A counterexample concerning nontangential convergence for the solution to the time-dependent Schrödinger equation

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Abstract. Considering the Schrödinger equation $\Delta_x u = i\partial u / \partial t$, we have a solution $u$ on the form

$$u(x, t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{it|\xi|^2} \hat{f}(\xi) \, d\xi, \quad x \in \mathbb{R}^n, t \in \mathbb{R}$$

where $f$ belongs to the Sobolev space. It was shown in [2], that assuming $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ being a strictly increasing function, with $\gamma(0) = 0$ and $u$ and $f$ as above, there exists an $f \in H^{\gamma/2}(\mathbb{R}^n)$ such that $u$ is continuous in $\{(x, t); t > 0\}$ and

$$\limsup_{(y,t) \to (x,0), |y-x| < \gamma(t), t > 0} |u(y, t)| = +\infty$$

for all $x \in \mathbb{R}^n$. This theorem was proved by choosing

$$\hat{f}(\xi) = \hat{f}_a(\xi) = |\xi|^{-n} (\log |\xi|)^{-3/4} \sum_{j=1}^{\infty} \chi_j(\xi) e^{-t(x_j \cdot \xi + t_j|\xi|^a)}$$

where $\chi_j$ is the characteristic function of shells $S_j$ with the inner radius rapidly increasing with respect to $j$. The purpose of this essay is to explain the proof given in article [2], by first showing that the theorem is true for $\gamma(t) = t$, and to investigate the result when we use

$$S^a f_a(x, t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{it|\xi|^a} \hat{f}_a(\xi) \, d\xi$$

instead of $u$. 

0. Introduction

0.1. Purpose of this paper. Let

\[ u(x, t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{it|\xi|^2} \hat{f}(\xi) \, d\xi, \quad x \in \mathbb{R}^n, \ t \in \mathbb{R} \]

be the solution of the Schrödinger equation \((\Delta_x - i\partial_t)u = 0\) and \(u(x, 0) = f(x)\). Here \(\hat{f}(\xi)\) is the Fourier transform (see Definition 1.2 below). It was shown by Sjögren and Sjölin [2] that near the vertical line \(t \mapsto (x, t)\) through an arbitrary point \((x, 0)\) there can exist points accumulating at \((x, 0)\) such that \(u\) takes values far from \(f\). Therefore we can not consider regions of convergence. The purpose of this paper is to explain the theorem which shows this and to study if the same is true when we have

\[ S^a f(x, t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{it|\xi|^a} \hat{f}(\xi) \, d\xi, \quad x \in \mathbb{R}^n, \ t \in \mathbb{R} \]

instead of \(u\).

The theorem is a counterexample and is constructed for \(f \in H^s(\mathbb{R}^n)\) with \(s = n/2\). When \(s > n/2\) there exist no counterexample and therefore we can consider regions of convergence. In fact, by straightforward computation we obtain \(f(x, t) \rightarrow f(x)\) as \(t \rightarrow 0\). This is true since

\[
|S^a f(x, t)| \leq (2\pi)^{-n} \int_{\mathbb{R}^n} |\hat{f}(\xi)| \, d\xi
\]

\[
= (2\pi)^{-n} \int_{\mathbb{R}^n} |\hat{f}(\xi)|(1 + |\xi|^2)^{s/2} (1 + |\xi|^2)^{-s/2} \, d\xi
\]

\[
\leq (2\pi)^{-n} \left( \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s \, d\xi \right)^{1/2} \left( \int_{\mathbb{R}^n} (1 + |\xi|^2)^{-s} \, d\xi \right)^{1/2}
\]

and the last integral above is convergent if and only if \(s > n/2\). For \(f \in H^s(\mathbb{R}^n)\) with \(s > n/2\) we see that \(|S^a f(x, t)| < \infty\). Therefore convergence along vertical lines can be widened to convergence regions when \(s > n/2\) and \(f \in H^s(\mathbb{R}^n)\). The statement now follows by compactness arguments.

0.2. Some earlier results. Existence of regions of convergence have been studied before, for different functions. In Stein and Weiss [5] nontangential convergence has been considered for the Poisson integral

\[(0.1) \quad u(x, y) = \int_{\mathbb{R}^n} P(x', y) f(x - x') \, dx', \]

where \(P(x', y)\) is the Poisson kernel

\[ P(x', y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-iz \cdot x'} e^{-|z|^2 y} \, dz. \]
The Poisson integral is the solution of \((\Delta_x u)(x, y) + (\partial^2_y u)(x, y) = 0\).

Let \(x_0 \in \mathbb{R}^n\) and \(\Gamma_\alpha(x_0) = \{(x, y) \in \mathbb{R}_+^{n+1} : |x - x_0| < \alpha y, \alpha, y > 0\}\) which is an infinite cone in \(\mathbb{R}_+^{n+1}\). If for every \(\alpha > 0\),

\[
\lim_{(x,y)\to(x_0,0), (x,y)\in \Gamma_\alpha(x_0)} u(x, y) = l,
\]

then \(l\) is the nontangential limit of \(u\) at \(x_0 \in \mathbb{R}^n\).

**Theorem 0.1.** (\([5]\) Theorem 3.16, p. 62) If \(f \in L^p(\mathbb{R}^n), 1 \leq p \leq \infty\) then the Poisson integral (0.1) has the nontangential limit \(f(x_0)\) at each \(x_0\) which belong to the Lebesgue set of \(f\). In particular at almost all points of \(\mathbb{R}^n\) the Poisson integral have these limits.

In \([4]\) \(\sup_{|x-y|<t}|F(x, y)|\) is studied when \(F\) is defined as \(F(x, y) = (f * \Phi_t)(x)\). It is proved that

**Theorem 0.2.** (\([4]\) Proposition, p. 57) Assume that \(\Phi\) has a radial majorant that is nonincreasing, bounded, and integrable. Then

\[
\sup_{|x-y|<t} |F(x, y)| \leq c Mf(x).
\]

Here \(Mf\) is the maximal function defined by

\[
(Mf)(x) = \sup_{r>0} c_n r^{-n} \int_{|y|<r} |f(x - y)| dy.
\]

For \(S^a f(x, t)\) convergence along vertical lines has been studied. In Sjölin [3] it is proved that

\[
\left( \int_B (\sup_{0<t<1} |S^a f(x, t)|)^2 dx \right)^{1/2} \leq C_B \|f\|_{H^s}
\]

where \(s > 1/2\) for \(n \geq 3\) and \(s \geq n/4\) for \(n \leq 2\) and \(B\) is an arbitrary ball in \(\mathbb{R}^n\). This was then used to study almost everywhere convergence along the vertical lines. Sjölin also shows that if \(f \in H^s\) has compact support, then \(\lim_{t \to 0} S^a f(x, t) = f(x)\) for almost every \(x \in \mathbb{R}^n\).

