

On the linearization of
non-Archimedean holomorphic functions
near an indifferent fixed point

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On the linearization of
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near an indifferent fixed point

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Abstract

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It is known since the work of Herman and Yoccoz in 1981, that Siegel's linearization theorem is true also for non-Archimedean fields. As a consequence, in characteristic zero, linearization is always possible near a non-resonant, indifferent fixed point. However, they also showed that the condition in Siegel's theorem is 'usually' not satisfied over fields of prime characteristic. Indeed, we prove that there exist power series f , such that the associated conjugacy function diverges. On the other hand, we also prove that if the degrees of the monomials of f are co-prime to p , then f is analytically linearizable. We find a lower, sometimes the best, bound of the size of the corresponding linearization disc. In the cases where we find the exact size of the linearization disc, we show, using the Weierstrass degree of the conjugacy, that f has an indifferent periodic point on the boundary.

In the p -adic case we give lower bounds for the size of linearization discs in \mathbb{C}_p . For quadratic maps, and certain power series containing a 'sufficiently large' quadratic term, we find the exact size of the linearization disc.

Moreover, we show that analytic linearization has applications to ergodicity. In particular, we prove that a power series f over a non-Archimedean field K , is uniquely ergodic on all spheres inside a linearization disc about a fixed point in K if and only if K is isomorphic to \mathbb{Q}_p for some prime p , and the multiplier is a generator of the group of units modulo p^2 .

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Key words: dynamical system, linearization, non-Archimedean field.

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Preface

This thesis is organized in the following way. First, an introduction to the subject is presented, together with a summary of the papers that this thesis is based upon. Secondly, the papers are given in fulltext, and they are:

- I. *On Siegel's linearization theorem for fields of prime characteristic*
K.-O. Lindahl
Nonlinearity, Vol. 17, No. 3, 745–763, 2004.
- II. *Divergence and convergence of solutions to the Schröder functional equation in fields of prime characteristic*
K.-O. Lindahl
Revised version of preprint 04015, MSI, Växjö University, 2004.
- III. *On ergodic behavior of p -adic dynamical systems*
M. Gundlach, A. Khrennikov, and K.-O. Lindahl
Inf. Dim. Anal. Quantum Prob. Rel. Top., Vol. 4, No. 4, 569–577, 2001.
- IV. *Estimates of linearization discs in p -adic dynamics with application to ergodicity*
K.-O. Lindahl
Revised version of preprint 04098, MSI, Växjö University, 2004.

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1 Introduction

The Archimedean theory of complex dynamics, i.e. iteration of holomorphic functions of a complex variable, has been intensively studied over the last hundred years, see e.g. [7, 24, 31, 46] and references therein. Over the last decades, there has been an increasing interest in non-Archimedean dynamics and its relationship, similarities, and differences with respect to the Archimedean theory of complex dynamics, see for example the early work of Herman and Yoccoz [32] and the work in [2, 3, 11–15, 17–19, 33, 34, 36, 38, 39, 41–43, 45, 50, 57, 58, 62, 65, 66]. Non-Archimedean fields and their dynamics have many applications; e.g. the counting of periodic points over global fields using local non-Archimedean fields [45, 47, 48, 53, 54, 68, 70], coding theory and computer oriented theories [21–23, 49, 63, 69], biochemistry and physics [4, 5, 35, 55, 56], and cognitive sciences [37]. Non-Archimedean dynamical systems can also be considered as a part of an extended domain of research in number theory and dynamical systems, see e.g. [6, 48].

Recall that a non-Archimedean field is a field K equipped with a non-trivial absolute value $|\cdot|$, satisfying the following ultrametric triangle inequality:

$$|x + y| \leq \max\{|x|, |y|\}, \quad \text{for all } x, y \in K.$$

Standard examples of non-Archimedean fields include the p -adics, and various function fields. The p -adics include the p -adic numbers and their extensions. These fields are all of characteristic zero. The most important function fields include fields of formal Laurent series over various fields. These can be of zero or prime characteristic.

As in complex dynamics, the study of non-Archimedean dynamics begins with the description of the local behavior near fixed points. We are concerned with the existence of a linearization at indifferent fixed points. Recall that a fixed point is said to be indifferent if the absolute value of the derivative at the fixed point is equal to one. Let $f \in K[[x - x_0]]$ be a power series over K of the form

$$f(x) = x_0 + \lambda(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \dots, \quad |\lambda| = 1.$$

We say that f is locally linearizable at the fixed point $x = x_0$, if there is a convergent power series g , such that $g \circ f \circ g^{-1}(x) = \lambda x$, that is, such that

$$g(f(x)) = \lambda g(x), \tag{1.1}$$

on some non-empty disc about x_0 . The function g is called the conjugacy and plays an important role in the dynamics of f . The idea of analytic conjugation was introduced by Ernst Schröder in 1971 to study iteration of complex rational functions. Equation (1.1) and its variants are therefore often referred to as the Schröder functional equation.

Note that in the special case that f is a power function x^λ , then the conjugacy g is the logarithm. Loosely speaking, a solution g , of the Schröder functional equation, can then be viewed as a ‘logarithm’ of the function f .

In fact, the functional equation (1.1) plays a central role in local arithmetic geometry, see e.g. Lubin [44] for a discussion on the relationship between the construction of conjugacies in p -adic dynamics and well-established techniques of local arithmetic geometry. Similarly, in fields of prime characteristic, the construction of conjugacies is related to the construction of so called Drinfeld Modules which play a central role in the arithmetic of function fields, see e.g. Goss [29] and references therein.

Provided that λ is not a root of unity, there always exist a formal solution the Schröder functional equation, as shown by Herman and Yoccoz [32]. By the ansatz of a power series we can then obtain a recursive formula for the coefficients of g . Explicit formulas for the conjugacy are known in a few cases, e.g. in characteristic zero for power functions. In addition, we obtain an explicit formula for the conjugacy of $\lambda x + a_p x^p$ in characteristic p .

From a dynamical point of view it is interesting to study the convergence properties of the conjugacy and its inverse. Indeed, by definition

$$g \circ f^n \circ g^{-1}(x) = \lambda^n x$$

on the disc where g converges and is one-to-one. The maximal such disc will be referred to as the linearization disc. Hence, there is a one-to-one correspondence between iterations of f and repeated multiplication by λ inside the linearization disc. In other words, the local dynamics under f can be determined solely in terms of λ .

In this thesis we use majorant series to estimate the radius of convergence of the conjugacy. This technique was used by Siegel [61] in the complex field case. We then use a generalization, by Benedetto [15], of the Weierstrass preparation theorem to estimate the maximal disc on which the conjugacy is one-to-one. As manifest in this thesis, although explicit formulas for the conjugacy are rare to find, it may still be possible to find the exact radius of the linearization disc, or at least a ‘decent’ estimate.

Finally, as noted in the last paper of this thesis, ergodicity of f is preserved under analytic conjugation. Applying our results on conjugation, we then show that f is (uniquely) ergodic on an entire sphere, inside a linearization disc, if and only if the field K is isomorphic to a field of p -adic numbers and the multiplier is a generator of the group of units modulo p^2 .

2 Background on non-Archimedean fields

Definition 2.1. Let K be a field. By an **absolute value** (or **valuation**) $|\cdot|$ on K we mean a function

$$K \ni x \mapsto |x| \in \mathbb{R}^+ \cup \{0\}$$

such that

- (i) $|x| = 0$ if and only if $x = 0$,
- (ii) $|xy| = |x| \cdot |y|$ for every $x, y \in K$, and
- (iii) there is a real number $C \geq 1$ such that $|x + y| \leq C \cdot \max\{|x|, |y|\}$ for every $x, y \in K$.

An absolute value is said to be **non-trivial** if it is not identically 1 on $K^* = K \setminus \{0\}$. If an absolute value is defined on K , then K is said to be a **valued field**.

Note that if $K = \mathbb{R}$ and $|\cdot|$ is the ‘usual’ absolute value, then we may choose $C = 2$. If it is possible to choose $C = 1$, then K is said to be non-Archimedean:

Definition 2.2. A valued field K is said to be **non-Archimedean** if it is equipped with a non-trivial absolute value $|\cdot|$, which in addition, satisfies the ultrametric triangle inequality:

$$|x + y| \leq \max\{|x|, |y|\}, \quad \text{for all } x, y \in K. \quad (2.1)$$

From now on, K will be assumed to be non-Archimedean. One useful consequence of ultrametricity is that for any $x, y \in K$ with $|x| \neq |y|$, the inequality (2.1) becomes an equality. In other words, if $x, y \in K$ and we have $|x| < |y|$, then $|x + y| = |y|$. As a consequence, all integers $n \in \mathbb{Z}$ have absolute value less than or equal to 1 in K . Hence, the Archimedean Property (given $x, y \in K$, $x \neq 0$, then there exists a positive integer n such that $|nx| > |y|$) does not hold in K .

