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Regular Articles

Spectral properties of a class of operator functions with applications to the Moore-Gibson-Thompson equation with memory

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A B S T R A C T

In this study, we present spectral enclosures and accumulation of eigenvalues of a class of operator functions with several unbounded operator coefficients. Our findings have direct relevance to the third-order Moore-Gibson-Thompson equation with memory and additional damping. The new results include sufficient conditions for the accumulation of branches of eigenvalues to the essential spectrum and new spectral enclosures for operator functions with several unbounded operator coefficients. To illustrate the analytical results, we apply the abstract findings to concrete equations of the Moore-Gibson-Thompson type. Additionally, we employ numerical computations to further elucidate the analytical results.

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1. Introduction

The distribution of complex eigenvalues of differential operators in physics has been studied in great detail [25, 39, 7, 5, 36, 8]. Importantly, integro-differential operators with memory terms and additional damping are essential in many areas of physics, including electromagnetics [24, 9, 20], viscoelasticity [40], and modeling of high-intensity ultrasound [32, 31].

In this paper, we consider spectral properties of operator functions related to third-order in-time equations with memory, which can be written in the form

\[ u_{ttt} + d u_{tt} + T_b u_t + a T_1 u - \int_0^t k(t - s) T_c u(s) ds = 0, \]  

(1.1)

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where $T_1$ is a positive self-adjoint operator in a separable Hilbert space $\mathcal{H}$, $a > 0$ and $d \geq 0$ are constants, $k$ is a Laplace transformable function, and $T_b, T_c$ are closed operators. Furthermore, we specialize the general results to the Moore-Gibson-Thompson type of equations, which may include additional damping \cite{10} and space-dependent material properties. The study is part of our interest in the mathematical foundations of acoustic wave propagation and predictive digital twins.

After formally applying the Laplace transform $\hat{f}(\lambda) = \int_0^\infty f(t)e^{-\lambda t} \, dt$ to (1.1) with homogeneous initial conditions, we obtain the operator function

$$T(\lambda) = \lambda^3 + d\lambda^2 + \lambda T_b + aT_1 - \hat{k}(\lambda)T_c$$  

in $\mathcal{H}$. Hence, the dependence on the spectral parameter $\lambda$ in (1.2) is nonlinear. Analysis of integro-differential equations in the form (1.1) can under some conditions be based on the reformulation $w_t = Sw$, where the system operator $S$ is a non-self-adjoint block operator \cite{13,12,26}. However, it is for some classes of operator functions possible to obtain tight spectral enclosures directly from the operator function \cite{35,1,22,21,17}.

Accumulation of eigenvalues to the essential spectrum was for a particular Moore-Gibson-Thompson equation without memory proved in \cite{34,37}. Furthermore, in \cite{28,43} the authors studied wave equations with memory (viscoelastic damping). They derived the essential spectrum of the system operator and studied the asymptotics of branches of eigenvalues. The approach for proving accumulation of eigenvalues in those papers is based on the assumption that the nonlinear eigenvalue problem can be written in the form $Lu = h(\lambda)u$, where $h$ is a given function of the spectral parameter $\lambda$. The solutions of the nonlinear eigenvalue problem are then directly related to the eigenvalues of the operator $L$. However, it is also important to study cases where the nonlinear eigenvalue problem cannot be written in the form $Lu = h(\lambda)u$. Common situations where a different approach is needed are when the material properties are space-dependent and when the equation contains several different differential operators with respect to space. In addition, the closeness of the operator function and the determination of the essential spectrum are more complicated in those cases.

A target application in this paper is the Moore-Gibson-Thompson equation with an exponentially decaying memory term, space-dependent material properties, and possibly additional lower-order (differential) operators with respect to space.

Rational operator functions share spectral properties with block operator matrices \cite{27,4,42,18}. In particular cases, the operator matrix is self-adjoint and the eigenvalues below the essential spectrum are characterized by a min-max principle. Moreover, utilizing the theory for self-adjoint operator functions \cite{23}, conditions have been established for the accumulation of branches of eigenvalues to all poles of a specific rational function with a real spectrum \cite{22}.

However, the theory on self-adjoint operator functions \cite{23} that was applied in \cite{22} is not applicable to our case. Therefore, we base our approach on factorization of bounded holomorphic operator functions that were introduced by M. Krein and H. Langer \cite{30} and further developed by several authors including I. Gohberg, V. I. Matsaev, A. S Markus; see \cite{35} and the references therein.

The abstract factorization results were in \cite{1} applied to a quadratic operator polynomial from elasticity theory. However, the results in \cite{35} are unfortunately difficult to apply to many concrete cases, such as higher-degree operator polynomials or rational functions, since particular properties of the numerical range must be known to prove the accumulation of eigenvalues. Therefore, we used in \cite{21,20} the enclosure of the numerical range introduced in \cite{19} and further developed in \cite{41} to prove the accumulation of eigenvalues for a class of rational functions with particular $\lambda$-dependent complex coefficients.

In this paper, we build on the same abstract results \cite{35,21} but study a class of operator functions with significant additional difficulties and different applications to integro-differential equations in physics.

Throughout this article $\mathcal{L}(\mathcal{H})$ denotes the collection of linear operators in the Hilbert space $\mathcal{H}$ and the essential spectrum of a closed operator $A$ is the set such that $A - \lambda$ is not Fredholm. The domain $\text{dom} \, T$ of
the closed operator function $T : \mathcal{D} \to \mathcal{L}(\mathcal{H})$, with $\mathcal{D} \subset \mathbb{C}$, is in general $\lambda$-dependent. For a closed operator function $T$, the spectrum and the essential are defined as

$$\sigma(T) = \{ \lambda \in \mathcal{D} : 0 \in \sigma(T(\lambda)) \}, \quad \sigma_{\text{ess}}(T) = \{ \lambda \in \mathcal{D} : 0 \in \sigma_{\text{ess}}(T(\lambda)) \}.$$ 

The remainder of this paper is organized as follows. In section 2 we present a characterization of the essential spectrum and an enclosure of the numerical range. In section 3, we prove the accumulation of eigenvalues to the essential spectrum in the case of one rational term. Finally, in section 4 we apply the new results to Moore-Gibson-Thompson equations with memory and illustrate the findings with finite element computations.

2. Spectral properties

Throughout this article, $T : \mathcal{D} \to \mathcal{L}(\mathcal{H})$ is assumed to be defined on the maximal set $\mathcal{D} \subset \mathbb{C}$ and $\dim \mathcal{H} = \infty$, if not otherwise stated. Moreover, $\Omega$ denotes a Lipschitz domain $\Omega \subset \mathbb{R}^n$, and $C^k(\Omega)$ denotes the space of $k$-times continuously differentiable functions.

2.1. The numerical range

Assume that $T : \mathcal{D} \to \mathcal{L}(\mathcal{H})$ can be written in the form (1.2) and that the operator function $T$ is densely defined on its natural domain

$$\text{dom } T = \text{dom } T_1 \cap \text{dom } T_b \cap \text{dom } T_c.$$ 

The numerical range of $T : \mathcal{D} \to \mathcal{L}(\mathcal{H})$ is the set

$$W(T) = \{ \lambda \in \mathcal{D} : \exists u \in \text{dom } T, \|u\| = 1, \text{so that } (T(\lambda)u, u) = 0 \},$$

where $(\cdot, \cdot)$ and $\| \cdot \|$ denote the inner product and norm in $\mathcal{H}$, respectively. Note that the numerical range of an operator (e.g. the system operator that is frequently used in the semigroup approach [34]) is convex, which is insufficient for our purposes. The numerical range of an operator function contains in general several components that can be bounded or unbounded. However, it is difficult to directly determine properties of the numerical range that are important for proving accumulation of eigenvalues, such as the number of separated and bounded components. Therefore, we consider an enclosure of the numerical range of $T$ when the operator $T_1$, $T_b$, and $T_c$ can be written in a particular form.