Let \(P = -p(D), D = (D_1, \ldots, D_n)\) and \(D_k = -i \partial_k\) where \(P\) is an elliptic operator. If \(f \in \mathcal{S}(\mathbb{R}^n)\), then it follows by Fourier’s inversion formula that

\[
u(x, t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{it \rho(\xi)} \hat{f}(\xi) d\xi, \quad x, t \in \mathbb{R}
\]

is the solution of the Cauchy problem \(Pu = i \partial_t u\), with \(u(x, 0) = f(x)\). In Sjögren and Sjölin [2] it was proved that

\[
\| \sup_{0<t<1} u(x, t) \|_{L^2(B)} \leq C_B \|f\|_{H^s}, \quad f \in \mathcal{S}, \quad s > 1/2
\]

for any ball \(B \subset \mathbb{R}^n\). This was used to prove convergence along almost every line \(t \mapsto (x + at, t), a \in \mathbb{R}^n\) through \((x, 0)\).
0.3. **Plan of this paper.** In Section 2 we prove a special case of Sjögren and Sjölin’s result, in the case $\gamma(t) = t$ (cf. the abstract or Theorem 2.1 below). In Section 3 we consider the general result (cf. Theorem 3.1), and in Section 4 we prove generalizations of Sjögren and Sjölin’s result to any $a > 1$ (cf. Theorem 4.1).
1. Definitions and notations

We start with some definitions which can be useful for the theorems.

**Definition 1.1.** The open ball in $\mathbb{R}^n$ with center at the origin and radius $\delta$ is denoted $B_\delta(0)$ and defined as

$$B_\delta(0) = \{ x \in \mathbb{R}^n; |x| < \delta \}.$$

**Definition 1.2.** The Fourier transform is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) \, dx$$

where $x \cdot \xi = x_1 \xi_1 + x_2 \xi_2 + \cdots + x_n \xi_n$.

**Definition 1.3.** The set of all tempered distributions $f$ (see e.g. [1]) such that $\hat{f}$ is locally integrable and

$$\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s \, d\xi < \infty$$

is called the Sobolev space of order $s$ and is denoted by $H^s(\mathbb{R}^n)$.

The function $u$ defined as

$$(1.1) \quad u(x, t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{it|\xi|^2} \hat{f}(\xi) \, d\xi, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}$$

is a solution of the Schrödinger equation $\Delta_x u = i\partial u / \partial t$.

We define the operator $S^a$ acting on a function $f$ to be

$$S^a f(x, t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{it|\xi|^a} \hat{f}(\xi) \, d\xi, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}.$$  

Taking $a = 2$ we get the special case $S^2 f(x, t) = u(x, t)$.

Numbers denoted by $C$, $c$ or $C'$ may be different at each occurrence.
2. Preparation for the theorem

We start with the following theorem which is a special case of Theorem 3 in [2].

**Theorem 2.1.** Let the relation between $u$ and $f$ be as in equation (1.1). Then there exists a function $f \in H^{n/2}(\mathbb{R}^n)$ such that $u$ is continuous in $\{(x, t); t > 0\}$ and

$$
\limsup_{(y, t) \to (x, 0), |y-x|<t} |u(y, t)| = +\infty
$$

for all $x \in \mathbb{R}^n$

**Proof.** (1) Let $\delta_k = 1/((k+1)\sqrt{n})$, $k = 0, 1, 2, 3, \ldots$. Let $m_0 = 0$ and let $T_k = B_{k+1}(0) \cap \delta_{k+1}\mathbb{Z}^n = \{x_{m_{k+1}}, \ldots, x_{m_k+1}\}$. Let $T = \bigcup_{k=0}^{\infty} T_k$. Now we choose $(t_j)_{j=1}^{\infty}$ such that $1 > t_1 > t_2 > \cdots > 0$ and

$$
\frac{1}{k+2} < t_j < \frac{1}{k+1}
$$

where $m_k + 1 \leq j \leq m_{k+1}$.

(2) Let $x$ be fixed. We want to show that for all $j \geq 1$ there exists $n_j \in \{m_k + 1, \ldots, m_{k+1}\}$ such that $|x_{n_j} - x| < t_j$. Every $x \in \mathbb{R}^n$ belongs to a hypercube with vertices in $\delta_{k+1}\mathbb{Z}^n$ and the sides of length $1/((k+2)\sqrt{n})$. Take a vertex $x' \in T$ in the hypercube and its diagonal $1/((k+2)$ as center and radius of a ball respectively. This ball $B_{1/((k+2))}(x')$ contains the whole hypercube. Hence the union of such balls equals $\mathbb{R}^n$. Therefore there exists $x_{n_j}$ for every $j$ such that $x \in B_{1/((k+2))}(x_{n_j}) \subset B_{t_j}(x_{n_j})$, where $1/(k+2) < t_j < 1/(k+1)$. Note that $(x_{n_j}, t_j) \to (x, 0)$ as $j \to \infty$.

(3) Let $(R_j)_{j=1}^{\infty}$ and $(R'_j)_{j=1}^{\infty}$ be two sequences such that $2 = R_1 < R'_1 < R_2 < R'_2 < \cdots$. In the remaining part of the proof we let $S_j = \{\xi \in \mathbb{R}^n; R_j < |\xi| < R'_j\}$ and

\[ (2.1) \quad \widehat{f}(\xi) = |\xi|^{-n} (\log |\xi|)^{-3/4} \sum_{j=1}^{\infty} \chi_j(\xi) e^{-i(x_{n_j}\xi + t_j|\xi|^2)} \]

where $\chi_j$ is the characteristic function of $S_j$.

Let $\sigma$ be the surface measure of the $n$-dimensional unit ball induced by the Lebesgue measure. Using Definition 1.3 it is easy to verify that $f \in H^{n/2}$. We just have to calculate the integral in Definition 1.3 which
for \( n > 1 \) gives us

\[
\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2(1 + |\xi|^2)^{n/2} d\xi
= \int_{\mathbb{R}^n \setminus B_2(0)} |\xi|^{-2n(\log|\xi|)^{-3/2}}(1 + |\xi|^2)^{n/2} d\xi
= \int_{2}^{\infty} \left( \int_{|\omega|=1} |r\omega|^{-2n(\log|r\omega|)^{-3/2}}(1 + |r\omega|^2)^{n/2} d\sigma(\omega) \right) r^{n-1} dr
= \int_{2}^{\infty} \left( \int_{|\omega|=1} d\sigma(\omega) \right) r^{-n-1}(\log r)^{-3/2}(1 + r^2)^{n/2} dr.
\]

Since \( r > 2 \) we have that \((1 + r^2)^{n/2} < (r^2 + r^2)^{n/2} = 2^{n/2}r^n\) and therefore

\[
\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2(1 + |\xi|^2)^{n/2} d\xi
< 2^{n/2} \int_{2}^{\infty} \left( \int_{|\omega|=1} d\sigma(\omega) \right) r^{-n-1}(\log r)^{-3/2} r^n d\xi
= 2^{n/2} \left( \int_{|\omega|=1} d\sigma(\omega) \right) \left( \int_{2}^{\infty} r^{-1}(\log r)^{-3/2} d\xi \right)
\leq C \left[ - (\log r)^{-1/2} \right]_{2}^{\infty} \leq C < \infty.
\]

For \( n = 1 \) we get

\[
\int_{\mathbb{R}} |\hat{f}(\xi)|^2(1 + |\xi|^2)^{1/2} d\xi
= \int_{2 \leq |\xi| < \infty} |\xi|^{-2(\log|\xi|)^{-3/2}}(1 + |\xi|^2)^{1/2} d\xi
\leq 2\sqrt{2} \int_{2}^{\infty} \xi^{-2(\log \xi)^{-3/2}} d\xi
= 2\sqrt{2} \int_{2}^{\infty} \xi^{-1} d\xi
= 4\sqrt{2} \left[ - (\log \xi)^{-1/2} \right]_{2}^{\infty} = 4\sqrt{2} \log 2 < \infty.
\]

This gives us that the integral

\[
\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2(1 + |\xi|^2)^{n/2} d\xi
\]

is convergent for \( n \geq 1 \), so \( f \in H^{n/2} \).