By ultrametricity, it is easy to see that the closed unit disc

$$\mathcal{O} = \{x \in K : |x| \leq 1\}$$

is a subring of K . The elements of \mathcal{O} are called the **integers** of K . Moreover, the open unit disc

$$\mathcal{M} = \{x \in K : |x| < 1\}$$

is an ideal of \mathcal{O} . An element $x \in K$ is called a **unit** if $|x| = 1$. Hence, \mathcal{M} consists exactly of the non-units of \mathcal{O} , and thereby \mathcal{M} has to be maximal. In fact, \mathcal{M} is the only maximal ideal of \mathcal{O} . Since \mathcal{M} is maximal, the factor ring

$$k = \mathcal{O}/\mathcal{M}$$

is a field — the **residue class field** of K .

Note that if K has positive characteristic p , then also $\text{char } k = p$; but if $\text{char } K = p$, then k could have characteristic 0 or p . Note also that if $x, y \in \mathcal{O}$ reduce to **residue classes** $\bar{x}, \bar{y} \in k$, then $|x - y|$ is 1 if $\bar{x} \neq \bar{y}$, and it is strictly less than 1 otherwise.

For a non-Archimedean field K with absolute value $|\cdot|$ we define the **value group** as the image

$$|K^*| = \{|x| : x \in K^*\}. \quad (2.2)$$

Note that, since $|\cdot|$ is multiplicative, $|K^*|$ is a multiplicative subgroup of the positive real numbers. We will also consider the full image $|K| = |K^*| \cup \{0\}$. The absolute value $|\cdot|$ is said to be **discrete** if the value group is cyclic, that is if there is a **uniformizer**¹ $\pi \in K$ such that $|K^*| = \{|\pi|^n : n \in \mathbb{Z}\}$.

Standard examples of non-Archimedean fields include the p -adics, and various function fields defined below. The p -adics include the p -adic numbers and their extensions. These fields are all of characteristic zero. The most important function fields include fields of formal Laurent series over various fields. These can be of any characteristic.

There is a vast literature on the theory of non-Archimedean fields of which we mention only a few in the subsequent sections. For instance, a nice introduction to p -adic numbers can be found in Gouvêa [30]. In Cassels [25], the topic is treated from an algebraic point of view, while for example Schikhof [60] have a more analytic approach to the concept.

p -adic numbers \mathbb{Q}_p

The primary examples of K with $\text{char } K = 0$ and $\text{char } k = p > 0$, sometimes referred to as the **mixed characteristic case**, are the p -adic numbers. The p -adic numbers were invented by Kurt Hensel in the last decade of the nineteenth century. His purpose was to provide a treatment of algebraic numbers. In order to do this, Hensel used an arithmetical analogue of Karl Weierstrass' method of developing analytical functions of a complex variable into power series. Combined with general field theory, this later on led to the theory of valuations, introduced by Józef Kürschák in 1912. Then in 1918, Alexander Ostrowski determined the set of all valuations on the field \mathbb{Q} of rational numbers.

Just like the real numbers, the p -adic numbers \mathbb{Q}_p are obtained as a completion of the rational numbers \mathbb{Q} with respect to an absolute value. More precisely, let p be a prime number and let n be an integer. The p -adic absolute value of n is defined by

$$|n|_p = p^{-\nu(n)},$$

where $\nu(n)$ denotes the order of p in n . If $n = 0$, then we set $\nu(0) = \infty$. In the same way, the p -adic absolute value of a rational number a/b is given by

$$|a/b|_p = p^{-\nu(a) + \nu(b)}.$$

¹Some authors use the terminology **prime element** or **local coordinate**.

The p -adic numbers \mathbb{Q}_p are obtained as the completion of the rational numbers with respect to $|\cdot|_p$.

Theorem 2.1 (Ostrowski). *Any non-trivial absolute value on \mathbb{Q} is equivalent to one of the absolute values $|\cdot|_p$ for some prime number p , or to the ordinary absolute value on \mathbb{Q} .*

In other words, there is basically only two kinds of analysis on \mathbb{Q} : real and p -adic. These have quite distinctive properties and the essential difference between \mathbb{R} and \mathbb{Q}_p , from a geometrical point of view, is that the p -adic numbers are non-Archimedean.

We shall often use the shorter notation $|\cdot|$ instead of $|\cdot|_p$. Given a prime p , a p -adic number x can be expressed in base p as

$$x = \sum_{i=\nu}^{\infty} x_i p^i, \quad x_i \in \{0, 1, \dots, p-1\},$$

for some integer ν . The absolute value of x is given by $|x| = p^{-\nu}$, provided that $x_\nu \neq 0$. If x is an integer, its p -adic expansion contains no negative powers of p and hence $|x| \leq 1$. Note that the positive integers always have a finite p -adic expansion whereas for example

$$-1 = \sum_{i=0}^{\infty} (p-1)p^i.$$

The set $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x| \leq 1\}$ is called the **p -adic integers** and consists of elements whose p -adic expansion contains only nonnegative powers of p . The set of integers \mathbb{Z} is dense in \mathbb{Z}_p .

Note that the p -adic absolute value is discrete and that the value group $|\mathbb{Q}_p^*|$ consists of all integer powers of p .

Function fields

The case $\text{char } K = \text{char } k$ is sometimes referred to as the **equal characteristic case**. The main type of such K are function fields as described below.

Example 2.1 ($\text{char } K = \text{char } k = p$). Let F_p be a field of characteristic $p > 0$, e.g. F_p could be the finite field \mathbb{F}_{p^n} of p^n elements, an infinite field like the algebraic closure of \mathbb{F}_{p^n} , or the infinite field of rational functions over \mathbb{F}_{p^n} . Let $K = F_p((T))$ be the field of all formal Laurent series in variable T , with coefficients in the field F_p . An element $x \in K$ is of the form

$$x = \sum_{i \geq i_0} x_i T^i, \quad x_{i_0} \neq 0, \quad x_i \in \mathbb{F}, \quad (2.3)$$

for some integer $i_0 \in \mathbb{Z}$. Given $0 < \epsilon < 1$, we define an absolute value $|\cdot|$ on K such that $|T| = \epsilon$ and

$$\left| \sum_{i \geq i_0} x_i T^i \right| = \epsilon^{i_0}. \quad (2.4)$$

Hence, $\pi = T$ is a uniformizer of K . Furthermore, K is complete with respect to $|\cdot|$ and, analogously to the p -adic numbers, can be viewed as the completion of the field of rational functions $F_p(T)$ over F_p with respect to the absolute value defined by (2.4). Note that i_0 is the order of the zero (or if negative, the order of the pole) of x at $T = 0$. Note also that in this case, K and its residue field $k = F_p$ have the same characteristic, namely $\text{char } F_p = p$. Moreover, $|\cdot|$ is the trivial absolute value on F_p , the subfield of K consisting of all constant series in K . As for the p -adic numbers, we can construct a completion \widehat{K} of an algebraic closure of K with respect to an extension of $|\cdot|$. Then \widehat{K} is a complete, algebraically closed non-Archimedean field, and its residue field \widehat{k} is an algebraic closure of $k = F_p$. It follows that \widehat{k} has to be infinite. The value group $|\widehat{K}^*|$, that is the set of real numbers which are actually absolute values of non-zero elements of \widehat{K} , will consist of all rational powers of ϵ , rather than just integer powers of ϵ as in $|K^*|$. In particular, the absolute value is discrete on K but not on \widehat{K} .

Example 2.2 ($\text{char } K = \text{char } k = 0$). Let F be a field of characteristic zero, e.g. F could be either \mathbb{Q} , \mathbb{Q}_p , \mathbb{R} or \mathbb{C} . Then define $K = F((T))$ and $|\cdot|$ as in the previous example. One major difference is that in this case the residue field $k = F$ must be infinite since F is of characteristic zero.

For information on the arithmetic of function fields, the reader may consult Goss [29].

Locally compact fields

Recall that a topological space is said to be **locally compact** if each of its points has a neighborhood with compact closure. We note the following two results. The proofs can be found in Cassels [25].

Proposition 2.1. *Let K be a non-Archimedean field with absolute value $|\cdot|$. Then K is locally compact (w.r.t. $|\cdot|$) if and only if all three of the following conditions are satisfied: (i) K is complete, (ii) $|\cdot|$ is discrete, and (iii) the residue field k is finite.*

In a locally compact non-Archimedean field, every element has a unique representation as a Laurent series in π with coefficients in the residue field k .