Note that $\lambda \in W(T)$, where $T$ denotes the operator function (1.2), if it exists a normalized vector $u \in \text{dom } T$ such that

$$t_{\alpha, \beta, \gamma}(\lambda) := \lambda^3 + d\lambda^2 + \lambda\beta_u + a\alpha_u - \tilde{k}(\lambda)\gamma_u = 0,$$ (2.1)

where

$$\alpha_u = (T_1u, u), \quad \beta_u = (T_bu, u), \quad \gamma_u = (T_cu, u).$$

It is important to derive bounds on $\beta_u$ and $\gamma_u$ as functions of $\alpha_u$. Then, we define a set of the form

$$W_{\alpha, \beta, \gamma}(T) := \{ \lambda \in \mathcal{D} : t_{\alpha, \beta, \gamma}(\lambda) = 0, \alpha \in W(T_1), \beta \in \Gamma_\beta(\alpha), \gamma \in \Gamma_\gamma(\alpha) \},$$ (2.2)

which is an enclosure of $W(T)$. In [19,21], we derived enclosures of the numerical range of rational operator functions with only one unbounded operator coefficient. It was then sufficient to derive $\alpha$- independent
$$\Gamma_{\beta, \Gamma_\gamma}. \text{However, those types of bounds are in general too large when the operator function has several unbounded operator coefficients.}$$

The construction below is from [29, Chapter VI]. Assume that $\mathcal{H}$, $\hat{\mathcal{H}}$ are infinite-dimensional separable Hilbert spaces. Let $D : \mathcal{H} \to \hat{\mathcal{H}}$ with domain $\text{dom} \, D$ denote a densely defined closed operator. Then we define the operator $T_L$ for a given bounded operator $L : \hat{\mathcal{H}} \to \hat{\mathcal{H}}$ by

$$(T_L u, v) = (LDu, Dv)_{\hat{\mathcal{H}}}, \quad u \in \text{dom} \, T_L, \quad v \in \text{dom} \, D,$$

(2.3)

where $\text{dom} \, T_L = \{ u \in \text{dom} \, D ; LDu \in \text{dom} \, D^* \}$ is a dense subspace of $\text{dom} \, D$. In the following, we assume that $T_1 = D^* D \geq \alpha_0 > 0$ is self-adjoint.

**Lemma 2.1.** Let $T_L$ be defined as in (2.3) and take $u \in \text{dom} \, T$. Then

$$(T_L u, u) \in (T_1 u, u) W(L).$$

**Proof.** Let $u \in \text{dom} \, T$ then

$$(T_L u, u) = (LDu, Du)_{\hat{\mathcal{H}}} = (Du, Du)_{\hat{\mathcal{H}}} (LDu, Du)_{\hat{\mathcal{H}}} = (T_1 u, u) (LDu, Du)_{\hat{\mathcal{H}}}.$$

The result follows from the definition of $W(L)$. $\Box$

Set

$$T_b := T_B + L_b, \quad T_c := T_C + L_c,$$

(2.4)

where the $T_B, T_C$ are defined in (2.3) and the not necessarily bounded pertubations $L_b, L_c$ satisfies the conditions in Lemma 2.2. In Lemma 2.2 “Conv” denotes the “convex hull”.

**Lemma 2.2.** Assume that the two operators $Y_b := T_1^{-\frac{p}{2}} L_b T_1^{-\frac{p}{2}}$ and $Y_c := T_1^{-\frac{q}{2}} L_c T_1^{-\frac{q}{2}}$ are bounded for given $0 \leq p \leq 1$ and $0 \leq q \leq 1$. Set $\alpha_u = (T_1 u, u)$, and

$\hat{W}(Y_b) := \text{Conv}(\{0\} \cup W(Y_b)), \quad \hat{W}(Y_c) := \text{Conv}(\{0\} \cup W(Y_c)).$

Then

$$(L_b u, u) \in \alpha_u^p \hat{W}(Y_b), \quad (L_c u, u) \in \alpha_u^q \hat{W}(Y_c)$$

for all $u \in \text{dom} \, T_1$.

**Proof.** For $p \in \{0, 1\}$ the result is trivial. Otherwise, assume take $u \in \text{dom} \, (T_1) \subset \text{dom}(T_1^p)$. Then, the spectral resolution [6, page 142] of $T_1$ gives

$$\|T_1^{\frac{p}{2}} u\|^2 = (T_1^{p} u, u) = \int_{\sigma(T_1)} \lambda^p \, d\mu_u(\lambda)$$

and from Jensen’s inequality, we have

$$\|T_1^{\frac{p}{2}} u\|^2 \leq \left( \int_{\sigma(T_1)} \lambda d\mu_u(\lambda) \right)^{\frac{p}{2}} = (T_1 u, u)^p = \|T_1^{\frac{1}{2}} u\|^{2p}.$$  

(2.5)
Set $z_u = T_1^{\frac{p}{2}} u$. Then

$$(L_b u, u) = ((T_1^{\frac{p}{2}} L_b T_1^{\frac{p}{2}}) T_1^{\frac{p}{2}} u, T_1^{\frac{p}{2}} u) = (Y_b z_u, z_u)$$

and

$$(L_b u, u) = (Y_b z_u, z_u) = (z_u, z_u) \frac{(Y_b z_u, z_u)}{(z_u, z_u)} = \|(T_1^{\frac{p}{2}} u)\|^2 \frac{(Y_b z_u, z_u)}{(z_u, z_u)}.$$

From (2.5) we conclude that

$$[0, 1] \ni s := \frac{\|(T_1^{\frac{p}{2}} u)\|^2}{\|(T_1^{\frac{p}{2}} u)\|^2}.$$ 

Hence,

$$(L_b u, u) = \|(T_1^{\frac{p}{2}} u)\|^2 \frac{(Y_b z_u, z_u)}{(z_u, z_u)} s = \|(T_1^{\frac{p}{2}} u)\|^2 \frac{(Y_b z_u, z_u)}{(z_u, z_u)} = s \alpha u p \frac{(Y_b z_u, z_u)}{(z_u, z_u)}.$$

Since

$$\frac{(Y_b z_u, z_u)}{(z_u, z_u)} \in W(Y_b) \quad \text{and} \quad s \in [0, 1]$$

it follows directly that $(L_b u, u) \in \alpha u p \hat{W}(Y_b)$ and $(L_c u, u) \in \alpha q u \hat{W}(Y_c)$ follows by analog reasoning. \(\Box\)

The function (1.2) with (2.4) takes the form

$$T(\lambda) = \lambda^3 + d\lambda^2 + \lambda(T_B + L_b) + aT_1 - \hat{k}(\lambda)(T_C + L_c) \quad (2.6)$$

and an enclosure of the numerical range $W(T)$ is given in Theorem 1.