(4) Let \( R_1 = 2 \) and \( R'_1 = 3 \) and for \( k < j \) choose

\[
R_j > \max \left( \frac{2^{3/2}}{(t_k - t_j)^{1/2}}, \frac{|x_{n_k} - x_{n_j}| + 1}{t_k - t_j}, R'_{j-1} \right).
\]
\[
R_j^2 > \frac{2^j}{t_k - t_j},
\]

(2.4) \((t_k - t_j)R_j > |x_{n_k} - x_{n_j}| + 1\)

and \(R_j > R'_{j-1}\). We also let \(R'_{j} = R^K_j\) where \(K\) is large.

(5) Define
\[
u_m(x, t) = (2\pi)^{-n} \int_{|\xi| < R'_m} e^{ix\cdot\xi} e^{it|\xi|^2} \hat{f}(\xi) \, d\xi.
\]

Then
\[
u_m(x, t) = \sum_{j=1}^{m} A_j(x, t),
\]

where
\[
A_j(x, t) = (2\pi)^{-n} \int_{S_j} \left( e^{i(x-x_{nj})\cdot\xi} e^{i(t-t_j)|\xi|^2} |\xi|^{-n} (\log |\xi|)^{-3/4} \right) \, d\xi.
\]

We want to estimate \(\nu_m(x_{n_k}, t_k)\) by something smaller which goes to infinity as \(k\) goes to infinity.

(6) In the following calculations we use the triangle inequality and \(\log(|\xi|) > 0\) since \(|\xi| > 2\). We get
\[
\left| \sum_{j=1}^{k-1} A_j(x, t) \right| = \sum_{j=1}^{k-1} (2\pi)^{-n} \int_{S_j} \left| e^{i(x-x_{nj})\cdot\xi} e^{i(t-t_j)|\xi|^2} |\xi|^{-n} (\log |\xi|)^{-3/4} \right| \, d\xi \leq \sum_{j=1}^{k-1} (2\pi)^{-n} \int_{S_j} |\xi|^{-n} (\log |\xi|)^{-3/4} \, d\xi \leq (2\pi)^{-n} \int_{2 \leq |\xi| \leq R'_{k-1}} |\xi|^{-n} (\log |\xi|)^{-3/4} \, d\xi.
\]

For \(n > 1\) a change of variable gives us that
\[
(2\pi)^{-n} \int_{2 \leq |\xi| \leq R'_{k-1}} |\xi|^{-n} (\log |\xi|)^{-3/4} \, d\xi = C' \left[ 4(\log r)^{1/4} \right]_{2}^{R'_{k-1}} = C' \left( (\log R'_{k-1})^{1/4} - (\log 2)^{1/4} \right) \leq C' (\log R'_{k-1})^{1/4}.
\]
When \( n = 1 \) we have that
\[
\left| \sum_{j=1}^{k-1} A_j(x, t) \right| \leq (2\pi)^{-1} \int_{2 \leq |\xi| \leq R_{k-1}'} |\xi|^{-1} (\log |\xi|)^{-3/4} d\xi
\]
\[
= \pi^{-1} \int_{2}^{R_{k-1}'} \xi^{-1} (\log \xi)^{-3/4} d\xi = \pi^{-1} \left[ 4 (\log \xi)^{1/4} \right]_{2}^{R_{k-1}'}
\]
\[
= 4\pi^{-1} \left( (\log R_{k-1}')^{1/4} - (\log 2)^{1/4} \right) \leq 4\pi^{-1} (\log R_{k-1}')^{1/4}.
\]
From the calculations above we can see that for \( n \geq 1 \)
\[
\left| \sum_{j=1}^{k-1} A_j(x, t) \right| \leq C' (\log R_{k-1}')^{1/4}.
\] (2.5)

(7) Then we estimate \( A_k(x_{n_k}, t_k) \). For \( n > 1 \) we have that
\[
A_k(x_{n_k}, t_k) = (2\pi)^{-n} \int_{S_k} |\xi|^{-n} (\log |\xi|)^{-3/4} d\xi
\]
\[
= C \int_{R_k}^{R_k'} r^{-1} (\log r)^{-3/4} dr
\]
\[
= C' \left( (\log R_k')^{1/4} - (\log R_k)^{1/4} \right)
\]
\[
= C' \left( (\log R_k')^{1/4} - (\log (R_k')^{1/K})^{1/4} \right)
\]
\[
= C \left( 1 - \frac{1}{K^{1/4}} \right) (\log R_k')^{1/4} > c (\log R_k')^{1/4}.
\]
Since \( R_k' > 3 \), \( C > 0 \) and \( K \) is large \( c \) can be chosen such that \( c > 0 \).

For \( n = 1 \) we get
\[
A_k(x_{n_k}, t_k) = (2\pi)^{-1} \int_{S_k} |\xi|^{-1} (\log |\xi|)^{-3/4} d\xi
\]
\[
= \pi^{-1} \int_{R_k}^{R_k'} \xi^{-1} (\log \xi)^{-3/4} d\xi
\]
\[
= 4\pi^{-1} \left( (\log R_k')^{1/4} - (\log R_k)^{1/4} \right)
\]
\[
= 4\pi^{-1} \left( (\log R_k')^{1/4} - (\log (R_k')^{1/K})^{1/4} \right)
\]
\[
= 4\pi^{-1} \left( 1 - \frac{1}{K^{1/4}} \right) (\log R_k')^{1/4} > c (\log R_k')^{1/4}.
\]
From the calculations above we see that for \( n \geq 1 \)
\[
A_k(x_{n_k}, t_k) > c (\log R_k')^{1/4}
\] (2.6)
where \( c > 0 \).

