On time-frequency analysis and pseudo-differential operators for vector-valued functions
On time-frequency analysis and pseudo-differential operators for vector-valued functions

Patrik Wahlberg

Växjö University Press

Series editor: Kerstin Brodén
ISSN: 1404-4307
ISBN: 978-91-7636-612-7
Printed by: Intellecta Docusys, Göteborg 2008
Abstract

This thesis treats different aspects of time-frequency analysis and pseudodifferential operators, with particular emphasis on techniques involving vector-valued functions and operator-valued symbols. The vector (Banach) space is either motivated by an application as in Paper I, where it is a space of stochastic variables, or is part of a general problem as in Paper II, or arises naturally from problems for scalar-valued operators and function spaces, as in Paper V. Paper III and IV fall outside this framework and treat algebraic aspects of time-frequency analysis and pseudodifferential operators for scalar-valued symbols and functions that are members of modulation spaces. Paper IV builds upon Paper III and applies the results to a filtering problem for second-order stochastic processes.

Paper I treats the Wigner distribution of a Gaussian weakly harmonizable stochastic process defined on $\mathbb{R}^d$. Paper II extends recent continuity results for pseudodifferential and localization operators, with symbols in modulation spaces, to the vector/operator-framework, where the vector space is a Hilbert or a Banach space. In Paper III we give algebraic results for the Weyl product acting on modulation spaces. We give sufficient conditions for a weighted modulation space to be an algebra under the Weyl product, and we also give necessary conditions for unweighted modulation spaces. In Paper IV we discretize the results of Paper III by means of a Gabor frame defined by a Gaussian function. Finally, Paper V deals with pseudodifferential operators with symbols that are almost periodic in the first variable. We show that such operators may be transformed to Fourier multiplier operators with operator-valued symbols such that the transformation preserves positivity and operator composition.
Acknowledgments

I thank my supervisor and friend Joachim Toft for accepting me as a Ph.D. student and for the stimulating collaboration on Paper III of the thesis. I am grateful for his suggestion of the problem, which originates in one of his papers, and for his crucial contribution to the paper. Also Paper IV originates from our collaboration, and in fact we worked out the essential calculations together in Växjö in January 2006 while working on Paper III.

I would also like to express my gratitude to my mathematics teachers in Lund: Alexandru Aleman, Adrian Constantin, Nils Dencker, Sigmundur Gudmundsson, Henrik Kalisch, Anders Melin, Arne Meurman and Nils Nilsson, as well as my teachers in Vienna: Hans Feichtinger and Karlheinz Gröchenig. They have all been excellent and inspiring teachers of graduate and seminar courses. Moreover, I have enjoyed nice discussions with Mikael Signahl on many mathematical topics. Finally I thank Per-Anders Svensson for helping me with typographical questions.
Introduction
In the thesis we will present results on time-frequency analysis and pseudodifferential operators. Three of the five papers in the thesis concerns the framework of vector-valued functions and operator-valued symbols, where vector space means a Hilbert space or a Banach space. Two papers treat algebraic questions for pseudodifferential operators with scalar-valued symbols and functions, that are members of modulation spaces.

As we hope to show, the extension to vector-valued functions and operator-valued symbols may be done for several reasons.

In the first place, as in Paper I and II, it is sometimes of interest in applications of time-frequency analysis and pseudodifferential calculus to study the vector/operator-valued extension. In Paper I we discuss the Wigner distribution of certain Gaussian stochastic processes, which are considered Hilbert space valued functions on $\mathbb{R}^d$. In Paper II we generalize some recent continuity results for pseudodifferential operators with modulation space symbols to the vector/operator-valued situation.

Secondly, as in Paper V, a problem in the theory of scalar-valued functions and symbols may lead to problems in the vector/operator-valued framework. More precisely, Paper V treats pseudodifferential operators whose symbols are almost periodic in the first variable. Introduction of Fourier series techniques transforms such operators to Fourier multiplier operators with operator-valued symbols.