Proposition 2.2. *Suppose K is locally compact and let π be a uniformizer. Let \mathcal{R} be a complete system of representatives of the residue field k . Then every $x \in \mathcal{O}$ is uniquely of the form*

$$x = \sum_{i=0}^{\infty} x_i \pi^i, \quad x_i \in \mathcal{R}. \quad (2.5)$$

Conversely, the right hand side always converges to give an $x \in \mathcal{O}$.

Proof. Let $x \in \mathcal{O}$ be given. There is precisely one $x_0 \in k$, namely $x_0 = \bar{x}$, such that $|x - x_0| < 1$ and then $x = x_0 + \pi y_1$ for some $y_1 \in \mathcal{O}$. There is

precisely one $x_1 \in k$ such that $|y_1 - x_1| < 1$, namely $x_1 = \bar{y}_1$, and then $y_1 = x_1 + \pi y_2$ for some $y_2 \in \mathcal{O}$. Continuing in this way we get for every positive integer N

$$x = x_0 + x_1\pi + \dots + x_N\pi^N + y_{N+1}\pi^{N+1}$$

with $x_i \in k$ and $y_{N+1} \in \mathcal{O}$. Now $|y_{N+1}\pi^{N+1}| \rightarrow 0$, and so (2.5) holds. \square

As noted above, \mathcal{O} is certainly a ring. If K is locally compact with uniformizer π , then $\pi\mathcal{O}$ is a unique maximal ideal in K so that the residue class field

$$k = \mathcal{O}/\pi\mathcal{O}.$$

Moreover, the set

$$(\mathcal{O}/\pi^n\mathcal{O})^* = \{a : a = a_0 + a_1\pi + \dots + a_{n-1}\pi^{n-1}, a_i \in k, a_0 \neq 0\} \quad (2.6)$$

is a group under multiplication modulo π^n , for each integer $n \geq 1$. We will refer to this group as the **group of units modulo π^n** .

Finite extensions of locally compact fields

It is easy to see that no non-Archimedean locally compact field is algebraically closed; if K is locally compact and π is a uniformizer in K , then $x^2 - \pi \in K[x]$ is irreducible since $|\sqrt{x}| = |\pi|^{1/2} \notin |K|$. Thus there are non-trivial finite extensions of any non-Archimedean locally compact field.

Let K be a locally compact non-Archimedean field. If L is a finite extension of K , it is possible to prolong the absolute value $|\cdot|$ on K to an absolute value $|\cdot|_L$ on L , in such a way that L will also be locally compact. In fact, there is only one possible way to do this, see for example Gouvêa [30] for the p -adic case. In what follows, we will use the notation $|\cdot|$ for the valuation on K , as well as for its prolongation to any finite extension of K .

We define the corresponding residue field l of L in the same manner as what was done for K . Then, k may be considered as a subfield of l . The degree $f = [l : k]$, of the field extension l/k is finite whenever $[L : K]$ is, and is called the **residue class degree** of the extension L/K . We write $f = f(L/K)$.

Turning to the valuation groups $|K^*|$ and $|L^*|$, we see that $|K^*|$ is a subgroup of $|L^*|$. Let π_K and π_L be uniformizers in K and L , respectively. Then, there is an integer e such that $|\pi_L|^e = |\pi_K|$. The **ramification index** e is defined as the least integer with this property. We write $e = e(L/K)$. The notation of e and f for the ramification index and residue class degree, respectively, are standard in this context.

Proposition 2.3. *Let L be a finite extension of a locally compact non-Archimedean field K . Then*

$$n = e \cdot f,$$

where $n = [L : K]$, $e = e(L/K)$, and $f = f(L/K)$.

Sketch of proof. Choose $\alpha_1, \alpha_2, \dots, \alpha_f \in \mathcal{O}_L$ such that the corresponding residue classes $\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_f \in l$ becomes a basis of the vector space l over k . Let π_L be a uniformizer in L . Then, neither of the e elements $|\pi_L|^0, |\pi_L|^1, \dots, |\pi_L|^{e-1} \in |L^*|$ belong to the same coset of the subgroup $|K^*|$. The ef elements in the set

$$\{\alpha_i \pi_L^j : 1 \leq i \leq f, 0 \leq j \leq e-1\}$$

can now be shown to be a basis of the vector space L over K . See Cassels [25] for the details. □

Of special interest are those extensions of which the ramification index e equals one. A finite extension L/K of degree n is called **unramified**, if $e(L/K) = 1$ (or equivalently, $f(L/K) = n$), and **ramified**, if $e(L/K) > 1$ (or equivalently, $f(L/K) < n$).

***p*-adic fields**

Definition 2.3. Let K be a complete non-Archimedean field with absolute value $|\cdot|$. We say that K is a ***p*-adic field** if K has characteristic zero, $|\cdot|$ is discrete, and the residue class field k is finite.

Cassels [25] give two alternative characterizations:

Proposition 2.4. A non-Archimedean field K of characteristic zero is a *p*-adic field if and only if it is locally compact.

Proposition 2.5. A non-Archimedean field K is a *p*-adic field if and only if it is isomorphic (as a valued field) to a finite extension of \mathbb{Q}_p for some p .

As noted above, to any finite field extension K of \mathbb{Q}_p there is an associated ramification index $e = e(K/\mathbb{Q}_p)$ such that

$$|K^*| = \{p^{n/e} : n \in \mathbb{Z}\}. \tag{2.7}$$

For example, by adjoining \sqrt{p} to \mathbb{Q}_p we get a ramified extension with ramification index $e = 2$.

Complex *p*-adic numbers \mathbb{C}_p

Let \mathbb{Q}_p^a be the algebraic closure of \mathbb{Q}_p and let the **complex *p*-adic numbers** \mathbb{C}_p be the completion of \mathbb{Q}_p^a with respect to the *p*-adic metric. Let us note that whereas the complex numbers are obtained by adjoining the single root $\sqrt{-1}$ to the real numbers, \mathbb{Q}_p^a is an infinite algebraic extension of \mathbb{Q}_p . For \mathbb{C}_p , the value group is

$$|\mathbb{C}_p^*| = \{p^r : r \in \mathbb{Q}\}.$$

In other words, the *p*-adic absolute value is extended from all integer powers of the prime p to all rational powers of p . In particular, the absolute value on \mathbb{C}_p is not discrete.

Non-Archimedean discs

Given an element $x \in K$ and a real number $r > 0$ we denote by $D_r(x)$ the open disc of radius r about x , by $\overline{D}_r(x)$ the closed disc, and by $S_r(x)$ the sphere of radius r about x . If $r \in |K^*|$ (that is if r is actually the absolute value of some nonzero element of K), we say that $D_r(x)$, $\overline{D}_r(x)$, and $S_r(x)$ are **rational**. Note that $S_r(x)$ is non-empty if and only if it is rational. If $r \notin |K^*|$, then we will call $D_r(x) = \overline{D}_r(x)$ an **irrational disc**. In particular, if $a \in K$ and $r = |a|^s$ for some rational number $s \in \mathbb{Q}$, then $D_r(x)$ and $\overline{D}_r(x)$ are rational considered as discs in \widehat{K} . However, they may be irrational considered as discs in K . Note that all discs are both open and closed as topological sets, because of ultrametricity. However, as we will see in Section 2 below, power series distinguish between rational open, rational closed, and irrational discs.

Again by ultrametricity, any point of a disc can be considered as its center. In other words, if $b \in D_r(a)$, then $D_r(a) = D_r(b)$; the analogous statement is also true for closed discs. In particular, if two discs have nonempty intersection, then they are concentric, and therefore one must contain the other.

Holomorphic functions

Much of the study of mapping properties of complex holomorphic functions focuses on functions defined on a simply connected domain; by the Riemann mapping theorem, any such domain is bi-holomorphic either to a disc or to \mathbb{C} . Over a non-Archimedean field K , however, the notion of “simply connected” cannot be defined naively, because K itself is a totally disconnected topological space. Fortunately, it does make sense to discuss power series which converges on a disc; in fact such power series are fundamental objects of rigid analysis, see e.g. [20, 27, 28]. As stated below, the image of a disc under a power series is either a disc or all of K . In light of the Riemann mapping theorem, it is then reasonable to define a simply connected domain in a non-Archimedean field K to be either a disc or all of K .