(8) For \( j > k \geq 2 \) and \( n > 1 \) we have that

\[
A_j(x_{n_k}, t_k) = (2\pi)^{-n} \int_{S_j} \left( e^{i(x_{n_k} - x_{n_j}) \cdot \xi} e^{i(t_k - t_j) \cdot |\xi|} |\xi|^{-n} (\log |\xi|)^{-3/4} \right) d\xi
\]

\[
= (2\pi)^{-n} \int_{|\omega|=1} \left\{ \int_{R_j} r^{-1} (\log r)^{-3/4} e^{iF(r, \omega)} dr \right\} d\sigma(\omega)
\]

where

\[
F(r, \omega) = r(x_{n_k} - x_{n_j}) \cdot \omega + (t_k - t_j)r^2.
\]

Let \( F^{(p)}(r, \omega) = \partial^p_r F(r, \omega) \). Differentiation of \( F \) gives us

(2.7) \[
F'(r, \omega) = (x_{n_k} - x_{n_j}) \cdot \omega + 2(t_k - t_j)r
\]

and

\[
F''(r, \omega) = 2(t_k - t_j).
\]

By using that \( t_k - t_j > 0 \), \( R_j < r < R'_j \), (2.7), (2.4), the reverse triangle inequality and Cauchy-Schwarz inequality we get that

\[
|F'(r, \omega)| = |(x_{n_k} - x_{n_j}) \cdot \omega + 2(t_k - t_j)r|
\]

\[
\geq |2(t_k - t_j)r - |(x_{n_k} - x_{n_j}) \cdot \omega||
\]

\[
\geq 2(t_k - t_j)r - |(x_{n_k} - x_{n_j}) \cdot \omega|
\]

\[
\geq 2(t_k - t_j)r - |x_{n_k} - x_{n_j}|
\]

\[
> 2(t_k - t_j)r - ((t_k - t_j)R_j - 1)
\]

\[
\geq 2(t_k - t_j)r - ((t_k - t_j)r - 1)
\]

(2.8) \[
= (t_k - t_j)r + 1 > (t_k - t_j)r \geq (t_k - t_j)R_j.
\]

We use integration by parts to get

\[
= \int_{R_j}^{R'_j} \frac{1}{r(\log r)^{3/4}} e^{iF(r, \omega)} dr
\]

\[
= \int_{R_j}^{R'_j} \frac{1}{r(\log r)^{3/4} i F'(r, \omega)} i F'(r, \omega) e^{iF(r, \omega)} dr
\]

\[
= \left[ \frac{1}{r(\log r)^{3/4} i F'(r, \omega)} e^{iF(r, \omega)} \right]_{R_j}^{R'_j} - \int_{R_j}^{R'_j} \frac{d}{dr} \left( \frac{1}{r(\log r)^{3/4} i F'(r, \omega)} \right) e^{iF(r, \omega)} dr
\]

(2.9) \[
= A - B.
\]
Then we estimate each part from the integration above by using (2.8) and (2.3) and see that

\[ |A| = \left| \frac{1}{r \log r} \mathcal{F}(r, \omega) e^{i \mathcal{F}(r, \omega)} \right|_{R_j^j} \]

\[ \leq \left| \frac{C}{R_j F'(R_j, \omega)} \right| < \frac{C}{(t_k - t_j) R_j} < C 2^{-j}. \]

We also have

\[ \left| \frac{d}{dr} \left( \frac{1}{r \log r} \mathcal{F}'(r, \omega) \right) e^{i \mathcal{F}(r, \omega)} \right| = \left| \frac{d}{dr} \left( \frac{1}{r} \cdot \frac{1}{\log r} \cdot \frac{1}{\mathcal{F}'(r, \omega)} \right) \right| \]

\[ \leq \frac{1}{r^3(t - j)} + C(t_k - t_j) = \frac{C}{r^3(t_k - t_j)}. \]

This gives us

\[ |B| = \left| \int_{R_j}^{R_j} \frac{d}{dr} \left( \frac{1}{r \log r} \mathcal{F}'(r, \omega) \right) e^{i \mathcal{F}(r, \omega)} \right| \]

\[ \leq \int_{R_j}^{R_j} \frac{C}{r^3(t_k - t_j)} \leq \frac{C}{R_j^2(t_k - t_j)} < C 2^{-j}. \]

From the calculations above and the triangle inequality we get to the conclusion that for \( j > k \geq 2 \)

\[ |A_j(x_{nk}, t_k)| = \left| (2\pi)^{-n} \int_{|\omega|=1} \left\{ \int_{R_j}^{R_j^j} \frac{1}{r \log r} e^{i \mathcal{F}(r, \omega)} \right\} d\sigma(\omega) \right| \]

\[ \leq (2\pi)^{-n} \int_{|\omega|=1} \left| \int_{R_j}^{R_j^j} \frac{1}{r \log r} e^{i \mathcal{F}(r, \omega)} \right| d\sigma(\omega) \]

\[ = (2\pi)^{-n} \int_{|\omega|=1} |A - B| d\sigma(\omega) \]

\[ \leq (2\pi)^{-n} \int_{|\omega|=1} (|A| + |B|) d\sigma(\omega) < C 2^{-j}. \]
For $n = 1$ we have that
\[
A_j(x_n k, t_k) = (2\pi)^{-1} \int_{S_j} \left( e^{i(x_n k - x_n j)} r e^{i(t_k - t_j) r^2} |r|^{-1} (\log |r|)^{-3/4} \right) dr
\]
\[
= (2\pi)^{-1} \left( \int_{R'_j} r^{-1} (\log r)^{-3/4} e^{iF(r, 1)} dr + \int_{R'_j} r^{-1} (\log r)^{-3/4} e^{iF(r, -1)} dr \right).
\]
In (2.9) take $F(r, 1)$ and $F(r, -1)$ respectively instead of $F(r, \omega)$. Then from the calculations before we get that
\[
|A_j(x_n k, t_k)| < C 2^{-j}.
\]
From above we see that (2.10) is true for $n \geq 1$.

(9) Using the results from (2.5), (2.6), (2.10) in combination with the reverse triangle inequality, and recalling that $R'_j = R'_j K$ with $K$ large, now gives
\[
|u_m(x_n k, t_k)| = \left| \sum_{j=1}^{m} A_j(x_n k, t_k) \right|
\]
\[
\geq |A_k(x_n k, t_k)| - \left| \sum_{j=1}^{k-1} A_j(x_n k, t_k) + \sum_{j=k+1}^{m} A_j(x_n k, t_k) \right|
\]
\[
\geq |A_k(x_n k, t_k)| - \left| \sum_{j=1}^{k-1} A_j(x_n k, t_k) \right| - \left| \sum_{j=k+1}^{m} A_j(x_n k, t_k) \right|
\]
\[
\geq c(\log R'_k)^{1/4} - C'(\log R'_{k-1})^{1/4} - C \sum_{k+1}^{m} 2^{-j}
\]
\[
\geq c(\log R'_k)^{1/4} - C'(\log R_k)^{1/4} - C \sum_{k+1}^{m} 2^{-j}
\]
\[
= \left( c - \frac{C'}{K^{1/4}} \right)(\log R'_k)^{1/4} - C \sum_{k+1}^{m} 2^{-j}
\]
(2.11)
\[
\geq c(\log R'_k)^{1/4}
\]
when $m > k$ and $K \geq 700$.