As a background to the five papers in the thesis, this introduction is intended to give a brief survey of time-frequency analysis and pseudodifferential operators in the scalar-valued setup, necessarily incomplete due to the great scope and diversity of the subject. The account aims in particular at the relatively recent algebraic and continuity results for the nonsmooth symbol classes consisting of modulation spaces, that are essential background for Papers II, III and IV.

Time-frequency analysis is an interdisciplinary area of research, with branches in pure and applied mathematics, physics and signal theory. It has one root in the work by Wigner and Weyl on the mathematical foundations of quantum mechanics from the 1930s [10,11]. Wigner tried to create a probability density function for a particle in the phase (position-momentum) space by introduction of the Wigner distribution

$$W_{f,f}(x,\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x + \tau/2) \overline{f(x - \tau/2)} e^{-i\langle \xi,\tau \rangle} d\tau$$

associated to a function $f$ defined on $\mathbb{R}^d$. It gives a description of $f$ over the phase space $(x,\xi) \in \mathbb{R}^{2d}$, which in the quantum mechanical interpretation may be seen as a probability density function for a particle in phase space. However, the Wigner distribution is nonnegative only for a very particular class of functions, — Gaussians, according to Hudson’s theorem, so the probability density interpretation is dubious. This phenomenon is one aspect of the Heisenberg Uncertainty Principle, which says that a particle cannot be localized in small regions in the phase space. In fact, small regions in phase
space may not be assigned probability, so the fact that the Wigner distribution does not serve as a probability density is not surprising. Nevertheless, if the phase space resolution is decreased by convolving the Wigner distribution with a sufficiently wide Gaussian function, a nonnegative phase space distribution is obtained, which is consistent with the Uncertainty Principle due to the decrease of resolution.

In mathematics, Uncertainty Principles occur in several forms in harmonic analysis. One basic version is the inequality

\[
\left( \int_{\mathbb{R}} |(t - a)^2 f(t)|^2 dt \right)^{1/2} \left( \int_{\mathbb{R}} |(\xi - b)^2 \mathcal{F} f(\xi)|^2 d\xi \right)^{1/2} \geq \frac{1}{2} \|f\|_{L^2}^2
\]

for all \(a, b \in \mathbb{R}\) and \(f \in L^2(\mathbb{R})\), which is a rather immediate consequence of the Cauchy–Schwarz inequality and the Plancherel theorem. It says that a function \(f\) and its Fourier transform, here normalized for \(f \in S(\mathbb{R}^d)\) as

\[
\mathcal{F} f(\xi) = \hat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-i \langle \xi, x \rangle} dx,
\]

cannot be arbitrarily well localized simultaneously, which in the quantum interpretation means Heisenberg’s Uncertainty Principle.

One of the most fundamental ingredients in time-frequency analysis is the short-time Fourier transform (STFT)

\[
V_g f(t, \xi) = \mathcal{F} (f T_g)(\xi), \quad t \in \mathbb{R}^d, \quad \xi \in \mathbb{R}^d,
\]

where \(T_g(x) = g(x - t)\). It consists of a Fourier transform of \(f\) localized with a sliding cut-off function \(g\). This gives a function \(V_g f\) defined on the phase (time-frequency) space \((t, \xi) \in \mathbb{R}^{2d}\) that encodes many properties of both \(f\) and its Fourier transform. The Wigner distribution can be expressed as an STFT according to

\[
W_{f,f}(x, \xi) = 2^d e^{2i(x, \xi)} V_f(2x, 2\xi) \quad \text{where} \quad \hat{f}(x) = f(-x).
\]

If we choose \(f\) and \(g\) with \(\|f\|_{L^2(\mathbb{R}^d)} = \|g\|_{L^2(\mathbb{R}^d)} = 1\), then we get, on the one hand, by Plancherel’s theorem \(\|V_g f\|_{L^2(\mathbb{R}^{2d})} = 1\), and on the other hand, by the Cauchy–Schwarz inequality,

\[
|V_g f(t, \xi)| \leq (2\pi)^{-d/2} \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)} = (2\pi)^{-d/2}.
\]