Let K be a complete non-Archimedean field with absolute value $|\cdot|$. Let f be a power series over K of the form

$$f(x) = \sum_{i=0}^{\infty} a_i(x - \alpha)^i, \quad a_i \in K.$$

Then f converges on the open disc $D_{R_f}(\alpha)$ of radius

$$R_f = \frac{1}{\limsup |a_i|^{1/i}}, \quad (2.8)$$

and diverges outside the closed disc $\overline{D}_{R_f}(\alpha)$ in K . The power series f converges on the sphere $S_{R_f}(\alpha)$ if and only if

$$\lim_{i \rightarrow \infty} |a_i| R_f^i = 0.$$

Notice that, in contrast to the complex field case, what happens at the points on the ‘boundary’ of the region of convergence is rather simple; either the series is convergent at all such points or at none of them. Over the complex field \mathbb{C} , the set of points on the boundary for which the series converges can be rather complicated.

Definition 2.4. Let $U \subset K$ be a disc, let $\alpha \in U$ and let $f : U \rightarrow K$. We say that f is **holomorphic** on U if we can write f as a power series

$$f(x) = \sum_{i=0}^{\infty} a_i(x - \alpha)^i \in K[[x - \alpha]]$$

which converges for all $x \in U$.

Holomorphicity is well-defined since, contrary to the complex field case, it does not matter which $\alpha \in U$ we choose in the definition of holomorphicity.

Proposition 2.6. Let U be a disc in K and let $f : U \rightarrow K$ be holomorphic. Then, for each $\beta \in U$, there exist $b_0, b_1, \dots \in K$ such that the series expansion $f(x) = \sum b_i(x - \beta)^i$ holds for all $x \in U$.

Proof. By assumption, $f(x) = \sum a_i(x - \alpha)^i$ for all $x \in U$. Now

$$(x - \alpha)^i = (x - \beta + \beta - \alpha)^i = \sum_{j=0}^i \binom{i}{j} (x - \beta)^j (\beta - \alpha)^{i-j}.$$

Set $t_{ij} = a_i \binom{i}{j} (x - \beta)^j (\beta - \alpha)^{i-j}$ if $j \leq i$ and $t_{ij} = 0$ otherwise. Recall that by ultrametricity, $|\binom{i}{j}| \leq 1$, and $|x - \beta| \leq \max\{|x - \alpha|, |\alpha - \beta|\}$. Then, for all i, j we have $|t_{ij}| \leq |a_i| \max\{|x - \alpha|, |\beta - \alpha|\}^i$. Since both $\sum a_i(x - \alpha)^i$ and $\sum a_i(\beta - \alpha)^i$ converges, it follows that $\lim_{i \rightarrow \infty} t_{ij} = 0$ uniformly in j . By definition, $\lim_{j \rightarrow \infty} t_{ij} = 0$ for all i . We conclude that

$$f(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} t_{ij} = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} t_{ij} = \sum_{j=0}^{\infty} b_j(x - \beta)^j$$

where $b_j = \sum_{i=0}^{\infty} a_i \binom{i}{j} (\beta - \alpha)^{i-j}$. The theorem follows. \square

A peculiar consequence of this theorem is that the recipe of analytic continuation often used in complex function theory, to extend the domain of a holomorphic function, does not work in the non-Archimedean case. In fact, suppose we have a holomorphic function

$$f(x) = \sum_{i=0}^{\infty} a_i x^i$$

on a disc U , where U is the region of convergence of the power series $\sum a_i x^i$. If we choose any $\beta \in U$ and develop f in a neighborhood $V \subset U$ of β into a power series in $x - \beta$, then by Proposition 2.6, the region of convergence

of our new power series is again U . So by this procedure “we never get out of U ”.

The basic mapping properties of non-Archimedean power series on discs are given by the following theorem.

Proposition 2.7 (Weierstrass Preparation Theorem). *Let K be algebraically closed. Let f be a non-zero holomorphic function on $\overline{D}_1(0)$. Then there are finitely many points $\alpha_1, \dots, \alpha_d \in \overline{D}_1(0)$ (not necessarily distinct) and a function u such that both u and $1/u$ are holomorphic on $\overline{D}_1(0)$, and*

$$f(x) = u(x) \cdot \prod_{j=1}^d (x - \alpha_j).$$

Furthermore, if we write $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $s = \max\{|a_i|\}$, then

$$d = \max\{i \geq 0 : |a_i| = s\}.$$

Proof. See [20] or [27]. An alternative proof, using Newton polygons, can be found in [40]. \square

It follows that d measures the degree of f , counting multiplicity, and we will refer to d as the Weierstrass degree defined below. The Weierstrass Preparation Theorem says that non-Archimedean power series on rational closed discs are essentially polynomials. After all, the condition that u and $1/u$ are both holomorphic is equivalent to saying that u has no zeros in the disc. Although the result is stated for rational closed discs, it has the following generalization to arbitrary discs.

Proposition 2.8 (Benedetto [15] Lemma 2.2). *Let K be algebraically closed. Let $f(x) = \sum_{i=0}^{\infty} a_i (x - \alpha)^i$ be a nonzero power series over K which converges on a rational closed disc $U = \overline{D}_R(\alpha)$, and let $0 < r \leq R$. Let $V = \overline{D}_r(\alpha)$ and $V' = D_r(\alpha)$. Then*

$$\begin{aligned} s &= \max\{|a_i| r^i : i \geq 0\}, \\ d &= \max\{i \geq 0 : |a_i| r^i = s\}, \quad \text{and} \\ d' &= \min\{i \geq 0 : |a_i| r^i = s\} \end{aligned}$$

are all attained and finite. Furthermore,

- a. $s \geq |f'(x_0)| \cdot r$.
- b. if $0 \in f(V)$, then f maps V onto $\overline{D}_s(0)$ exactly d -to-1 (counting multiplicity).
- c. if $0 \in f(V')$, then f maps V' onto $D_s(0)$ exactly d' -to-1 (counting multiplicity).

Definition 2.5. *Let $U \subset K$ be a rational closed disc, and let f be a holomorphic function on U . For any disc $V \subseteq U$, the **Weierstrass degree** (or simply the degree) $\deg_V f$ of f on V is the number d (if V is closed) or d' (if V is open) in Proposition 2.8.*

Note that if $0 \in f(V)$, then the Weierstrass degree is the same as the notion of degree as ‘the number of pre-images of a given point, counting multiplicity’. If $0 \notin f(V)$, then the Weierstrass degree of f is 0, even though points in the image of f may have many pre-images.

A power series f over a complete non-Archimedean field K is always one-to-one and hence isometric on some non-empty disc about an indifferent fixed point $x_0 \in K$. This is a consequence of the local invertibility theorem, see [60]. As noted in paper IV, the maximal such disc is given by the following proposition.

Proposition 2.9. *Let K be algebraically closed. Let $f \in K[[x]]$ be convergent on some non-empty disc about $x_0 \in K$. Suppose that $f(x_0) = x_0$ and $|f'(x_0)| = 1$, and write*

$$f(x) = x_0 + \lambda(x - x_0) + \sum_{i \geq 2} a_i(x - x_0)^i, \quad a = \sup_{i \geq 2} |a_i|^{1/(i-1)}.$$

Let M be the largest disc, with $x_0 \in M$, such that $f : M \rightarrow M$ is bijective and isometric. Then $M = D_{1/a}(x_0)$ if either $\max_{i \geq 2} |a_i|^{1/(i-1)}$ is attained (as for polynomials) or f diverges on $S_{1/a}(x_0)$. Otherwise, $M = \overline{D}_{1/a}(x_0)$.

If $f \in K(x)$ is a non-polynomial rational function, then f may be one-to-one on a possibly larger, open connected affinoid. For example, the rational function

$$R(x) = \frac{1 + x^{p+1}}{x^p(1 + (px)^{p+1})} \in \mathbb{C}_p(x),$$

has indifferent fixed points on the sphere $\{|z| = \sqrt{p}\}$ and is one-to-one on the annulus $C = \{1 < |z| < p\}$, see Rivera-Letelier [57].

Affinoids are regions consisting of a disc from which a finite number of discs are removed. More precisely, let K be algebraically closed and define the **projective line** $\mathbb{P}(K) = K \cup \{\infty\}$. A **$\mathbb{P}(K)$ -disc** is either a disc in K or the complement in $\mathbb{P}(K)$ of a disc in K . A **connected affinoid** U is any non-empty subset of $\mathbb{P}(K)$ which may be written as

$$U = \mathbb{P}(K) \setminus (D_1 \cup \dots \cup D_n),$$

where the D_i ’s are pairwise disjoint $\mathbb{P}(K)$ -discs. If all the D_i ’s are open (resp., closed, rational, irrational), we say that U is open (resp., closed, rational, irrational). Affinoids arise naturally in the theory of rigid analysis, see e.g. [28].

Although connected affinoids are topologically disconnected, they behave like connected sets under the application of rational functions, or more generally, under the application of (appropriately defined) meromorphic functions, i.e. the quotient of two holomorphic functions.