This can be shown for $n > 1$ taking
\[
C' = 4(2\pi)^{-n} \int_{|\omega|=1} d\sigma(\omega), \quad c = 4 \left( 1 - \frac{1}{K^{1/4}} \right)(2\pi)^{-n} \int_{|\omega|=1} d\sigma(\omega),
\]
\[
C' = 5(2\pi)^{-n} \int_{|\omega|=1} d\sigma(\omega).
\]
and for $n = 1$ taking
\[
C' = 4\pi^{-1}, \quad c = 4 \left( 1 - \frac{1}{K^{1/4}} \right)\pi^{-1}, \quad C = 5\pi^{-1}.
\]

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Then for \( n \geq 1 \) let
\[
(\log R_k')^{1/4} \left( c - \frac{C'}{K^{1/4}} - \frac{C \sum_{k+1}^{n} 2^{-j}}{(\log R_k')^{1/4}} \right) \\
\geq (\log R_k')^{1/4} \left( c - \frac{C'}{K^{1/4}} - \frac{C 2^{-k}}{(\log R_k')^{1/4}} \right) > 0.
\]

Then
\[
4 - \frac{8}{K^{1/4}} - \frac{5 \cdot 2^{-1}}{(\log 3)^{1/4}} > 0 \\
\iff K^{1/4} > \frac{8}{4 - \frac{5 \cdot 2^{-1}}{(\log 3)^{1/4}}},
\]

which is satisfied for \( K > 700 \).

(10) We want to show that \( u_m \) is continuous on every compact set. Continuity of each \( u_m \) follows from the fact that
\[
e^{ix \xi} e^{it|\xi|^2} \hat{f}(\xi)
\]
is continuous and that
\[
\int_{|\xi|<R'_m} |e^{ix \xi} e^{it|\xi|^2} \hat{f}(\xi)| d\xi = \int_{|\xi|<R'_m} |\hat{f}(\xi)| d\xi < C,
\]
\( \forall t \in \mathbb{R}, \ x \in \mathbb{R}^n \). Here below we show the continuity of each \( u_m \) explicitly. Let \( k \in \mathbb{R}^n \) and \( h \in \mathbb{R} \) then
\[
|u_m(x+k, t+h) - u_m(x, t)| \\
\leq |u_m(x+k, t+h) - u_m(x, t+h)| + |u_m(x, t+h) - u_m(x, t)|.
\]

We calculate each part above and get that
\[
|u_m(x+k, t+h) - u_m(x, t+h)| \\
= \left| (2\pi)^{-n} \int_{|\xi|<R'_m} e^{i(x+k) \cdot \xi} e^{i(t+h)|\xi|^2} - e^{ix \xi} e^{i(t+h)|\xi|^2} \hat{f}(\xi) \right| d\xi \\
\leq (2\pi)^{-n} \int_{|\xi|<R'_m} \left| e^{i\frac{k \cdot \xi}{2}} - e^{-i\frac{k \cdot \xi}{2}} \right| |\hat{f}(\xi)| d\xi \\
= (2\pi)^{-n} \int_{|\xi|<R'_m} \left| 2 \sin \left( \frac{k \cdot \xi}{2} \right) \right| |\hat{f}(\xi)| d\xi \\
= (2\pi)^{-n} \int_{|\xi|<R'_m} \left| 2 \sin \left( \frac{k \cdot \xi}{2} \right) \right| k \cdot \xi |\hat{f}(\xi)| d\xi \\
\leq |k|(2\pi)^{-n} \int_{|\xi|<R'_m} |\xi| |\hat{f}(\xi)| d\xi.
\]
For every \( m \) the set \( |\xi| < R'_m \) is compact and therefore
\[
|u_m(x+k, t+h) - u_m(x, t+h)| \to 0 \text{ as } |k| \to 0.
\]
For $|u_m(x, t + h) - u_m(x, t)|$ the proof is constructed in a similar way
to conclude that $|u_m(x, t + h) - u_m(x, t)| \to 0$ as $|h| \to 0$. From this we
get that

$$|u_m(x + k, t + h) - u_m(x, t)| \to 0 \text{ as } |k|, |h| \to 0.$$  

So each $u_m$ is continuous in $\{(x, t); t > 0\}$.

(11) Now it remains to show that $u$ is continuous for $\{(x, t); t > 0\}$. Assume that

$$L \subset \{(x, t); t > 0, x \in \mathbb{R}^n\}.$$  

In (2.3) and (2.4) we want to replace $(x_n, t_k)$ with $(x, t) \in L$. This can
be done for all $j > j_0$, where $j_0 < \infty$ since $R_j$ is rapidly increasing.
Therefore (2.10) holds when $(x_n, t_k)$ is replaced by $(x, t) \in L$ and

$$j > j_0.$$  

We use (2.10) to conclude that

$$c(\log R_k^{1/4})^{1/4} \leq |u_m(x_n, t_k)| = |u_m(x_n, t_k) - u(x_n, t_k) + u(x_n, t_k)|$$

$$\leq |u_m(x_n, t_k) - u(x_n, t_k)| + |u(x_n, t_k)| < 1 + |u(x_n, t_k)|.$$  

This gives us that

$$|u(x_n, t_k)| > c(\log R_k^{1/4})^{1/4} - 1 \to +\infty \text{ as } k \to +\infty$$

and Theorem 2.1 is proved. □
3. Theorem

Now we are going to approach the same theorem but for an arbitrary function \( \gamma \) which is strictly increasing.

**Theorem 3.1.** (Theorem 3 in [2].) Assume that the function \( \gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is strictly increasing, continuous and that \( \gamma(0) = 0 \). Let the relation between \( u \) and \( f \) be as in equation (1.1). Then there exists a function \( f \in H^{n/2} (\mathbb{R}^n) \) such that \( u \) is continuous in \( \{(x,t); t > 0\} \) and

\[
\limsup_{(y,t) \to (x,0), |y-x| < \gamma(t), t>0} |u(y,t)| = +\infty
\]

for all \( x \in \mathbb{R}^n \).

**Proof.** (1) Let \( \delta_k = \gamma(1/(k+1))/\sqrt{n} \), \( k = 0, 1, 2, 3, \ldots \) since \( \gamma \) is strictly increasing it is clear that \( \delta_k \) is strictly decreasing. Let \( m_0 = 0 \) and let \( x_{m_k+1}, \ldots, x_{m_{k+1}} \) be all points in \( B_{k+1}(0) \cap \delta_{k+1}\mathbb{Z}^n \). Now we choose \( (t_j)_1^\infty \) such that \( 1 > t_1 > t_2 > \cdots > 0 \) and

\[
\frac{1}{k+2} < t_j < \frac{1}{k+1}
\]

where \( m_k + 1 \leq j \leq m_{k+1} \).