Thus if \(\varepsilon > 0\) is a small number and \(U \subset \mathbb{R}^{2d}\) is a subset that captures the fraction \(1 - \varepsilon\) of the energy of \(V_g f\), we have

\[
1 - \varepsilon = \int_U |V_g f(t, \xi)|^2 dt d\xi \leq (2\pi)^{-d} \text{meas}(U).
\]

Thus, \(V_g f\) cannot be arbitrarily well concentrated in the phase space, which is another instance of the Uncertainty Principle.

When a physical object is localized in the space-velocity phase space by means of the radar principle, a short smooth signal pulse \(\varphi(t)\) is multiplied
by a high frequency carrier $e^{i\xi_0 t}$ and transmitted towards a target. This technique is used by several species of animals, e.g. bats and whales, to find prey with high-frequency sound waves. Let us assume that $\text{supp} \tilde{\varphi}$ is much smaller than $\xi_0$, so the Fourier transform of the transmitted signal $f(t) = \varphi(t)e^{i\xi_0 t}$ is supported in a small neighborhood around $\xi_0$. The reflected signal propagates back to the receiver and is detected as $f_r(t)$. It is delayed $\Delta t$ in proportion to the distance to the target, and all frequencies are Doppler shifted by roughly the same amount $\Delta \xi$ in proportion to the radial velocity of the target. Therefore $f_r(t) \approx e^{i\Delta \xi t}f(t-\Delta t)$. At the receiver one compares the signal $e^{i\xi t}f(t-s)$ and the reflected signal by the inner product

$$\left| \int_{\mathbb{R}} e^{i\xi t}f(t-s)e^{i\Delta \xi t}f(t-\Delta t)\,dt \right| = (2\pi)^{1/2} |V f(\Delta t - s, \Delta \xi - \xi)|.$$

One localizes the target in the phase space by maximization of $|V f|$ over $(s, \xi)$, but since $V f$ is not well concentrated in phase space there is an uncertainty in the localization. Thus Uncertainty Principles explains resolution limits for phase space localization of microscopic as well as macroscopic particles.

In the late 1940s, Gabor suggested expansions of functions in series of the form

$$f = \sum_{k,n \in \mathbb{Z}^d} c_{k,n}M_{bn}T_{ak}\tilde{\varphi},$$

where $(M_{bn}f)(t) = e^{2\pi ib\langle n,t \rangle}f(t)$, $a, b > 0$, the coefficients $c_{k,n} = c_{k,n}(f) \in \mathbb{C}$, and $\varphi$ is a test (“window”) function such that both itself and its Fourier transform are well localized (Gabor used a Gaussian as $\varphi$).

Mathematical time-frequency analysis emerged during the 1980s, and developed along the two lines of wavelets and Gabor theory [11]. Janssen initiated a rigorous approach to Gabor’s idea, studied the Wigner distribution, made many fundamental contributions and started a strong and still ongoing trend in mathematics known as Gabor or Weyl–Heisenberg frames [1, 11]. A major goal was to find necessary and sufficient conditions on the test function $\varphi$ and the parameters $a, b$ such that the collection $\mathcal{G} = \{M_{bn}T_{ak}\varphi\}_{n,k \in \mathbb{Z}^d}$ constitute a frame or a Riesz basis for $L^2(\mathbb{R}^d)$. A Riesz basis consists of the image of a bounded invertible linear operator acting on an orthonormal basis, whereas the condition for a frame is the following relaxation: there exists $0 < A \leq B < \infty$ such that

$$A\|f\|_{L^2}^2 \leq \sum_{k,n \in \mathbb{Z}^d} |\langle f, M_{bn}T_{ak}\varphi \rangle_{L^2(\mathbb{R}^d)}|^2 \leq B\|f\|_{L^2}^2, \quad f \in L^2(\mathbb{R}^d). \quad (1)$$