For more information on non-Archimedean power series the reader can consult [26, 59, 60]. From a dynamical point of view, the paper [15] contains many useful results on non-Archimedean analogues of complex analytic mapping theorems relevant for dynamics.

3 Background on dynamical systems

Let K be a field, and let $R \in K(x)$ be a rational function defined over K . Denote by R^n the n -fold composition of R with itself. Thus, R^0 is the identity map, $R^1 = R$, $R^2 = R \circ R$, and so on. We will consider the dynamics of R meaning the action of $\{R^n\}_{n \geq 0}$ on the projective line $\mathbb{P}(K)$.

The first objects of study in such a dynamical system are the periodic points of R . Given a point $x \in \mathbb{P}(K)$, we say that x is **periodic of period** $n \geq 1$ if $R^n(x) = x$; the smallest positive period n of x is called the **minimal period** of x . If x has minimal period 1, we say that x is a **fixed point** of R . More generally, if a point $x \in \mathbb{P}(K)$ has the property that $R^m(x)$ is periodic for some $m \geq 0$, we say that x is **preperiodic**. Note that all periodic points are preperiodic.

For any $x \in \mathbb{P}(K)$, the set $\{R^n\}_{n \geq 1}$ is called the **(forward) orbit** of x . If x is periodic, the forward orbit of x is called a **(periodic) cycle**. Clearly, a point is preperiodic if and only if its forward orbit is finite.

If $x \in K$ is a periodic point of R of minimal period $n \geq 1$, we define the **multiplier** of x to be $\lambda = (R^n)'(x)$. Note that the multiplier is the same for all points in a given periodic cycle.

The character of a periodic point x is determined by the multiplier λ . If $|\lambda| > 1$, we say that x is **repelling**. If $|\lambda| < 1$, we say that x is **attracting**. Finally, if $|\lambda| = 1$ we say that x is **indifferent**. This last case can be divided into two subcases: if λ is a root of unity, we say that x is **resonant**; otherwise we say that x is **non-resonant**.

To analyze the various cases, we assume that x_0 is a fixed point of R with multiplier $\lambda = R'(x_0)$. Because R is rational, we know that its Taylor series

$$f(x) = x_0 + \lambda(x - x_0) + \sum_{i=2}^{\infty} a_i(x - x_0)^i,$$

at x_0 converges on a disk U about x_0 . The map f is said to be **(analytically) linearizable** at x_0 if there is a convergent power series solution g , with $g(x_0) = 0$ and $g'(x_0) = 1$, to the following form of the Schröder functional equation

$$g \circ f(x) = \lambda g(x). \quad (3.1)$$

There always exist a unique formal power series solution g of (3.1), see [32] for the non-Archimedean case. The coefficients of g can be obtained recursively by the formula

$$b_k = \frac{1}{\lambda(1 - \lambda^{k-1})} \sum_{l=1}^{k-1} b_l \left(\sum \frac{l!}{\alpha_1! \cdots \alpha_k!} a_1^{\alpha_1} \cdots a_k^{\alpha_k} \right) \quad (3.2)$$

where $\alpha_1, \alpha_2, \dots, \alpha_k$ are nonnegative integer solutions of

$$\begin{cases} \alpha_1 + \dots + \alpha_k = l, \\ \alpha_1 + 2\alpha_2 + \dots + k\alpha_k = k, \\ 1 \leq l \leq k-1. \end{cases} \quad (3.3)$$

We will refer to the formal solution g as the **conjugacy (function)**. In the attracting and repelling cases, the formal conjugacy always converges on some non-empty neighborhood of the fixed point, both in the complex and the non-Archimedean case. If λ is a root of unity, and if $\deg R \geq 2$, then R cannot be locally conjugate to multiplication by λ . The remaining case, that the corresponding fixed point is indifferent and non-resonant, has a different outcome depending on the field K . In fact, it is not completely solved yet.

It was proven in [43] that in characteristic $p > 0$, like in complex dynamics, the conjugacy function may diverge, due to the presence of small divisors, i.e. small divisor terms $1 - \lambda^n$ in the denominator of the coefficients of the conjugacy g . The problem is that $|1 - \lambda^n|$ may be small if the multiplier λ is ‘too close’ to a n th root of unity. In this way the problem is related to the topological and arithmetic structure of the roots of unity in K as described in [43]. In 1942 Siegel proved in his celebrated paper [61] that the condition

$$|1 - \lambda^n| \geq Cn^{-\beta} \quad \text{for some real numbers } C, \beta > 0, \quad (3.4)$$

is sufficient for convergence in the complex field case.

In 1981 Herman and Yoccoz [32] proved that Siegel’s theorem is true also for non-Archimedean fields:

Theorem 3.1 (Non-Archimedean Siegel Theorem [32]). *Let K be a complete non-Archimedean field, and let f be a power series over K of the form $f(x) = \lambda x + O(x^2)$, convergent on some non-empty disc about 0. If λ satisfies Siegel’s condition (3.4), then f is analytically linearizable at $x = 0$. If 0 is either attracting or repelling, then Siegel’s condition (3.4) is always satisfied. Moreover, if $\text{char } K = 0$, every λ not a root of unity satisfies the Siegel condition (3.4).*

Their general theorem applies to any finite dimension (with the modification that in dimension $d > 1$ there exist examples in characteristic zero where the corresponding Siegel condition is violated and the associated formal conjugacy diverges).

In what follows, we will focus on non-resonant indifferent fixed points.

Linearization discs and periodic points

We will use the following definition of a linearization disc.

Definition 3.1. *Let K be a complete algebraically closed non-Archimedean field, and let f be a power series over K . Suppose that f is analytically linearizable at an indifferent fixed point $x_0 \in K$. Then, the corresponding (**indifferent**) **linearization disc** of f about x_0 , denoted by $\Delta_f(x_0)$, is defined as the largest disc $U \subset K$, with $x_0 \in U$, such that the semi-conjugacy*

$$g(f(x)) = \lambda g(x),$$

*holds for all $x \in U$, and g converges and is one-to-one on U . We will refer to the maximal disk of semi-conjugacy as the (**indifferent**) **semi-disc**.*

Remark 3.1. This notion of a linearization disc is well-defined since there always exist a largest disc on which g is one-to-one (provided that g is convergent). Also note that f must certainly be one-to-one on a linearization disc $\Delta_f(x_0)$; if f fails to be one-to-one, so does g .

We will sometimes discuss linearization discs over a field L which is not algebraically closed. In this case we will refer to

$$\Delta_f(x_0, L) := \Delta_f(x_0) \cap L$$

as the linearization disc in L .

As noted in paper IV, both f and the conjugacy g are one-to-one and isometric on a non-Archimedean linearization disc.

Lemma 3.1. *Let K be algebraically closed. Suppose that $f \in K[[x]]$ has a linearization disc $\Delta_f(x_0)$ about $x_0 \in K$. Let g , such that $g(x_0) = 0$ and $g'(x_0) = 1$, be the corresponding conjugacy function. Then, the two mappings $g : \Delta_f(x_0) \rightarrow g(\Delta_f(x_0))$ and $f : \Delta_f(x_0) \rightarrow \Delta_f(x_0)$ are bijective and isometric. In particular, if $x_0 = 0$, then $g(\Delta_f(x_0)) = \Delta_f(x_0)$. Furthermore, $\Delta_f(x_0) \subseteq M \subseteq \overline{D}_{1/a}(x_0)$, where M and a are defined as in Proposition 2.9.*

As a consequence, the radius of a linearization disc $\Delta_f(x_0)$ is equal to that of $g(\Delta_f(x_0))$. In particular, the radius of a linearization disc is independent of the location of the fixed point x_0 . Therefore, we may, without loss of generality, assume that $x_0 = 0$.

Given K , we will estimate the radius of the linearization disc of power series in the two-parameter family

$$\mathcal{F}_{\lambda,a}(K) := \left\{ \lambda x + \sum a_i x^i \in K[[x]] : |\lambda| = 1, a = \sup_{i \geq 2} |a_i|^{1/(i-1)} \right\}. \quad (3.5)$$

As noted above, for all $f \in \mathcal{F}_{\lambda,a}(K)$, the corresponding linearization disc satisfies

$$\Delta_f(0) \subseteq \overline{D}_{1/a}(0).$$

The linearization disc and the semi-disc do not necessarily coincide; the linearization disc may be strictly contained in the semi-disc as manifest in our results in characteristic p , as well as for p -adic power functions, as shown by Arrowsmith and Vivaldi [3]. The reason for this is that the semi-disc may contain other periodic points, breaking the full conjugacy; if x belongs to a semi-disc about the origin and $x \neq 0$ is periodic of period n , then by the semi-conjugacy relation $g \circ f^n(x) = \lambda^n g(x)$ and the fact that λ is not a root of unity, the conjugacy g cannot be one-to-one at the point x , and consequently x cannot belong to the linearization disc. The semi-disc may in fact contain several linearization discs. In particular, Arrowsmith and Vivaldi showed that power functions over $K = \mathbb{C}_p$ (for which the conjugacy is the logarithm), have periodic points on the boundary of the linearization disc. We consider this example in more detail in section 6 on power functions in paper IV of this thesis.