(2) Let \( x \) be fixed. We want to show that for all \( j \geq 1 \) there exists \( n_j \in \{m_k+1, \ldots, m_{k+1}\} \) such that \( |x_{n_j} - x| < \gamma(t_j) \). Every \( x \in \mathbb{R}^n \) belongs to a hypercube with vertices in \( \delta_{k+1}\mathbb{Z}^n \) and the sides of length \( \gamma(1/(k+2))/\sqrt{n} \). Take a vertex \( x' \in T \) in the hypercube and its diagonal \( \gamma(1/(k+2)) \) as center and radius of a ball respectively. This ball \( B_{\gamma(1/(k+2))}(x') \) contains the whole hypercube. Hence the union of such balls equals \( \mathbb{R}^n \). Therefore there exists \( x_{n_j} \) for every \( j \) such that \( x \in B_{\gamma(1/(k+2))}(x_{n_j}) \subset B_{\gamma(t_j)}(x_{n_j}) \), where \( 1/(k+2) < t_j < 1/(k+1) \). Note that since \( \gamma(0) = 0 \) and \( \gamma \) is continuous and strictly increasing \( (x_{n_j}, t_j) \to (x,0) \) as \( j \to \infty \).

(3) Paragraph (3) – (12) in the proof of Theorem 2.1 hold independently of the choice of \( \gamma \). This completes the proof of Theorem 3.1.

\( \square \)
4. Generalization of the theorem

Now we want to see what happens if we instead of using (1.1), take

\[ (4.1) \quad S^a f(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{it|\xi|^a} \hat{f}(\xi) \, d\xi, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \quad a > 1. \]

If \( a = 2 \) we have the special case \( u(x,t) = S^2 f(x,t) \). In Theorem 3.1, which is a result proved by Sjörgen and Sjölin [2], the function \( u(x,t) = S^2 f(x,t) \) was considered. Now we are going to generalize this result so that we get a theorem for (4.1) where \( a > 1 \) instead of \( a = 2 \).

Some parts of the proof of the following theorem are different depending on the value of \( a \). We will therefore distinguish between the cases \( a \geq 2 \) and \( 1 < a < 2 \) when this is needed. We want to use a similar approach as in Theorem 2.1. We therefore define a function \( \hat{f}_a \) for \( a > 1 \) (see (4.2) in the proof of the theorem below). For \( a = 2 \) we have that \( \hat{f}_a = \hat{f} \).

For \( 1 < a < 2 \) we choose \( (R_j)_{j=1}^\infty \) and \( (R'_j)_{j=1}^\infty \) satisfying equation (4.3) instead of equation (2.2). My result, the following theorem, is a generalization of earlier results by Sjörgen and Sjölin [2].

**Theorem 4.1.** Assume that the function \( \gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is strictly increasing, continuous and that \( \gamma(0) = 0 \). Let the relation between \( S^a f \) and \( f \) be as in equation (4.1) under the restriction that \( a > 1 \). Then there exists a function \( f \in H^{n/2}(\mathbb{R}^n) \) such that \( S^a f \) is continuous in \( \{(x,t); t > 0\} \) and

\[
\limsup_{(y,t) \to (x,0), |y-x| < \gamma(t), t > 0} |S^a f(y,t)| = +\infty
\]

for all \( x \in \mathbb{R}^n \)

**Proof.** (1) We will use the same approach as when we proved Theorem 3.1. As before we let \( \delta_k = \gamma(1/(k+1))/\sqrt{n}, \ k = 0, 1, 2, 3, \ldots \). Since \( \gamma \) is strictly increasing it is clear that \( \delta_k \) is strictly decreasing. Let \( m_0 = 0 \) and let \( x_{m_0+1}, \ldots, x_{m_k+1} \) be all points in \( B_{k+1}(0) \cap \delta_{k+1} \mathbb{Z}^n \). We choose \( (t_j)_1^\infty \) such that \( 1 > t_1 > t_2 > \cdots > 0 \) and

\[
\frac{1}{k+2} < t_j < \frac{1}{k+1}
\]

where \( m_k + 1 \leq j \leq m_{k+1} \). From Theorem 3.1 paragraph (2) we can see that for all \( j \geq 1 \) there exist \( n_j \in \{m_k + 1, \ldots, m_{k+1}\} \) such that \( |x_{n_j} - x| < \gamma(t_j) \).

(2) Let \( (R_j)_{j=1}^\infty \) and \( (R'_j)_{j=1}^\infty \) be two sequences such that \( 2 = R_1 < R'_1 < R_2 < R'_2 < \cdots \). Let \( S_j = \{\xi \in \mathbb{R}^n; R_j < |\xi| < R'_j \} \) and instead of (2.1) we take

\[ (4.2) \quad \hat{f}_a(\xi) = |\xi|^{-n}(\log |\xi|)^{-3/4} \sum_{j=1}^\infty \chi_j(\xi) e^{-i(x_{n_j}, \xi + t_j |\xi|^a)} \]

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where $\chi_j$ is the characteristic function of $S_j$. Since

$$|\hat{f}_a(\xi)| = |\xi|^{-n}(\log |\xi|)^{-3/4} = |\hat{f}(\xi)|$$

for $\xi \in S_j$ for any $j$ and 0 otherwise, we have that $f_a \in H^{n/2}$.

Let $R_1 = 2$ and $R'_1 = 3$ and $R'_j = R^K_j$ where $K$ is large. We now have two different cases depending on the value of $a$.

For $a \geq 2$ choose $R_j > R'_{j-1}$ such that for $k < j$, (2.2) holds.

For $1 < a < 2$ choose $R_j > R'_{j-1}$ such that for $k < j$

$$R_j > \max \left( \left( \frac{2|x_{nk} - x_{nj}|}{a(t_k - t_j)} \right)^{1/(a-1)}, \left( \frac{2^j}{(t_k - t_j)} \right)^{1/a}, R'_{j-1} \right).$$

Then

$$R_j^a > \frac{2^j}{(t_k - t_j)},$$

$$R_j^{a-1} > \frac{2|x_{nk} - x_{nj}|}{a(t_k - t_j)}$$

and $R_j > R'_{j-1}$.

(3) Let

$$S_{m}^a f_a(x, t) = (2\pi)^{-n} \int_{|\xi| < R'_m} e^{ix\cdot\xi} e^{it|\xi|^a} \hat{f}_a(\xi) d\xi.$$ 

Then

$$S_{m}^a f_a(x, t) = \sum_{j=1}^{m} A^a_j(x, t),$$

where

$$A^a_j(x, t) = (2\pi)^{-n} \int_{S_j} \left( e^{i(x-x_{nj})\cdot\xi} e^{i(t-t_j)|\xi|^a} \hat{f}_a(\xi) \right) d\xi.$$ 

(4) By using similar arguments which lead to (2.5) and (2.6) we have

$$\left| \sum_{j=1}^{k-1} A^a_j(x, t) \right| \leq C(\log R'_{k-1})^{1/4}$$

and

$$A^a_k(x_{nk}, t_k) > c(\log R'_k)^{1/4}.$$
(5) For $A^q_j(x_{n_k}, t_k)$ where $j > k \geq 2$ we need some calculations. We define $F_a(r, \omega)$ as

$$F_a(r, \omega) = r(x_{n_k} - x_{n_j}) \cdot \omega + (t_k - t_j)r^a$$

and use $F_a(r, \omega)$ instead of $F(r, \omega)$ to see that for $n > 1$,

(4.10) $$A^q_j(x_{n_k}, t_k) = (2\pi)^{-n} \int_{|\omega|=1} \left\{ \int_{R_j}^{R_j'} r^{-1}(\log r)^{-3/4}e^{iF_a(r,\omega)}dr \right\} d\sigma(\omega).$$

Let $F_a^{(p)}(r, \omega) = \partial_r^p F_a(r, \omega)$. By differentiation we get

(4.11) $$F'_a(r, \omega) = (x_{n_k} - x_{n_j}) \cdot \omega + a(t_k - t_j)r^{a-1}$$

and

$$F''_a(r, \omega) = a(a - 1)(t_k - t_j)r^{a-2}.$$ 

When estimating $|F'_a(r, \omega)|$ we separate the cases $a \geq 2$ and $1 < a < 2$.