**Theorem 1.** Let $T$ denote the operator function (2.6) and assume that the conditions in Lemma 2.2 are fulfilled. Define the function

$$t_{\alpha, \beta, \gamma, \hat{\alpha}, \hat{\beta}, \hat{\gamma}}(\lambda) := \lambda^3 + d\lambda^2 + \lambda(\beta \alpha + \hat{\beta} \alpha^p) + a\alpha - \hat{k}(\lambda)(\gamma \alpha + \hat{\gamma} \alpha^q)$$

and set

$$\Gamma := W(T_1) \otimes W(B) \otimes W(C) \otimes \hat{W}(Y_b) \otimes \hat{W}(Y_c) \subset \mathbb{R} \otimes \mathbb{C}^4.$$ 

Then

$$W(T) \subset W_{\alpha, \beta, \gamma, \hat{\alpha}, \hat{\beta}, \hat{\gamma}}(T) := \{ \lambda \in D : t_{\alpha, \beta, \gamma, \hat{\alpha}, \hat{\beta}, \hat{\gamma}}(\lambda) = 0, \{\alpha, \beta, \gamma, \hat{\alpha}, \hat{\beta}, \hat{\gamma}\} \in \Gamma \}.$$ 

**Proof.** Follows directly from Lemma 2.1 and Lemma 2.2. \(\Box\)

Note that the enclosure $W_{\alpha, \beta, \gamma}(T)$ in (2.2) corresponds to $W_{\alpha, \beta, \gamma, 0, 0}(T)$ with $L_b = L_c = 0$. 

Example 1. Applications where Theorem 1 can be used to derive the enclosure (2.2) include analysis of integro-differential equations with additional damping. The Moore-Gibson-Thompson equation with an additional term $u_t$ was studied in [33] and damping terms in elastic systems that relate to $T_1^{1/2}$ have been studied extensively [10]. For example, we can apply the theory to equations of the form

$$u_{ttt} + du_{tt} + (T_1 + T_1^{1/2} + q)u_t + T_1u - c \int_0^t k(t - s)T_1u(s)ds = 0, \tag{2.7}$$

where $q : \Omega \to \mathbb{R}$ is a non-negative continuous function and $T_1$ is the Laplace operator or the biharmonic operator. Hence, the associated operator function (2.6) with e.g. the (Dirichlet) Laplace operator $T_1 = -\Delta \geq 0$ is given by

$$T_B = T_1, \quad L_b = T_1^{1/2} + q, \quad Y_b = I + T_1^{-1/4}qT_1^{-1/4},$$

$$T_C = cT_1, \quad L_c = 0, \quad a = 1.$$ Lemma 2.2 can be used to estimate $(T_bu, u)$ in terms of $(T_1u, u)$. In the example we obtain

$$1 \leq (Y_bu, u) \leq 1 + \frac{\max q}{\sqrt{\alpha_0}}.$$

Hence, the set $\Gamma$ in Theorem 1, with $p = 1/2$, is

$$\Gamma := W(T_1) \otimes \{1\} \otimes \{c\} \otimes \left[1, 1 + \frac{\max q}{\sqrt{\alpha_0}}\right] \otimes \{0\}.$$

2.2. The essential spectrum

In this subsection, we specify conditions such that (1.2) is a closed (closable) operator function and determine the essential spectrum. In the sequel, we will specialize in operator functions relevant to the study of Moore-Gibson-Thompson equations with (smooth) space-dependent material properties and memory terms. The conditions in Theorem 2.3 are then sufficient for our purposes.

Theorem 2.3. Assume that $T_1$ is a positive self-adjoint operator in $\mathcal{H}$ with compact resolvent. Moreover, assume that the closed operators $T_b$ and $T_c$ in the operator function (1.2) can be written in the form $T_b = \varphi_bT_1 + L_b$, $T_c = \varphi_cT_1 + L_c$ for some $\varphi_b, \varphi_c \in C(\bar{\Omega})$ and $\mathcal{C}_{0, \infty}(\bar{\Omega}) = \mathcal{H}$. Furthermore,

$$K_b = T_1^{-\frac{1}{2}}L_bT_1^{-\frac{1}{2}}, \quad K_c = T_1^{-\frac{1}{2}}L_cT_1^{-\frac{1}{2}}. \tag{2.8}$$

are assumed to be compact. Define the set

$$S = \{\lambda \in \mathbb{C} : F(\lambda, x) = 0 \text{ for some } x \in \Omega\}, \quad F(\lambda, x) = \lambda \varphi_b(x) + a - \hat{k}(\lambda)\varphi_c(x).$$

Then,

(i) $T : \mathcal{D} \to \mathcal{L}(\mathcal{H})$ with dom $T_1$ is closable,

(ii) $\sigma_{\text{ess}}(T) = S$,

(iii) Assume that $T$ is self-adjoint. Then $\sigma_r(T) \cap \mathbb{R} = \emptyset$. 

**Proof.** (i) The densely defined operator $T(\lambda)$, $\text{dom} T(\lambda) = \text{dom} T_1$ has the closed adjoint

$$T(\lambda)^* = \tilde{\lambda}^3 + d\tilde{\lambda}^2 + [\tilde{\lambda}\varphi_b + a - \tilde{k}(\lambda)\varphi_c]T_1 + \tilde{\lambda}L_b^* - \tilde{k}(\lambda)L_c^*, $$

where $\text{dom} T(\lambda)^* \supset \text{dom} T_1$. Hence, $T(\lambda)^*$ is densely defined, which implies the claim [38, Theorem VIII.1].

(ii) Define the bounded operator function $\hat{T} : D \to \mathcal{B}(\mathcal{H})$ as

$$\hat{T}(\lambda) := T_1^{-\frac{3}{2}}L_bT_1^{-\frac{1}{2}} = (\lambda^3 + d\lambda^2)T_1^{-1} + F(\lambda, \cdot) + \lambda K_b - \tilde{k}(\lambda)K_c. \quad (2.9)$$

Then, we obtain

$$\sigma_{\text{ess}}(\hat{T}) = \sigma_{\text{ess}}(\hat{T}),$$

where $\hat{T}(\lambda)$ is a Fredholm operator if the multiplication operator $F$ is non-zero for $x \in \bar{\Omega}$.

Take $\epsilon > 0$ and $x_0 \in \Omega$ such that $F(\lambda, x_0) = 0$ and $|F(\lambda, x)| < \epsilon$ for all $x \in \Omega_m$ with $\Omega_m := \{ x \in \bar{\Omega} : |x - x_0| < 1/m \}$ for some positive integer $m$. Let $\{u_m\}$ denote a sequence of orthonormal smooth function with support in $\Omega_m \setminus \Omega_{m+1}$. Then Bessel’s inequality implies $u_m \to 0$ weakly. We have

$$\|F(\lambda, \cdot)u_m\| \leq \epsilon,$$

which implies that $\{u_m\}$ is a singular sequence for the multiplication operator $F$ and we conclude that $\lambda \in \sigma_{\text{ess}}(\hat{T})$, since the set of all $\lambda$ for which there exists a singular sequence is in the set of all $\lambda$ such that $T(\lambda)$ is not Fredholm [15, Section 1.4]. (iii) Take $\lambda \in \sigma_r(\hat{T}) \cap \mathbb{R}$, and set $V := \text{ran} \hat{T}(\lambda)$. Since $\hat{T}$ is self-adjoint it follows from the projection theorem that

$$\{0\} \neq V^\perp = \text{Ker} \hat{T}(\lambda)^* = \text{Ker} \hat{T}(\lambda).$$

Hence, $\lambda$ is an eigenvalue of $\hat{T}$. However, we assumed that $\lambda \in \sigma_r(\hat{T})$ is real and the claim follows since $\lambda \in \sigma_r(\hat{T})$ is not an eigenvalue. □

We will in the remaining part of the article assume that the operator function $T$ is closed.