In fact, in view of the the results in paper II, it seems that the presence of periodic points on the boundary of an indifferent linearization disc is typical in the general setting when f is a power series defined over an algebraically closed non-Archimedean field K ; if the linearization disc Δ is rational open and the radius of the semi-disc of f is strictly greater than that of Δ , so that there is a change of the Weierstrass degree of the conjugacy on the boundary of Δ , then f has an indifferent periodic point on the boundary.

The local and global dynamics on the semi-disc and the larger domain of quasi-periodicity (i.e. the interior of the set of points in $\mathbb{P}(K)$ that are recurrent by R), containing $D_{1/a}(x_0)$ was classified for rational functions over $K = \mathbb{C}_p$ by Rivera-Letelier in his thesis [57]. We will mention some of his results in the next section.

Fatou components of rational functions

Let K be an algebraically closed field and let $R \in K(x)$ be a rational function over K . As in the complex field case we define the **Fatou set of R** to be

$$\mathcal{F} = \{x \in \mathbb{P}(K) : \{R^n\} \text{ is equicontinuous on some neighborhood of } x\}.$$

Here, equicontinuity is defined using the spherical metric on the projective line $\mathbb{P}(K)$. Recall that a family F of functions from a metric space X to a metric space Y is called **equicontinuous** at $x_0 \in X$, if for every $\epsilon > 0$, there is a $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \epsilon$ for all $f \in F$ and for all $x \in X$ such that $d_X(x, x_0) < \delta$. The key point is that ϵ is chosen independently of f . Intuitively, the Fatou set is the region of order, where the iterates $\{R^n\}$ are well behaved; the **Julia set** $\mathcal{J} = \mathbb{P}(K) \setminus \mathcal{F}$ is the region of chaos, where small errors become huge after many iterations of R .

Remark 3.2. In the complex dynamics litterature, the Fatou set is usually defined to be the set of points on the Riemann sphere $\mathbb{P}(\mathbb{C})$ having a neighborhood on which the family $\{R^n\}$ is normal (with respect to the spherical metric). By the Arzelà-Ascoli Theorem, the above definition is equivalent. However, for an algebraically closed non-Archimedean field K , the Arzelà-Ascoli Theorem does not apply, because K is not locally compact.

The Fatou set, being an open subset of $\mathbb{P}(\mathbb{C})$, may be subdivided into its connected components, and it is not difficult to show that if U is a connected Fatou component of \mathcal{F}_R , then $R(U)$ is also. Thus, R induces a discrete dynamical system on the set of connected components of \mathcal{F} . In particular, components may be fixed, periodic, or preperiodic; or if a component is not preperiodic, then it is called a **wandering domain**. By a famous theorem of Sullivan [64], \mathcal{F} has no wandering domains in complex dynamics. Sullivan's work completed a program of fully classifying all possible dynamics on complex Fatou components. The classification theorem then states that: if $R \in \mathbb{C}(z)$ is a rational function with Fatou set \mathcal{F} , and U is a periodic component of \mathcal{F} , then U is either an attracting component

(containing an attracting fixed point), a parabolic component (containing a resonant indifferent periodic point), a linearization disc (containing an indifferent non-resonant periodic point), or a Herman ring (containing no preperiodic point).

Because non-Archimedean fields are totally disconnected, it would be pointless to discuss the connected components of the Fatou set. However, there are several other definitions of ‘components’ of the Fatou set which are better analogues of complex Fatou components [9, 10, 14, 57]. The dynamics of non-Archimedean periodic components seems to have much more variety than that of Archimedean components. In particular, the constructions by Benedetto [16] of wandering domains over any algebraically closed complete non-Archimedean field suggest that Sullivan’s proof has no analogue for K .

However, we will only discuss what Rivera-Letelier [57] refers to as a **fixed analytical component** of R : the union of all connected affinoids U (open or closed) containing x_0 , on which R maps U into U and is one-to-one. These components are non-Archimedean analogues of Siegel discs and Herman rings. Rivera-Letelier gave a classification of analytical components. He showed that every such component C is one of two general types:

Quasi-periodic: Every point of C is recurrent (i.e. contained in the closure of its own forward orbit), and C contains infinitely many indifferent periodic points, which form a discrete subset of C . Every closed affinoid D contained in C contains at most finitely many periodic points. Moreover, C is an open connected affinoid (which is a big deal since the infinite union of intersecting connected affinoids may be a very complicated non-affinoid space. Nevertheless, such a space is called a **connected analytic space**, because it still behaves very much like a connected set). Conversely, given an open connected affinoid A , then there is a rational function R , such that A is a connected component of the domain of quasi-periodicity of R .

Attracting: C contains a unique attracting fixed point x_0 , and all the other points of C are attracted to x_0 under iteration. C is either a rational open disk or else a ‘space of Cantor type’ (a certain complicated type of connected analytic space).

Example 3.1 (Rivera-Letelier [57]). Let R be the rational function

$$R(x) = \frac{1 + x^{p+1}}{x^p(1 + (px)^{p+1})} \in \mathbb{C}_p(x).$$

Let $x_0 \in \{|z| = \sqrt[p]{p}\}$ be a fixed point of R . Then x_0 is indifferent and the corresponding fixed analytical component is $C(x_0) = \{1 < |z| < p\}$.

Example 3.2. If $f \in \mathcal{F}_{\lambda,a}(\mathbb{C}_p)$, where $\mathcal{F}_{\lambda,a}(K)$ is defined by (3.5), is a polynomial, then the corresponding analytical component $C(0) = D_{1/a}(0)$.

In the complex field case, a rational function R is locally linearizable near an indifferent fixed point z_0 if and only if the connected component U , of the Fatou set, containing z_0 is conformally isomorphic to the unit disc under an isomorphism which conjugates R on U to multiplication by λ on the disc, see e.g. Milnor [46, p. 116]. A complex linearization disc is then usually referred to as “a Fatou component on which f is conformally conjugate to multiplication by λ ”. This definition would not make sense in the non-Archimedean case; the component containing an indifferent fixed point x_0 , may contain infinitely many periodic points of R , breaking the conjugacy. In fact, the sometimes even smaller set M , the maximal topological disc on which R is one-to-one, may contain infinitely many periodic points, different from x_0 . In other words, in contrast to the complex field case, the non-Archimedean linearization disc is in general strictly smaller than the ‘Siegel disc’ (i.e. the fixed analytical component).

4 Summary of papers

Paper I: On Siegel's linearization theorem for fields of prime characteristic

In this paper we consider the problem of linearization of a power series of the form $f(x) = \lambda x + O(x^2)$, $|\lambda| = 1$, defined over a field K of prime characteristics p . We describe the problem and its relation to the problem of small divisors, arithmetic of the multiplier λ , geometry of the unit sphere, and the roots of unity in K . A brief review of related results in non-Archimedean and complex dynamics is also given.

We prove that the conjugacy may diverge due to the presence of small divisor terms in the coefficients of the conjugacy. When small divisor terms are present this brings about a problem of a combinatorial nature, where the convergence of the conjugacy is determined in terms of the characteristics of the state space K and the powers of the monomials of f , rather than in terms of the diophantine properties of the multiplier λ , as in the complex field case.

We prove that quadratic polynomials are always linearizable if the characteristics $\text{char } K = 2$. We obtain an explicit formula for the coefficients of the conjugacy and find the exact size of the corresponding linearization disc, and show the existence of a periodic point on the boundary. As noted in paper II, this result has the following generalization for arbitrary p :

Theorem 4.1. *Let $\text{char } K = p$, and let $f \in K[x]$ be polynomial of the form*

$$f(x) = \lambda x + a_p x^p, \quad a_p \neq 0. \quad (4.1)$$

Then, the conjugacy function is given by

$$g(x) = x + \sum_{j=1}^{\infty} b_{p^j} x^{p^j},$$

where

$$b_{p^j} = \frac{a_p^{(p^j-1)/(p-1)}}{\lambda^j (1 - \lambda^{p^j-1})(1 - \lambda^{p^{j-1}-1}) \dots (1 - \lambda^{p-1})}. \quad (4.2)$$

Moreover, g converges on the open disc of radius $\rho_p = 1/a$, where the number $a = |a_p|^{1/(p-1)}$, and diverges on the sphere $S_{\rho_p}(0)$. Let

$$\sigma_p = \frac{p^{m'} - 1}{a},$$

where $m' = 1$ if $m = 1$, and otherwise

$$m' = \min\{n \in \mathbb{Z} : n \geq 1, p^n \equiv 1 \pmod{m}\}.$$

Then, the linearization disc of f about the origin is the disc $D_{\sigma_p}(0)$. Moreover in the algebraic closure \widehat{K} we have $\deg(g, \overline{D}_{\sigma_p}(0)) = p^{m'}$. Furthermore, f has an indifferent periodic point of period $\kappa \leq p^{m'}$ on the sphere $S_{\sigma_p}(0)$ in \widehat{K} , with multiplier λ^κ .