If $a \geq 2$, then by using that $t_k - t_j > 0$, $R_j < r < R_j'$, (4.11), (2.4), it follows by the reverse triangle inequality and Cauchy-Schwarz inequality that

\[
|F'_a(r, \omega)| = |(x_{n_k} - x_{n_j}) \cdot \omega + a(t_k - t_j)r^{a-1}| \\
\geq |a(t_k - t_j)r^{a-1} - (x_{n_k} - x_{n_j}) \cdot \omega| \\
\geq a(t_k - t_j)r^{a-1} - |x_{n_k} - x_{n_j}| \\
> a(t_k - t_j)r^{a-1} - ((t_k - t_j)R_j - 1) \\
\geq a(t_k - t_j)r^{a-1} - ((t_k - t_j)r - 1) \\
\geq a(t_k - t_j)r^{a-1} - ((t_k - t_j)r^{a-1} - 1) \\
= (a - 1)(t_k - t_j)r^{a-1} + 1 > (a - 1)(t_k - t_j)r^{a-1} \\
\geq (a - 1)(t_k - t_j)R_j^{a-1}.\]

(4.12)
If $1 < a < 2$, then by using (4.11), (4.5), $R_j < r < R'_j$, it follows by the reverse triangle inequality and Cauchy-Schwarz inequality that

$$
|F_a'(r, \omega)| = \left| (x_{n_k} - x_{n_j}) \cdot \omega + a(t_k - t_j) r^{a-1} \right|
\geq \left| a(t_k - t_j) r^{a-1} - (x_{n_k} - x_{n_j}) \cdot \omega \right|
\geq a(t_k - t_j) r^{a-1} - |x_{n_k} - x_{n_j}|
> a(t_k - t_j) r^{a-1} - \frac{a}{2} (t_k - t_j) R_j^{a-1}
\geq a(t_k - t_j) r^{a-1} - \frac{a}{2} (t_k - t_j) R'_j^{a-1}
\geq \frac{a}{2} (t_k - t_j) r^{a-1} \geq \frac{a}{2} (t_k - t_j) R_j^{a-1}.
$$

(4.13)

We use integration by parts to get

$$
\int_{R_j}^{R'_j} \frac{1}{r(\log r)^{3/4}} e^{i F_a(r, \omega)} dr
= \int_{R_j}^{R'_j} \frac{1}{r(\log r)^{3/4} F'_a(r, \omega)} e^{i F_a(r, \omega)} e^{i F_a(r, \omega)} dr
= \left[ \frac{1}{r(\log r)^{3/4} F'_a(r, \omega)} e^{i F_a(r, \omega)} \right]_{R_j}^{R'_j} - \int_{R_j}^{R'_j} \frac{d}{dr} \left( \frac{1}{r(\log r)^{3/4} F'_a(r, \omega)} \right) e^{i F_a(r, \omega)} dr
= A_a - B_a.
$$

(4.14)

Then we estimate each part from the integration above to see that

$$
|A_a| = \left| \left[ \frac{1}{r(\log r)^{3/4} F'_a(r, \omega)} e^{i F_a(r, \omega)} \right]_{R_j}^{R'_j} \right|
\leq \left| \frac{C}{R_j F'_a(R_j, \omega)} \right|.
$$

For $a \geq 2$ we use (4.12) and (2.3) to see that

$$
|A_a| \leq \left| \frac{C}{R_j F'_a(R_j, \omega)} \right| \leq \frac{C}{R_j(t_k - t_j) R_j^{a-1}} \leq \frac{C}{(t_k - t_j) R_j^{a}} \leq C 2^{-j}.
$$
For $1 < a < 2$ we use (4.13) and (4.4) to see that
\[
|A_a| \leq \left| \frac{C}{R_j F'_a(R_j, \omega)} \right| \leq \frac{C}{R_j^a(t_k - t_j) R_j^{a-1}} = \frac{2C}{a(t_k - t_j) R_j^a} \leq C 2^{-j}.
\]

By using (4.12) for $a \geq 2$ and (4.13) for $1 < a < 2$ we get the following estimation
\[
\left| \frac{d}{dr} \left(\frac{1}{r (\log r)^{3/4} F'_a(r, \omega)} \right) e^{i F_a(r, \omega)} \right| = \left| \frac{d}{dr} \left(\frac{1}{r} \cdot \frac{1}{(\log r)^{3/4} \cdot F'_a(r, \omega)} \right) \right|
\leq \frac{1}{r^2 F'_a(r, \omega)} \left| \frac{1}{(\log r)^{3/4}} \left(1 + \frac{3}{4 \log r} \right) \right| + \frac{|F''_a(r, \omega)|}{r |F'_a(r, \omega)|^2} \frac{1}{(\log r)^{3/4}}
\leq \frac{C}{r^2 |F'_a(r, \omega)|} + \frac{C |F''_a(r, \omega)|}{r |F''_a(r, \omega)|^2}
\leq \frac{C}{r^{a+1}(t_k - t_j)} + \frac{C}{r^{a+1}(t_k - t_j)^2} = \frac{C}{r^{a+1}(t_k - t_j)}.
\]

Because of this we have that
\[
|B_a| = \left| \int_{R_j}^{R'_j} \frac{d}{dr} \left(\frac{1}{r (\log r)^{3/4} F'_a(r, \omega)} \right) e^{i F_a(r, \omega)} \, dr \right|
\leq \int_{R_j}^{R'_j} \frac{C}{r^{a+1}(t_k - t_j)} \, dr
\leq \frac{C}{R_j^{a}(t_k - t_j)}.
\]

For $a \geq 2$ we use (2.3) to see that
\[
|B_a| \leq \frac{C}{R_j^a(t_k - t_j)} \leq \frac{C}{R_j^2(t_k - t_j)} \leq C 2^{-j}.
\]

For $1 < a < 2$ we use (4.4) to see that
\[
|B_a| \leq \frac{C}{R_j^a(t_k - t_j)} \leq C 2^{-j}.
\]

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From the calculations above we get to the conclusion that for $j > k \geq 2$

$$|A_a^j(x_{nk}, t_k)| = |(2\pi)^{-n} \int_{|\omega|=1} \left\{ \int_{R_j} \frac{1}{r(|\log r|)^{3/4}} e^{iF_a(r, \omega)} dr \right\} d\sigma(\omega)|$$

$$\leq (2\pi)^{-n} \int_{|\omega|=1} \left| \int_{R_j} \frac{1}{r(|\log r|)^{3/4}} e^{iF_a(r, \omega)} dr \right| d\sigma(\omega)$$

$$= (2\pi)^{-n} \int_{|\omega|=1} |A_a - B_a| d\sigma(\omega)$$

$$\leq (2\pi)^{-n} \int_{|\omega|=1} (|A_a| + |B_a|) d\sigma(\omega) < C2^{-j}.$$  

For $n = 1$ we have that

$$A_a^j(x_{nk}, t_k) = (2\pi)^{-1} \int_{S_j} \left(e^{i(x_{nk} - x_{nj})r} e^{i(t_k - t_j)|r^n| |r|^{-1}(|\log |r||)^{-3/4}} \right) dr$$

$$= (2\pi)^{-1} \left( \int_{R_j} r^{-1}(|\log r|)^{-3/4} e^{iF_a(r, 1)} dr + \int_{R_j} r^{-1}(|\log r|)^{-3/4} e^{iF_a(r, -1)} dr \right).$$

In (4.14) take $F_a(r, 1)$ and $F_a(r, -1)$ respectively instead of $F_a(r, \omega)$.