**Example 2.** Theorem 2.3 can be applied in several different settings and we exemplify with an equation of the form

$$u_{ttt} + du_{tt} - \text{div} \varphi_b \nabla u_t - a\Delta u - \int_0^t k(t - s)\text{div} \varphi_c \nabla u(s)ds = 0, \quad (2.10)$$

for given non-negative $C^1$-functions $\varphi_b$ and $\varphi_c$. Assume that $\Omega \subset \mathbb{R}^n$ is bounded with a Lipschitz continuous boundary $\partial \Omega$. The operator $T_1 : L^2(\Omega) \to L^2(\Omega)$ is taken as the self-adjoint operator whose quadratic form is the closure of

$$t_1[u] = \int_{\Omega} |\nabla u|^2dx$$

with domain $C_0^\infty(\Omega)$, [29, p. 331]. Set

$$T_b = \varphi_b T_1 + L_b \text{ with } L_b u := -\nabla \varphi_b \cdot \nabla u, \quad (2.11)$$

$$\tilde{\lambda}$$
and
\[ T_c = \varphi_c T_1 + L_c \text{ with } L_c u := -\nabla \varphi_c \cdot \nabla u. \] (2.12)

The first order differential operators, \( L_b \) and \( L_c \) satisfy condition (2.8) and it follows from Theorem 2.3 that the essential spectrum of the corresponding closed operator function \( T \) is
\[ \sigma_{\text{ess}}(T) = \{ \lambda \in \mathbb{C} : \lambda \varphi_b(x) + a - \tilde{k}(\lambda)\varphi_c(x) = 0 \text{ for some } x \in \Omega \}. \]

Note that the essential spectrum depends upon the regularity of \( \varphi_b, \varphi_c \); see [11,2] for related results for Helmholtz and Maxwell’s equations.

Several authors have studied the system (2.10) in the case when \( \varphi_c = 1 \) and \( \varphi_b = b \) for some positive constant \( b \) [14,32]. The system is then called non-critical if \( a < bd \) and the system is critical if \( a = bd \) [32]. A straightforward generalization to the case
\[ T_b = \varphi_b T_1 + L_b, \quad T_c = \varphi_c T_1 + L_c, \]
where \( L_b, L_c \) satisfies condition (2.8) and \( \varphi_b \) and \( \varphi_c \) are non-negative \( C^2 \)-functions is to call the system non-critical if \( a < \varphi_b(x)d \) for all \( x \in \bar{\Omega} \) and critical if \( a = \varphi_b(x)d \) for some \( x \).

2.3. The memory term

It is common to model the memory effects in \( k \) with the sum of exponential terms:
\[ k(t) = \sum_{j=1}^{N} \eta_j e^{-\theta_j t}, \quad \tilde{k}(\lambda) := \sum_{j=1}^{N} \frac{\eta_j}{\lambda + \theta_j}. \] (2.13)

Several important spectral properties, such as accumulation of eigenvalues require quite detailed knowledge of \( k \). Therefore, we will assume that \( k \) is a finite sum of exponential terms (2.13) with \( \eta_j > 0 \) and \( 0 < \theta_1 < \theta_2 < \cdots < \theta_N \).

Example 3. Consider the operator function \( T \) associated with (2.10). Let \( \Omega = \{ x \in \mathbb{R}^2 : |x| < 1 \} \) and set
\[ \varphi_b(x) = \frac{\pi}{4} + \arctan(|x|), \quad \varphi_b(\bar{\Omega}) = \left[ \frac{\pi}{4}, \frac{\pi}{2} \right], \] (2.14)
\[ \varphi_c(x) = \frac{3\pi}{16} + \frac{\pi}{16} \cos(\pi|x|), \quad \varphi_c(\bar{\Omega}) = \left[ \frac{\pi}{8}, \frac{\pi}{4} \right]. \] (2.15)

Furthermore, let \( a = 1 \) and \( d = 4/\pi \), which implies \( a = \varphi_b(0)d \). Hence, the system is critical. Let \( T_1 = -\Delta \) denote the Dirichlet Laplace operator on \( \Omega \). Then, \( W(T_1) = [j_{0,1}^2, \infty) \), where \( j_{0,1}^2 \) denotes the first zero of the Bessel function \( J_0 \). For the enclosure of the numerical range, we use that \( T \) is of the form (2.6) with \( L_b = L_c = 0 \). The set \( \Gamma \) in Theorem 1 is then
\[ \Gamma := W(T_1) \otimes \varphi_b(\bar{\Omega}) \otimes \varphi_c(\bar{\Omega}) \otimes \{0\} \otimes \{0\}, \]
with the enclosure \( W_{\alpha,\beta,\gamma}(T) := W_{\alpha,\beta,\gamma,0,0}(T) \).

Fig. 1 illustrates the enclosure of the numerical range for two cases with three rational terms in \( \tilde{k} \).

**Theorem 2.4.** Assume that conditions in Theorem 2.3 are fulfilled.
Case 1. Assume that $F(0, x) > 0$ for all $x \in \Omega$. Then

(i) $\sigma_{\text{ess}}(T) \subset (-\infty, 0)$

(ii) Assume that $\varphi_b(x_0) = 0$, $\varphi_c(x_0) > 0$, and $F(\lambda, x_0) = 0$ for some $x_0 \in \Omega$. Then, $\lambda \in (-\theta_N, 0)$. Moreover, $\varphi_b(x_0) = \varphi_c(x_0) = 0$ implies that $F(\lambda, x_0) \neq 0$.

Case 2. Assume that $F(0, x_0) \leq 0$ for some $x_0 \in \Omega$. Then $\sigma_{\text{ess}}(T)$ contains points with non-negative real part.

**Proof.** Case 1. Note that $F(\lambda, x_0) \prod_{j=1}^{N} (\lambda + \theta_j)$ for a fixed $x_0 \in \Omega$ is a $(N + 1)$-th degree polynomial in $\lambda$. Hence, $F(\lambda, x_0) = 0$ has at most $N + 1$ solutions. By assumption is $F(0, x_0) > 0$, $F(\lambda, x_0) \to +\infty$, $\lambda \to -\theta_j^+$, and $F(\lambda, x_0) \to -\infty$, $\lambda \to -\theta_j^−$. Moreover, $F(\lambda, x_0) \to \pm\infty$, $\lambda \to \pm\infty$. Hence, $F(\lambda, x_0) = 0$ has $N + 1$ real solutions in $(-\infty, 0)$.

(ii) $F(\lambda, x_0) \prod_{j=1}^{N} (\lambda + \theta_j)$ is an $N$-th degree polynomial in $\lambda$ and $F(\lambda, x_0) = 0$ has $N$ real solutions in $(-\theta_N, 0)$.

Case 2. $F(\lambda, x_0) \to \infty$, $\lambda \to \infty$ and the claims follow from the condition at $\lambda = 0$. □

**Theorem 2.5.** Assume $\alpha \geq \alpha_0 > 0$ and that the function $t_{\alpha, \beta, \gamma}$ in (2.1) has real coefficients with $\hat{k}$ rational (2.13). Then

(i) $W_{\alpha, \beta, \gamma}(T)$ is symmetric with respect to $\mathbb{R}$

(ii)

$$0 \in W_{\alpha, \beta, \gamma}(T) \iff a - c \sum_{j=1}^{N} \frac{\eta_j}{\theta_j} = 0 \text{ for some } c \in [\min \varphi_c, \max \varphi_c].$$

(iii) Assume that

$$a \neq c \sum_{j=1}^{N} \frac{\eta_j}{\theta_j} \text{ for all } c \in [\min \varphi_c, \max \varphi_c].$$

Then, $W_{\alpha, \beta, \gamma}(T) \cap i\mathbb{R} = \emptyset$ if $a < bd$ for all $b \in [\min \varphi_b, \max \varphi_b]$ or $a = bd$, $\min \varphi_c > 0$, and $T_1$ is bounded.

