\mu_\beta}(\Phi_{BB'})(f) \cdot S_{\mu_\beta}(\exp(i\lambda(Bw, \xi))(B^*f)d\lambda.
\]
Now we obtain
\[
S_{\mu_\beta}(\Phi_{BB'} \cdot \delta(\langle w, \xi \rangle))(f) = S_{\mu_\beta}(\Phi_{BB'})(f) \int_{\mathbb{R}} S_{\mu_\beta}(\exp(i\lambda(w, B^*\xi))(B^*f)d\lambda
\]
which gives
\[
E_\beta(\langle B^*f, B^*f \rangle) \cdot S_{\mu_\beta}(\delta(\langle w, B^*\xi \rangle))(B^*f)
\]

References


W. Bock: bock@mathematik.uni-kl.de
Department of Mathematics, TU Kaiserslautern (TUK), D-67663 Kaiserslautern, Germany

A. Gumanoy: angelyn.gumanoy@g.usm.edu.ph
Department of Mathematics, MSUIIT, Iligan, The Philippines

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