Then from the calculations before we get that

$$|A_a^j(x_{nk}, t_k)| < C2^{-j}.$$  

From above we see that (4.15) is true for $n \geq 1$.

(6) Using the results from (4.8), (4.9), (4.15) in combination with the reverse triangle inequality, and recalling that $R_j' = R_j^K$ with $K$ large, now

$$|S_m a f_a(x_{nk}, t_k)| = \left| \sum_{j=1}^m A_a^j(x_{nk}, t_k) \right|$$

$$\geq |A_a^k(x_{nk}, t_k)| - \left| \sum_{j=1}^{k-1} A_a^j(x_{nk}, t_k) \right| - \left| \sum_{j=k+1}^m A_a^j(x_{nk}, t_k) \right|$$

$$\geq c(\log R_k')^{1/4} - C'(\log R_{k-1}')^{1/4} - C \sum_{k+1}^m 2^{-j}$$

$$\geq c(\log R_k')^{1/4} - C'(\log R_k')^{1/4} - C \sum_{k+1}^m 2^{-j}$$

$$= \left( c - \frac{C'}{K^{1/4}} \right) (\log R_k')^{1/4} - C \sum_{k+1}^m 2^{-j}$$

$$\geq c(\log R_k')^{1/4}$$  

(4.16)
when \( m > k \) and \( K \geq 700 \) for \( a \geq 2 \) and \( K \geq 5800 \) for \( 1 < a < 2 \). To estimate \( K \) we can use the similar calculations as in the end of paragraph (9) in the proof of Theorem 2.1.

(7) In order to show that \( S^a_n f_a \) is continuous on every compact set we use the same calculations as in Theorem 2.1 paragraph (10), but with \( |\xi|^a \) instead of \( |\xi|^2 \).

(8) Now it remains to show that \( S^a f_a \) is continuous for \( \{(x, t); t > 0\} \). Assume that

\[
L \subset \subset \{(x, t); t > 0, x \in \mathbb{R}^n\}.
\]

For \( a \geq 2 \) we can see from Theorem 2.1 paragraph (11) that (2.3), (2.4) hold for \( j > j_0 \) with \( (x, t) \in L \) instead of \( (x_n, t_k) \). We can therefore replace \( (x_n, t_k) \) by \( (x, t) \in L \) in (4.15). By replacing \( |\xi|^2 \) by \( |\xi|^a \) the same calculations as in Theorem 2.1 paragraph (11) can be used to show that \( S^a_n f_a \) converges locally uniformly in \( t > 0 \).

For \( 1 < a < 2 \) we want to replace \( (x_n, t_k) \) with \( (x, t) \in L \) in (4.4) and (4.5). This can be done for all \( j > j_0 \), where \( j_0 < \infty \) since \( R_j \) is rapidly increasing. Therefore (4.15) holds when \( (x_n, t_k) \) is replaced by \( (x, t) \in L \) and \( j > j_0 \). By replacing \( |\xi|^2 \) by \( |\xi|^a \) the same calculations as in Theorem 2.1 paragraph (11) can be used to show that \( S^a_n f_a \) converges locally uniformly in \( t > 0 \).

(9) Since each \( S^a_n f_a \) is a continuous function which converges locally uniformly to \( S^a f_a \), it follows that \( S^a f_a \) is continuous in \( \{(x, t); t > 0\} \). The same arguments as in Theorem 2.1 paragraph (12) hold for \( S^a f_a \). So for \( a > 1 \), \( |S^a f_a(x_n, t_k)| \geq c(\log R'_k)^{1/4} - 1 \rightarrow +\infty \) as \( k \rightarrow +\infty \) and Theorem 4.1 is proved.
5. Conclusion

We started with the theorem given in [2] to see that for
\[ u(x, t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{it|\xi|^2} \hat{f}(\xi) \, d\xi, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R} \]
which is as solution to the Schrödinger equation it is possible to choose a function \( f \in H^{n/2} \) such that
\[ \limsup_{(y, t) \to (x, 0), |y - x| < \gamma(t), t > 0} |u(y, t)| = +\infty. \]
We wanted to be able to extend this theorem and therefore we started by looking at
\[ S^a f_a(x, t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{it|\xi|^a} \hat{f}_a(\xi) \, d\xi, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R} \]
where \( a \neq 2 \) to see if the corresponding theorem holds. When looking at \( S^a f(x, t) \) we see that using similar arguments and calculations as in Theorem 3.1 we can prove that a similar theorem holds when \( a > 1 \).

We start with defining \( S_j = \{ \xi \in \mathbb{R}^n; R_j < |\xi| < R'_j \} \) and let
\[ \hat{f}_a(\xi) = |\xi|^{-n} (\log |\xi|)^{-3/4} \sum_{j=1}^{\infty} \chi_j(\xi) e^{-i(x_j \xi + t_j |\xi|^a)} \]
where \( \chi_j \) is the characteristic function of \( S_j \). Then we choose the sequences \( (R_j)_{j=1}^{\infty} \) and \( (R'_j)_{j=1}^{\infty} \) satisfying the conditions \( R_j < R'_j \) and
\[ R_j > \max \left( \frac{2^{a/2}}{a(t_k - t_j)^{1/2}}, \frac{|x_{n_k} - x_{n_j}| + 1}{(t_k - t_j)} \right), \quad R'_j \]
when \( a \geq 2 \) and
\[ R_j > \max \left( \frac{2^j |x_{n_k} - x_{n_j}|}{a(t_k - t_j)} \right)^{1/(a-1)}, \left( \frac{2^j}{(t_k - t_j)} \right)^{1/a}, R'_j \]
when \( 1 < a < 2 \). In this essay we have showed that \( f_a \) is a function such that \( f_a \in H^{n/2}(\mathbb{R}^n) \), \( S^a f_a \) is continuous in \( \{ (x, t); t > 0 \} \) and that
\[ \limsup_{(y, t) \to (x, 0), |y - x| < \gamma(t), t > 0} |S^a f_a(y, t)| = +\infty \]
for \( a > 1 \).
REFERENCES