(iv) Assume that $\min \varphi_c = 0$ and $a/d \in \varphi_b(\bar{\Omega})$. Then

$$W_{\alpha, \beta, \gamma}(T) \cap i\mathbb{R} \neq \emptyset.$$

**Proof.** (i) follows directly since $\prod_{j=1}^{N} (\lambda + \theta_j)t_{\alpha, \beta, \gamma}(\lambda)$ is a polynomial with real coefficients. Set $\lambda = i\tilde{y}$, $\tilde{y} \in \mathbb{R}$. Then $t_{\alpha, \beta, \gamma}(i\tilde{y}) = 0$ if and only if
\[ \alpha a - \hat{y}^2 d - \alpha c \sum_{j=1}^{N} \frac{\eta_j \theta_j}{y_j^2 + \theta_j^2} = 0 \]  
(2.16)

and

\[ -\hat{y}^3 + aby + \hat{y}\alpha c \sum_{j=1}^{N} \frac{\eta_j}{y_j^2 + \theta_j^2} = 0. \]  
(2.17)

Claim (ii) follows directly from those equations. Assume \( y \neq 0 \). By combining the equations (2.16) and (2.17) we obtain the condition

\[ (a - bd) - dc \sum_{j=1}^{N} \frac{\eta_j}{y_j^2 + \theta_j^2} - c \sum_{j=1}^{N} \frac{\eta_j \theta_j}{y_j^2 + \theta_j^2} = 0, \]

which implies (iii). To prove (iv), we let \( \gamma = 0 \) and \( \beta = \alpha a/d \). Then

\[ t_{\alpha, \beta, 0}(\lambda) := \lambda^3 + d\lambda^2 + \lambda \alpha \frac{a}{d} + a\alpha = (\lambda^2 + \alpha \frac{a}{d})(\lambda + d) \]

Hence, \( t_{\alpha, \beta, 0}(\lambda) = 0 \) when

\[ \lambda = -d, \quad \lambda = \pm i \sqrt{\frac{a}{d}}, \quad \alpha \in W(T_1). \]

Example 4. Let

\[ k(\lambda) = \frac{\eta}{\lambda + \theta}, \quad F(\lambda, x) = \lambda \varphi_b(x) + a - \varphi_c(x)k(\lambda). \]

Then, \( F(0, x) = a - \varphi_c(x)\eta/\theta \) and Theorem 2.4 implies that the essential spectrum is contained in \((\infty, 0)\) if \( a > \varphi_c(x)\eta/\theta \) for all \( x \in \Omega \). Assume that \( \varphi_c(x) = 0 \) if and only if \( x \in \Omega_c \subset \Omega \), let \( \Omega_0 = \Omega \setminus \Omega_c \), and assume that \( \min \varphi_b > 0 \). Then

\[ \sigma_{\text{ess}}(T) = \{\lambda_c(\Omega_c), \lambda_+(\Omega_0), \lambda_-(\Omega_0)\}, \]

where

\[ \lambda_c(x) = -\frac{a}{\varphi_b(x)}, \quad \lambda_\pm(x) = -\frac{a + \varphi_b(x) \theta \pm \sqrt{(a - \varphi_b(x) \theta)^2 + 4\varphi_b(x)\varphi_c(x)}}{2\varphi_b(x)}. \]

A frequently studied case is when \( T_b = bT_1 \) and \( T_c = cT_1 \) for some constants \( b \) and \( c \) [14,32]. Define for \( \lambda \in \mathbb{C} \setminus \{-\theta_1, \ldots, -\theta_N\} \) the closed operator \( T(\lambda) : \mathcal{H} \to \mathcal{H} \) as

\[ T(\lambda) = f(\lambda) + g(\lambda)T_1, \quad \text{dom} \ T(\lambda) = \begin{cases} \mathcal{H} & \text{if } g(\lambda) = 0, \\ \text{dom} T_1 & \text{otherwise}, \end{cases} \]  
(2.18)

where \( f(\lambda) = \lambda^3 + d\lambda^2 \) and \( g(\lambda) = a + b\lambda - \hat{k}(\lambda)c \).

Let \( W_\alpha(T) \) denote the enclosure of the numerical range, which is given by the zeros of

\[ t_\alpha(\lambda) := \lambda^3 + d\lambda^2 + F_\alpha(\lambda), \quad F_\alpha(\lambda) := \lambda b\alpha + a\alpha - \hat{k}(\lambda)c\alpha. \]
That is

$$W_\alpha(T) := \{ \lambda \in \mathcal{D} : t_\alpha(\lambda) = 0, \alpha \in W(T_1) \}.$$  \hfill (2.19)

The function $F_\alpha$ is the $x$-independent version of the function in Theorem 2.3. Set $\lambda = \tilde{x} + i\tilde{y}$, $\tilde{x}, \tilde{y} \in \mathbb{R}$. Then $t_\alpha(\tilde{x} + i\tilde{y}) = 0$ if and only if

$$\tilde{x}^3 - 3\tilde{x}\tilde{y}^2 + d\tilde{x}^2 - dy^2 + ba\tilde{x} + ac \sum_{j=1}^{N} \frac{\eta_j (\tilde{x} + \theta_j)}{\tilde{y}^2 + (\tilde{x} + \theta_j)^2} = 0 \quad (2.20)$$

and

$$3\tilde{x}^2\tilde{y} - \tilde{y}^3 + 2\tilde{x}\tilde{y} + \alpha b\tilde{y} + \tilde{y}ac \sum_{j=1}^{N} \frac{\eta_j}{\tilde{y}^2 + (\tilde{x} + \theta_j)^2} = 0.$$ \hfill (2.21)

**Lemma 2.6.** The following enclosures hold

(i) $S_1 \cap W_\alpha(T) = \emptyset$, where

$$S_1 = \{ z = x + iy : x \in (-\infty, -\frac{2}{3}) \cup (0, \infty), y \in (-\sqrt{\alpha_0 b}, 0) \cup (0, \sqrt{\alpha_0 b}) \}.$$

(ii) Assume that $\alpha_0 b > 1/3$. Then $S_2 \cap W_\alpha(T) = \emptyset$, where

$$S_2 = \{ z = x + iy : x \in (-\infty, \infty), y \in (-\sqrt{\alpha_0 b - \frac{1}{3}}, 0) \cup (0, \sqrt{\alpha_0 b - \frac{1}{3}}) \}.$$

**Proof.** (i) Assume that $x(2 + 3x) > 0$ and $0 < y < \sqrt{\alpha_0 b}$. Then the left hand side of (2.21) is positive and the statement follows then from Theorem 2.3 (i). The proof of the second part (ii) is similar to (i). \hfill \Box

We characterize below the behavior of the zeros of $t_\alpha$ when $\alpha \to \infty$.

**Theorem 2.7.** The function $t_\alpha$ has in the limit $\alpha \to \infty$, $N + 1$ real zeros $\lambda_1, \lambda_2, \ldots, \lambda_{N+1}$ and two non-real zeros that approach $\tilde{x}_0 \pm i\infty$ with

$$\tilde{x}_0 = -\frac{1}{2}(d + \theta_1 + \cdots + \theta_N + \lambda_1 + \cdots + \lambda_{N+1}). \quad (2.22)$$

**Proof.** Let $p_\alpha$ denote the $(N + 3)$-th degree polynomial

$$p_\alpha(\lambda) := \prod_{j=1}^{N} (\lambda + \theta_j) t_\alpha(\lambda) \quad (2.23)$$

The leading terms in $p_\alpha$ are

$$\lambda^{N+3} + \lambda^{N+2}(d + \theta_1 + \cdots + \theta_N) + \cdots$$

We will use that the zeros of a monic polynomial depend continuously on its coefficients. Theorem 2.3 shows that $\sigma_{\text{ess}}(T)$ is given by the zeros of $F_\alpha$ and $\sigma_{\text{ess}}(T) \subset W_\alpha(T)$. Moreover, Theorem 2.4 shows that $F_\alpha$ has $N + 1$ real zeros, $\lambda_1, \lambda_2, \ldots, \lambda_{N+1}$. Hence, the remaining two zeros can be written in the form $\tilde{x}_0 \pm i\tilde{y}_0$ and (2.22) follows from Vieta’s formulas. \hfill \Box
In the following theorem, we state sufficient conditions for \( W(T) \subset \mathbb{C}_- \) also in the critical case \( a = bd \).