In the case $\text{char } K = p > 2$ we give a sufficient condition for divergence of the conjugacy for quadratic maps as well as for certain class of power series containing a quadratic term.

These results unveils a connection between convergence of the conjugacy and the mutual relation between the powers of the monomials of f and the characteristic p . ‘Good’ powers for convergence being those co-prime to p , ‘bad’ powers being those prime to p . However, the blend of prime and co-prime powers may sometimes yield convergence, at least for power series, as shown by the following example.

Example 4.1. Let $\text{char } K = 3$ and let $g(x) = x + x^2$, then

$$g^{-1}(x) = x - x^2 + 2x^3 + x^4 + \dots$$

Moreover, g is the conjugacy for the power series f defined by the equation $f(x) = g^{-1}(\lambda g(x))$. Hence f is a power series of the form

$$f(x) = \lambda x + (\lambda - \lambda^2)x^2 + 2(\lambda^3 - \lambda^2)x^3 + (\lambda^4 - \lambda^2)x^4 + \dots,$$

which is certainly linearizable.

This brings about a problem of a combinatorial nature of seemingly great complexity and a complete description seems difficult. However, there might be a sharp distinction for polynomials.

Open problem 4.1. *Let K be of characteristic $p > 0$. Is there a polynomial of the form $f(x) = \lambda x + O(x^2) \in K[x]$, with λ not a root of unity satisfying $|1 - \lambda^n| < 1$ for some $n \geq 1$, and containing a monomial of degree prime to p , such that the formal conjugacy g converges?*

Paper II: Divergence and convergence of solutions to the Schröder functional equation in fields of prime characteristic

In this paper we develop further the ideas in paper I and prove some more general results. First we give a class of polynomials containing a monomial of degree prime to p , such that the conjugacy diverges:

Theorem 4.2. *Let $\text{char } K = p > 0$ and let $\lambda \in K$, $|\lambda| = 1$. Then, polynomials of the form*

$$f(x) = \lambda x + a_{p+1}x^{p+1} \in K[x], \quad a_{p+1} \neq 0,$$

are not analytically linearizable at the fixed point at the origin if $|1 - \lambda| < 1$.

Then we prove that if the degrees of the monomials of a power series f are co-prime to p , then f is analytically linearizable:

Theorem 4.3. *Let $\text{char } K = p > 0$ and let $\lambda \in K$, $|\lambda| = 1$, but not a root of unity. Then, convergent power series of the form*

$$f(x) = \lambda x + \sum_{p \nmid i} a_i x^i \in K[[x]],$$

are linearizable at the origin.

As noted in paper I, for the same multiplier λ , the the blend of monomials of degree prime and co-prime to char $K = p$, may sometimes yield divergence sometimes not, and a complete description seems difficult.

The rest of the paper is devoted to estimates of semi-discs and linearization discs for power series that satisfy the convergence condition of Theorem 4.3.

Let K be a field of prime characteristic p . Let $\lambda \in K$, not a root of unity, be such that the integer

$$m = m(\lambda) = \min\{n \in \mathbb{Z} : n \geq 1, |1 - \lambda^n| < 1\}, \quad (4.3)$$

exists. Given λ and hence m , the integer k' is defined by

$$k' = k'(\lambda) = \min\{k \in \mathbb{Z} : k \geq 1, p|k, m|k - 1\}. \quad (4.4)$$

Let $a > 0$ be a real number. We will associate with the pair (λ, a) , the family of power series

$$\mathcal{F}_{\lambda,a}^p(K) = \left\{ \lambda x + \sum_{p|i} a_i x^i \in K[[x]] : a = \sup_{i \geq 2} |a_i|^{1/(i-1)} \right\}, \quad (4.5)$$

and the real numbers

$$\rho = \rho(\lambda, a) = \frac{|1 - \lambda^m|^{\frac{1}{mp}}}{a}, \quad (4.6)$$

and

$$\sigma = \sigma(\lambda, a) = \frac{|1 - \lambda^m|^{\frac{1}{k'-1}}}{a}, \quad (4.7)$$

respectively. Our most general estimate on the semi-disc and linearization disc for $f \in \mathcal{F}_{\lambda,a}^p(K)$ can then be stated in the following way.

Theorem 4.4. *Let $f \in \mathcal{F}_{\lambda,a}^p(K)$. Then, the semi-conjugacy holds on the open disc $D_\rho(0)$. The full conjugacy $g \circ f \circ g^{-1}(x) = \lambda x$ holds on $D_\sigma(0)$.*

Under further assumptions on f , the linearization disc may contain the larger disc $D_\rho(0)$.

Theorem 4.5. *Let $f \in \mathcal{F}_{\lambda,a}^p(K)$ be of the form*

$$f(x) = \lambda x + \sum_{i \geq i_0} a_i x^i,$$

for some integer $i_0 > k'$. Then, the full conjugacy $g \circ f \circ g^{-1}(x) = \lambda x$ holds on a disc larger than or equal to $D_\rho(0)$ or the closed disc $\overline{D}_\rho(0)$, depending on whether g converges on $\overline{D}_\rho(0)$ or not.

Note that the estimate of the linearization disc in Theorem 4.3 is maximal in the sense that in \widehat{K} , the completion of the algebraic closure of K , quadratic polynomials have a fixed point on the sphere $S_\sigma(0)$ if $m(\lambda) = 1$, breaking the conjugacy there. In fact, the estimate is maximal in a broader sense, according to the following theorem.

Theorem 4.6. *Let $f \in \mathcal{F}_{\lambda,a}^p(K)$. Suppose $a = |a_{k'}|^{1/(k'-1)}$ and $|a_i| < a^{i-1}$ for all $i < k'$. Then, $D_\sigma(0)$ is the linearization disc of f about the origin. In \widehat{K} we have $\deg(g, \overline{D}_\sigma(0)) = k'$. Furthermore, f has an indifferent periodic point in \widehat{K} on the sphere $S_\sigma(0)$ of period $\kappa \leq k'$, with multiplier λ^κ .*

The existence of a periodic point on the boundary follows from Lemma 6.1 in this paper which states that if there is a shift of the value of the Weierstrass degree from 1 to $d > 1$, of the conjugacy function on a sphere S , then there is an indifferent periodic point of period $\kappa \leq d$, on the sphere S . This is true in any non-Archimedean field K . In other words, we have the following theorem.

Theorem 4.7. *Let K be a complete algebraically closed non-Archimedean field. Let $f \in K[[x]]$ have a linearization disc Δ about an indifferent fixed point. Suppose that Δ is rational open, and that the radius of the corresponding semi-disc of f is strictly greater than that of Δ , then f has an indifferent periodic point on the boundary of Δ .*

Paper III: On ergodic behavior of p -adic dynamical systems

This paper is a joint work with Matthias Gundlach and Andrei Khrennikov. We study the ergodic properties of p -adic monomial maps of the form

$$f(x) = x^s,$$

defined on spheres about the fixed point $x = 1$. It is shown that f is minimal (every forward orbit is dense) on the sphere $S_{p^{-l}}(1)$, $l \geq 1$, if and only if s is a generator of the group of units $(\mathbb{Z}/p^2\mathbb{Z})^*$.

Although not explicitly stated, the proof makes extensive use of the fact that the p -adic logarithm conjugates x^s to the multiplier map $s \mapsto sx$ near $x = 1$. The minimality of x^s is then shown to be equivalent to its unique ergodicity. Recall that a continuous map $T : X \rightarrow X$ is uniquely ergodic if there exists only one probability measure μ , defined on the Borel σ -algebra $\mathcal{B}(X)$ of X , such that T is measure-preserving with respect to μ . In our case the unique invariant measure is the normalized Haar measure (the measure of a disc is equal to its radius).

The paper ends with the formulation of the following problem.