**Theorem 2.8.** Assume that \( c > 0, a > c \sum_{j=1}^{N} \frac{y_j}{b_j}, a \leq bd, \text{ and } x_0 < 0. \) Then
\[
W_\alpha(T) \subset \mathbb{C}_-.
\]

**Proof.** Theorem 2.3 and Theorem 2.7 implies that in the limit \( \alpha \to \infty \) all zeros are in \( \mathbb{C}_- \). Moreover, from Theorem 2.5 follows that no zeros are on the imaginary axis. The conclusion follows from the continuity of the zeros of (2.23) on its coefficients. \( \square \)

3. Accumulation of eigenvalues

The proof of accumulation of eigenvalues is based on analyzing a bounded operator polynomial that is related to the unbounded rational operator function \( T \). Below, we state known results [35] for bounded operator polynomials on \( \mathcal{H} \) in the form
\[
P(\lambda) := \sum_{i=0}^{M} P_i \lambda^i, \quad P_0 = (I + K)H, \quad P_k = I + \hat{K},
\]
where \( k \leq M \). Assume that \( K, \hat{K}, \text{ and } P_i \) for \( i \in \{0, \ldots, k - 1\} \) are compact. Furthermore, assume that \( H \) is a normal operator in the Schatten-von Neumann class \( \mathcal{S}(\mathcal{H}), p \geq 1 \) with spectrum on a finite number of rays.

Recall that the bounded operator polynomials \( P \) and \( R \) are called equivalent if there exist bounded operators \( E(\lambda) \) and \( F(\lambda) \) that are invertible for all \( \lambda \in \mathcal{D} \) such that \( P(\lambda) = E(\lambda)R(\lambda)F(\lambda) \) for all \( \lambda \in \mathcal{D} \). Moreover, \( R \) is called a spectral divisor of order \( k \) if there exists a bounded operator polynomial \( Q \) such that
\[
P(\lambda) = Q(\lambda)R(\lambda) = \left( \sum_{i=0}^{M-k} \lambda^i Q_i \right) \left( \lambda^k + \sum_{i=0}^{k-1} \lambda^i R_i \right),
\]
where \( \sigma(R) = U \cap \sigma(P) \) and \( \sigma(Q) \subset \mathbb{C} \setminus \mathcal{U} \) for some open \( U \subset \mathbb{C} \).

Let \( \mathcal{U} \) denote the disk \( \mathcal{U} = \{ \lambda : |\lambda - \lambda_0| < r \} \). Then, it follows from [35, Theorem 26.13] that the operator polynomial \( P \) has a spectral divisor \( R \) of order \( k \) with \( \sigma(R) = U \cap \sigma(P) \) if there is some open bounded \( U \subset \mathbb{C} \) such that \( 0 \in U \) and

i. \( \overline{W(P)} \cap \partial \mathcal{U} = \emptyset \),

ii. \( (P(\lambda)u, u) \) has exactly \( k \) roots in \( U \) for all \( u \in \mathcal{H} \setminus \{0\} \).

Furthermore, if \( k = 1 \), then it follows from [35, Theorem 26.13] that \( \partial \mathcal{U} \) can be any simple closed rectifiable curve separating \( \mathbb{C} \) into a bounded domain \( \mathcal{U} \) with an unbounded complement.

3.1. The studied class of rational operator function

In this section, we consider operator functions in the form
\[
\tilde{T}(\lambda) = \lambda^3 + d\lambda^2 + \lambda T_b + aT_1 - \hat{k}(\lambda)T_c,
\]
where \( T_1 \) is a normal operator with resolvent in the Schatten von Neumann class \( \mathcal{S}(\mathcal{H}), p \geq 1, T_b = bT_1 + L_b, \) and \( T_c = cT_1 + L_c \) for some constants \( a > 0 \) and \( b, c \geq 0 \). Furthermore, \( L_b \) and \( L_c \) satisfies condition (2.8).
In this section, we consider spectral properties in the case when
\[ \hat{k}(\lambda) = \frac{\eta}{\lambda + \theta}, \quad \eta, \theta > 0, \]
and investigate the accumulation of eigenvalues.

Our goal is to transform the operator function \( \hat{T} \) in Theorem 2.3 to an operator polynomial with one pole mapped to the origin. Define the bounded operator polynomial
\[ P(\lambda) := \begin{cases} \hat{T}(\lambda)(\lambda + \theta) & \lambda \neq -\theta, \\ -\eta(c + K_c) & \lambda = -\theta, \end{cases} \tag{3.2} \]
where the compact operator \( K_c \) (and \( K_b \)) is defined in (2.9).

**Lemma 3.1.** Let \( \hat{T} \) be defined by (2.9) and \( P \) by (3.2), then
\[ \sigma(P) \setminus \{ -\theta \} = \sigma(\hat{T}), \quad \sigma_{\text{ess}}(P) \setminus \{ -\theta \} = \sigma_{\text{ess}}(\hat{T}), \quad \sigma_{\text{disc}}(P) \setminus \{ -\theta \} = \sigma_{\text{disc}}(\hat{T}). \]
Furthermore, \( P(\lambda) = C(\lambda) + p(\lambda) \) where
\[ C(\lambda) = (\lambda^4 + (d + \theta)\lambda^3 + d\theta\lambda^2)T_1^{-1} + (\lambda^2 + \lambda\theta)K_b - \eta K_c, \quad p(\lambda) = b\lambda^2 + (a + b\theta)\lambda + a\theta - c\eta, \]
and \( \sigma(P) \setminus \{ -\theta \} = \sigma(\hat{T}) \).

**Proof.** The statement follows from the definition of \( \hat{T} \) and \( P \) and straightforward computing. \( \square \)

**Corollary 3.2.** Let \( P \) be defined by (3.2) and \( b > 0 \). Then
\[ \sigma_{\text{ess}}(P) = \{ \lambda_+, \lambda_- \}, \quad \lambda_\pm = -a + b\theta \pm \sqrt{(a - b\theta)^2 + 4bc\eta}, \]
\[ \frac{2b}{2b} \]

**Proof.** Using Lemma 3.1 and Theorem 2.3 the result is obtained by solving \( b\lambda^2 + (a + b\theta)\lambda + a\theta - c\eta = 0 \). \( \square \)

**Lemma 3.3.** Assume that \( bd \geq 1 \) and \( \alpha_0 > \eta c_2/(\theta - c\eta) \). Then \( W(\hat{T}) \cap i\mathbb{R} = \emptyset \).

**Proof.** The proof is similar to the proof of Theorem 2.5. \( \square \)

In the following steps, we apply a linear shift to map one of the points in the essential spectrum to the origin, enabling the use of results from [35] and [21].

Define the bounded operator functions
\[ Q_\pm(\omega) = T_1^{-1}\omega^4 + (4\lambda_\pm + d + \theta)T_1^{-1}\omega^3 + Q^\pm_2\omega^2 + Q^\pm_1\omega + Q^\pm_0, \tag{3.3} \]
where
\[ Q^\pm_0 := (\lambda^2_\pm + (d + \theta)\lambda^2_\pm + d\theta\lambda^2_\pm)T_1^{-1} + (\lambda^2_\pm + \lambda_\pm\theta)K_b - \eta K_c, \]
\[ Q^\pm_1 := (4\lambda^2_\pm + 3(d + \theta)\lambda^2_\pm + 2d\theta\lambda_\pm)T_1^{-1} + (2\lambda_\pm + \theta)K_b + (2b\lambda_\pm + a + b\theta), \]
\[ Q^\pm_2 := (6\lambda^2_\pm + 3(d + \theta)\lambda_\pm + 2d\theta)T_1^{-1} + K_b + b \]

**Lemma 3.4.** Let \( Q_\pm \) be the operator function (3.3) and \( P \) be given by (3.2) then it follows that
\[ Q^\pm(\omega) = P(\lambda), \quad \omega = \lambda - \lambda_\pm, \]
where \( \lambda_{\pm} \) are given as in Corollary 3.2. In particular \( Q^\pm(0) = P(\lambda_{\pm}) \).

**Proof.** By straightforward computations, evaluating \( Q^\pm(\lambda - \lambda_{\pm}) \).

**Corollary 3.5.** \( \sigma(Q^\pm) + \lambda_{\pm} = \sigma(P) \) and \( 0 \in \sigma_{\text{ess}}(Q^\pm) \).

3.2. Two points in the essential spectrum

First, we consider the case when \( \lambda_{\pm} \neq \lambda_{-} \), implying that the essential spectrum of \( Q_{\pm} \) consists of two points. This is the most natural case and holds unless \( a = b\theta \) and \( bc\eta = 0 \).

Define the operator polynomial

\[
R(\omega) = \sum_{i=0}^{4} \omega^i R_i := \frac{Q_{\pm}(\omega)}{2b\lambda_{\pm} + a + b\theta}.
\]

**Lemma 3.6.** Let \( R \) be the operator polynomial defined by (3.4) then it follows that

\[
R_0 = \frac{(\lambda_{\pm}^4 + (d + \theta)\lambda_{\pm}^3 + d\theta\lambda_{\pm}^2)T_{1}^{-1} + (\lambda_{\pm}^2 + \lambda_{\pm}\theta)K_b - \eta K_c}{2b\lambda_{\pm} + a + b\theta},
\]

and

\[
R_1 = I + \frac{(4\lambda_{\pm}^3 + 3(d + \theta)\lambda_{\pm}^2 + 2d\theta\lambda_{\pm})T_{1}^{-1} + (2\lambda_{\pm} + \theta)K_b}{2b\lambda_{\pm} + a + b\theta}.
\]

**Proof.** By straightforward computations.

We will in the following assume that the \( T_1 \) is sufficiently large such that it exists a spectral divisor and prove in Section 4 that \( T_1 \) is sufficiently large for an equation of Moore-Gibson-Thompson type. A more general explicit lower bound on \( T_1 \) can be derived as in [21].

**Theorem 3.7.** Assume that \( T_1, K_b \) and \( K_c \) are self-adjoint operators and either \( a \neq b\theta \) or \( bc\eta > 0 \). Let \( R \) be defined as the operator polynomial (3.4) and define \( \mathcal{H}_0 = \ker(R_0) \) and \( \mathcal{H}_1 = \mathcal{H} \perp \mathcal{H}_0 \). Furthermore, assume that \( T_1 \) is large enough.

Then, the origin is an accumulation point of a branch of eigenvalues if \( \dim \mathcal{H}_1 = \infty \). Additionally, there is an open set \( U \) around the origin in which the set of eigenvectors and associated vectors corresponding to the eigenvalues of \( R \) is complete and minimal in \( \mathcal{H} \). Moreover, if \( \dim \mathcal{H}_1 < \infty \) then the number of eigenvalues in \( U \setminus \{0\} \) (repeated according to multiplicity) is \( \dim \mathcal{H}_1 \).

**Proof.** From the condition \( a \neq b\theta \) or \( bc\eta > 0 \) it follows that \( \lambda_{\pm} \neq \lambda_{-} \) and therefore \( 2b\lambda_{\pm} + a + b\theta \neq 0 \) and consequently \( R \) is well-defined. Furthermore, \( R_0 \) is compact and since \( \lambda_{\pm} \in \mathbb{R} \) it is also self-adjoint which means that its spectrum is at a finite number of rays from the origin. Additionally, \( R_1 \) is a compact perturbation of the identity operator. Finally, provided that \( T_1 \) is large enough there will be an open set \( U \) such that \( 0 \in U \) and \( (R(\lambda)u, u) = 0 \) has exactly one solution in \( U \) and no solutions in \( U \) for all \( u \in \mathcal{H} \). This implies that there will be a spectral divisor of order 1 (also known as spectral root) around 0; see [35, Theorem 26.19]. The result now follows from Theorem 3.9, Theorem 3.10, and Lemma 3.6 in [21].

**Remark 3.8.** We do not need \( R_0 \) selfadjoint, only that it has its spectrum on a finite number of rays from the origin. But given the structure of \( R_0 \), the condition will rarely hold unless it is selfadjoint or a similar condition, such as skew-selfadjointness.
3.3. Single point in the essential spectrum

Now we consider the case when $\lambda_+ = \lambda_-$, implying that the essential spectrum of $Q_\pm$ consists of one point.

**Lemma 3.9.** Assume that $\lambda_+ = \lambda_-$. Then $a = b\theta > 0$, $c = 0$ and $\lambda_\pm = -\theta$.

**Proof.** From Corollary 3.2 it follows directly that $a = b\theta$ and $4bc\eta = 0$. Since $a$, $\eta$, $\theta > 0$ it follows that $b\theta > 0$ and thus $c = 0$. This implies that $\lambda_\pm = -\theta$. \qed

**Corollary 3.10.** Assume that $\lambda_+ = \lambda_-$. Then the operator polynomial $Q := Q_+ = Q_-$ given in (3.3) simplifies to

$$Q(\omega) = T_1\omega^4 + (d-3\theta)T_1\omega^3 + ((3\theta^2 - 2d\theta)T_1^{-1} + K_b)b\omega^2 + ((d\theta^2 - \theta^3)T_1^{-1} - \theta K_b)\omega - \eta K_c. \quad (3.5)$$

The operator polynomial $Q(\omega)/b$ with $Q$ defined in Corollary 3.10 can be written in the form

$$\frac{Q(\omega)}{b} = R(\omega) = \sum_{i=0}^{4} \omega^i R_i,$$ \quad (3.6)

where

$$R_4 = \frac{T_1^{-1}}{b}, \quad R_3 = \frac{d-3\theta}{b} T_1^{-1}, \quad R_2 = I + \frac{(3\theta^2 - 2d\theta)T_1^{-1} + K_b}{b}, \quad R_1 = \frac{(d\theta^2 - \theta^3)T_1^{-1} - \theta K_b}{b}, \quad R_0 = -\frac{\eta}{b} K_c.$$

**Theorem 3.11.** Assume that $K_c$ has spectrum on a finite number of rays, $a = b\theta$, and $c = 0$. Let $R$ be defined as the operator polynomial (3.6) and define $H_0 = \ker(R_0)$, $H_1 = H_\perp H_0$. Furthermore, assume that $T_1$ is large enough.

Then, the origin is an accumulation point of a branch of eigenvalues if $\dim H_1 = \infty$.