Problem 4.1 (Lindahl, Khrennikov, and Gundlach). *Is a polynomial of the form $f(x) = x^s + q(x)$, with q ‘small’, ergodic if and only if $x \mapsto x^s$ is?*

The answer is affirmative if we add the assumption that q contains no linear term. A solution was obtained by Anashin [1] for polynomials and certain locally analytic functions. In paper IV we obtain a generalization of this result, using analytic conjugation. See Theorem 4.12 below.

Paper IV: Estimates of linearization discs in p -adic dynamics with application to ergodicity

We study linearization discs in p -adic dynamics, i.e. over the p -adic numbers \mathbb{Q}_p , finite extensions of \mathbb{Q}_p , as well as over the algebraic closure \mathbb{C}_p . As stated above, it is known since the work of Herman and Yoccoz, that p -adic power series are always linearizable near an indifferent non-resonant periodic point.

A mainly unresolved issue which we consider in this paper concerns the radius of the corresponding linearization disc. Estimates were previously obtained for quadratic polynomials over \mathbb{Q}_p by Ben-Menahem [8], and Thiran, Versteegen, and Weyers [67]. Pettigrew, Roberts, and Vivaldi [52], obtained estimates for power series over the p -adic integers, \mathbb{Z}_p , with maximal multipliers. Some related estimates of small divisors in \mathbb{C}_p were also obtained by Khrennikov [36].

We generalize these results and give lower bounds for the size of linearization discs for power series over \mathbb{C}_p . The expressions for these estimates take a more complicated form, compared to that of the characteristic p case studied in paper II. This stems from the fact that the geometry of the roots of unity is more complex in \mathbb{C}_p .

Given λ , not a root of unity, and a real number a , let $\mathcal{F}_{\lambda,a}(\mathbb{C}_p)$ be the family

$$\mathcal{F}_{\lambda,a}(\mathbb{C}_p) := \left\{ \lambda(x - x_0) + \sum a_i(x - x_0)^i \in \mathbb{C}_p[[x]] : a = \sup_{i \geq 2} |a_i|^{1/(i-1)} \right\},$$

and let

$$R(t) := \begin{cases} 0, & \text{if } t = 0, \\ p^{-\frac{1}{p^{t-1}(p-1)}}, & \text{if } t \geq 1. \end{cases} \quad (4.8)$$

Then, there is an integer m

$$m = m(\lambda) = \min\{n \in \mathbb{Z} : n \geq 1, |1 - \lambda^n| < 1\}, \quad (4.9)$$

and a unique nonnegative integer $s = s(\lambda)$ such that

$$R(s) \leq |1 - \lambda^m| < R(s + 1),$$

and a root of unity $\alpha = \alpha(\lambda)$ in \mathbb{C}_p such that

$$|\alpha - \lambda^m| \leq |\gamma - \lambda^m|$$

for every root of unity $\gamma \in \mathbb{C}_p$.

Theorem 4.8 (General estimate). *Let $f \in \mathcal{F}_{\lambda,a}(\mathbb{C}_p)$. Then, the linearization disc $\Delta_f(x_0) \supseteq D_\sigma(x_0)$ where*

$$\sigma = \sigma(\lambda, a) := a^{-1} R(s + 1)^{\frac{1}{m}} |1 - \lambda^m|^{\frac{1}{m}(1 + \frac{p-1}{p}s)} \left(\frac{|\alpha - \lambda^m|}{|1 - \lambda^m|} \right)^{1/mp^s}. \quad (4.10)$$

Moreover, if the conjugacy function converges on the closed disc $\overline{D}_\sigma(x_0)$, then $\Delta_f(x_0) \supseteq \overline{D}_\sigma(x_0)$.

Note that the estimate $\sigma = \sigma(\lambda, a)$ depends only on λ and the real number a . To find the exact size of the linearization disc we do in general need more information about the coefficients of f . However, for a large class of quadratic polynomials, and certain power series containing a ‘sufficiently large’ quadratic term, we prove that

$$\tau = |1 - \lambda|^{-1/p} \sigma(\lambda, a) \quad (4.11)$$

is the exact radius of the linearization disc. More precisely, our main result can be stated in the following way.

Theorem 4.9 (Quadratic case). *Let p be an odd prime and let $\lambda \in \mathbb{C}_p$, not a root of unity, belong to the annulus $\{z : p^{-1} < |1 - z| < 1\}$. Let f be a quadratic polynomial of the form $f(x) = \lambda x + a_2 x^2 \in \mathbb{C}_p[x]$. Then, the coefficients of the conjugacy g satisfy*

$$|b_k| = \frac{|a_2|^{k-1} |1 - \lambda|^{\lfloor \frac{k-1}{p} \rfloor}}{\prod_{n=1}^{k-1} |1 - \lambda^n|}. \quad (4.12)$$

Moreover, the linearization disc about the origin, $\Delta_f(0) = D_\tau(0)$, where $\tau = |1 - \lambda|^{-1/p} \sigma(\lambda, a)$.

This result is then extended to power series containing a ‘sufficiently large’ quadratic term. We also give sufficient conditions, there being a fixed point on the ‘boundary’ of the linearization disc, i.e. the sphere $S_\tau(x_0)$ about x_0 of radius τ .

Note that $\tau(\lambda, a) < 1/a$. Hence, at least in this case, the linearization disc cannot contain the maximal disc $D_{1/a}(x_0)$ on which f is one-to-one.

Theorem 4.10 (Asymptotic behavior of σ). *Let $|\alpha - \lambda^m|$ be fixed. Then, the estimate σ of the radius of the linearization disc goes to $1/a$ as m or s goes to infinity. If s and m are fixed, then $\sigma \rightarrow 0$ as $|\alpha - \lambda^m| \rightarrow 0$.*

For finite extensions of \mathbb{Q}_p , we give a sufficient condition on the multiplier under which the corresponding linearization disc is maximal, i.e. its radius coincides with that of the maximal open disc $D_{1/a}(0)$ in \mathbb{C}_p on which f is one-to-one.

Theorem 4.11 (Maximal linearization discs in extensions of \mathbb{Q}_p). *Let K be a finite extension of \mathbb{Q}_p of degree n , with ramification index e , residue field k of degree $[k : \mathbb{F}_p] = n/e$, and uniformizer π . Let $f \in \mathcal{F}_{\lambda, a}(\mathbb{C}_p) \cap K[[x]]$ be a power series. Suppose that λ has a maximal cycle modulo π^2 and*

$$\log_p e \leq (p^{n/e} - 3)p/(p - 1) - \nu \left(\frac{\alpha - \lambda^{p^{n/e} - 1}}{1 - \lambda^{p^{n/e} - 1}} \right) + \log_p(p - 1),$$

where $\nu(x)$ is the order p in x . Then, the corresponding linearization disc $\Delta_f(x_0, K) = \Delta_f(x_0) \cap K$ is maximal in the sense that $\Delta_f(x_0, K)$ is either the open or closed disc of radius $1/a$. In particular, if either $\max_{i \geq 2} |a_i|^{1/(i-1)}$ is attained (as for polynomials) or f diverges on $S_{1/a}(x_0)$, then $\Delta_f(x_0, K) = D_{1/a}(x_0, K)$.

Moreover, we also show that, for any complete non-Archimedean field, transitivity is preserved under analytic conjugation into a linearization disc.

Using a result by Oxtoby [51], we note that minimality is equivalent to the unique ergodicity of f on compact subsets of a non-Archimedean linearization disc. In particular, a power series f over \mathbb{Q}_p is minimal, hence uniquely ergodic, on all spheres inside a linearization disc about a fixed point if and only if the multiplier is a generator of the group of units $(\mathbb{Z}/p^2\mathbb{Z})^*$. We also note that in finite extensions of \mathbb{Q}_p , a power series cannot be ergodic on an entire sphere, that is contained in a linearization disc, and centered at the corresponding fixed point. The same is true for fields of prime characteristic. Hence, ergodicity on spheres inside a non-Archimedean linearization disc is only possible for fields of p -adic numbers.

Theorem 4.12 (Ergodic non-Archimedean linearization spheres).

Let K be a complete non-Archimedean field and let f be holomorphic on a disc U in K . Suppose that f has a linearization disc $\Delta \subset U$ and $S \subset \Delta$ is a sphere about the corresponding fixed point $x_0 \in K$. Then $f: S \rightarrow S$ is ergodic if and only if K is isomorphic to \mathbb{Q}_p and the multiplier is a generator of the group of units $(\mathbb{Z}/p^2\mathbb{Z})^$. Furthermore, if $K = \mathbb{Q}_p$ and λ is a generator of the group of units $(\mathbb{Z}/p^2\mathbb{Z})^*$, then the radius of Δ is $1/a$ (considered as a disc in \mathbb{Q}_p).*

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