**Proof.** From $a = b\theta$ and $c = 0$ it follows that $\lambda_+ = \lambda_- = -\theta$ from (3.9), so the operator polynomial only has one point in the essential spectrum. The proof then follows in a similar vein as the proof of Theorem 3.7. The notable exception is that due to that $R_0$ and $R_1$ are compact and that $R_2$ is a compact perturbation of the identity operator, a spectral divisor of order 2 is used. In the end, the result follows from Theorem 3.10 and Lemma 3.6 in [21]. \qed

4. The existence of a spectral divisor

In this section, we apply the results from the previous sections to a Moore-Gibson-Thompson type of equation in $\Omega \subset \mathbb{R}^n$. The equation can then formally be written in the form

$$u_{ttt} + du_{tt} + bT_1u_t + L_bu_t + aT_1u - c \int_0^t k(t-s)T_1u(s)ds = 0, \quad (4.1)$$

with $u(x,t) = 0$, $x \in \partial \Omega$ and $u(x,0) = u_0(x)$, $u_t(x,0) = u_1(x)$, $u_{tt}(x,0) = u_2(x)$. 
Spectral properties for the Moore-Gibson-Thompson equation without a memory term \((c = 0 \text{ and } L_b = 0)\) were considered in [34,37]. The essential spectrum of the corresponding system operator is then the point \([-a/b]; \text{ see } [34, \text{ Theorem 3.2}]. The function \(F\) in Theorem 2.3 is in that case reduced to \(F(\lambda) = \lambda b + a\). Hence, as expected, we obtain the same essential spectrum \(\sigma_{\text{ess}}(T) = \{-a/b\}\). Note that the relation between the spectrum of a class of rational operator functions and its linearization was considered in [27,18]; see also [17] for a discussion related to waves in viscoelastic materials with a long memory.

In the following subsection, we consider numerical computations in the case \(L_c = c T_1\) and \(L_b = z I\), with \(z \in \mathbb{C}\). A positive \(z\) corresponds to an additional damping term. However, \(L_b\) could also be a differential operator of order one. Furthermore, we assume that

\[
\hat{k}(\lambda) = \frac{\eta}{\lambda + \theta}, \quad \eta, \theta > 0.
\]

From a derivation identical to the proof of Theorem 2.7 follows that the numerical range contains points with arbitrary large imaginary parts and the real parts of those eigenvalues approach

\[
\hat{x}_0 = -\frac{1}{2} (\theta + d + \lambda_+ + \lambda_-) = -\frac{1}{2} (d - \frac{a}{b}),
\]

where \(\lambda_\pm\) is given in Corollary 3.2. Hence, \(x_0 = 0\) in the critical case \(a = bd\) and from Theorem 2.7 we expect that there exist sequences of eigenvalues that approach \(\pm \infty\).

**Example 5.** Consider for \(T_1 \geq 2\pi^2\) the accumulation of eigenvalues in the critical case \(b = 2\), \(c = 1\), \(d = 1\), \(a = bd\), \(\theta = 1\), \(\eta = 1\), and \(z = 0\). Note that Theorem 2.5, (iii) implies that no numerical eigenvalues can be on the imaginary axis. Theorem 3.7 can only be applied under the condition that \(T_1\) is large enough. It is possible but technical to derive general sufficient conditions for the accumulation of eigenvalues as in [21, \text{ Theorem 5.13}]. However, we can for a given set of parameters use (2.20), (2.21) to show that \(T_1\) is large enough such that an spectral divisor exists. From Lemma 2.6, (ii) follows that \(S_2 \cap \sigma_a(T) = \emptyset\), where

\[
S_2 \approx \{z = x + iy : x \in (-\infty, \infty), y \in (-6.26, 0) \cup (0, 6.26)\}
\]

The parts of the numerical range that contains the essential spectrum \(\{\lambda_-, \lambda_+\}\) are therefore on the real axis. By straightforward computations, it can be shown that (2.20) with \(y = 0\) implies that a line segment centered at \(x = -1\) is in a gap of \(\sigma_a(T)\) and the two line segments containing \(\lambda_-\) and \(\lambda_+\) are bounded. Since, \(\lambda_+ \neq \lambda_-\), the theory in subsection 3.2 apply and \(\partial U\) can be any simple closed rectifiable curve separating \(\mathbb{C}\) into a bounded domain \(U\) with an unbounded complement. The corresponding operator polynomial \(P\) has spectral divisors for \(\{\lambda_-, \lambda_+\}\). Hence, \(T_1\) is large enough for the existence of a spectral divisor, and Theorem 3.7 implies that it exists branches of eigenvalues that accumulate to the essential spectrum. It is clear that the branches of eigenvalues that accumulate at \(\{\lambda_-, \lambda_+\}\) are real. The numerical computations in the following subsection will indicate if the accumulation of eigenvalues is from the right or from the left.

Note that Theorem 3.7 or Theorem 3.11 can be applied to a variety of applications, but we need to show the existence of a spectral divisor for the particular parameters.

### 4.1. Numerical illustrations with a finite element method

For illustration, we consider Example 5 discussed in the previous subsection. Let \(V_h^p\) denote the space of piece-wise polynomials of degree \(p\) on a triangulation \(T_h\) of \(\Omega\), where \(\Omega\) is a square with side one. The operator function \(T\) corresponding to (4.1) is discretized with the symmetric interior penalty method [3]. This results in a nonlinear-matrix eigenvalue problem of the form \(T^h(\lambda^h)v^h = 0\). The approximate eigenpairs \((\lambda^h, v^h) \in \mathbb{R} \times V_h^p\) are then computed as in [16].
The numerically computed complex eigenvalues for a non-selfadjoint case with $T_1 = -(1 + 0.1i)\Delta$ and $L_b = (5 + i)I$.

Fig. 2 visualizes for $T_1 = -\Delta \geq 2\pi^2$, where $2\pi^2$ is the smallest eigenvalue when $\Omega = (0,1)^2$, the accumulation of eigenvalues for the same parameters as in the previous subsection: $b = 2$, $c = 1$, $d = 1$, $a = bd$, $\theta = 1$, $\eta = 1$, and $z = 0$.

The eigenvalues of the real axis in Fig. 2 closest to the real axis have imaginary parts $\pm 6.397$, which as expected, are not points in the set $(4.2)$.

We know from the previous subsection that there are sequences of eigenvalues that accumulate at $\lambda_\pm$ and we add below explicit numerically computed enclosures of the numerical range. We use that $\lambda_+, \lambda_- \in W_\alpha$ and that the roots of a monic polynomial depend continuously on its coefficients. The numerical result is that

$$\overline{W(T)} \cap \mathbb{R} \subset \bar{W}_1 \cup \bar{W}_2, \quad \bar{W}_1 = [\lambda_+, -1.683], \quad \bar{W}_2 = [\lambda_-, -0.292],$$

where $\bar{W}_1$ and $\bar{W}_2$ contain bounded and separated parts of the numerical range. From Theorem 3.7 follows then that the eigenvalues of the infinite-dimensional problem accumulate to $\lambda_+ = -1 - 2^{-1/2}$ and to $\lambda_- = -1 + 2^{-1/2}$.

In Fig. 3 we illustrate the validity of Theorem 3.7 also for non-selfadjoint operators $T_1$ and $L_b$. For simplicity, we consider the case $T_1 = -(1 + 0.1i)\Delta$ and $L_b = (5 + i)I$ when $a = 0.5$, $b = 2$, $c = 1$, $d = 1$, $\theta = 1$, and $\eta = 1$. The spectrum is then not in $C_-$ and we cannot expect that there are sequences of eigenvalues that approach $\tilde{x}_0 \pm i\infty$ for any $\tilde{x}_0$. However, we have also in that case accumulation of eigenvalues to $\lambda_\pm$, and the figure shows sequences of real numerical eigenvalues that cluster to those points.

Many of the results presented in this article are applicable to other types of settings that are important in physics and engineering. One interesting setting is structured containing functionally graded material properties, where the Moore-Gibson-Thompson equation is of the form $(2.10)$. 
References