The theory of frames for Hilbert spaces was systematically developed by Duffin and Schaeffer already in 1952. The condition (1) leads to the unconditionally convergent Gabor expansion

$$f = \sum_{k,n \in \mathbb{Z}^d} \langle f, M_{bn}T_{ak}\varphi \rangle M_{bn}T_{ak}\varphi, \quad (2)$$
where $\tilde{\phi}$ denotes the canonical dual window function, defined by $\tilde{\phi} = S^{-1} \phi$ where $S = \sum_{k,n} \langle \cdot, M_{bn} T_{ak} \phi \rangle M_{bn} T_{ak} \phi$ is the frame operator defined by $\phi$ and $a, b$. A Gabor frame that is not a Riesz basis is redundant, that is, the expansion coefficients are nonunique. The condition $ab \leq 1$ is necessary for $G$ to be a frame. Moreover, Balian and Low proved the important negative result that if $ab = 1$ ("critical density") and $G$ is a frame, then both $\phi$ and $\hat{\phi}$ have poor localization properties [11]. An interpretation of this result is that Gabor frames should be redundant, since otherwise the lack of time and frequency localization of the window functions hampers the interpretation of the coefficient $c_{k,n}$ as representing the phase space contribution from the time-frequency point $(ak, bn)$ to $f$. Nonunique expansion coefficients may thus actually be desirable.

Pseudodifferential calculus was founded in the 1960s by Hörmander, Kohn and Nirenberg among others. It originates in problems in elliptic linear partial differential equations with variable coefficients and the then recently developed theory of singular integral operators. Ever since the 1960s it has been a rapidly growing part of analysis, with fundamental applications in several branches of mathematics and mathematical physics. Pseudodifferential operators has developed into a powerful and indispensable language to understand and solve linear partial differential equations. Its applications are far-reaching: from a tool for deep problems in pure mathematics, e.g. in differential geometry, topology, Hodge theory and the Atiyah–Singer index theorem, to mathematical modeling in physics, signal and image processing, and engineering, for example mobile radio communications. The literature is consequently enormous and difficult to survey. The books [15, 19] are standard references and [10, 11] contain presentations that clarify the connection to time-frequency analysis.

In pseudodifferential calculus one defines an operator using a symbol function $a$, defined on the phase space, by

$$a_t(x, D)f(x) = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} a(1-t)x + ty, \xi) e^{ix(1-t)x} f(y) dy d\xi. \quad (3)$$

Here $t$ is a real parameter which gives the Kohn–Nirenberg calculus when $t = 0$ and the Weyl calculus $a_{1/2}(x, D) = a^{w}(x, D)$ when $t = 1/2$. The fact that pseudodifferential operators contain partial differential operators with variable coefficients is most easily seen in the Kohn–Nirenberg calculus: since $\tilde{\partial}^\alpha f(\xi) = (i\xi)\alpha \hat{f}(\xi)$ for any multiindex $\alpha$, it follows from (3) and Fourier’s inversion formula that the symbol for a partial differential operator $P = \sum_{|\alpha| \leq m} a_{\alpha}(x) \partial^\alpha$ is

$$a(x, \xi) = \sum_{|\alpha| \leq m} a_{\alpha}(x)(i\xi)^\alpha. \quad (4)$$

If the symbol $a$ does not depend on $\xi$ then $a_t(x, D)$ is a multiplication operator, and if $a$ does not depend on $x$ then $a_t(x, D)$ is a convolution (Fourier multiplier) operator. Pseudodifferential operators may be defined
Pseudodifferential calculus concerns the correspondence between the symbol $a$ and the operator $a(x, D)$, sometimes also denoted $Op_t(a)$ (or $Op(a)$ when the parameter $t = 0$ or $t$ is understood or unessential). The idea of defining an operator from a phase space symbol function, called quantization, was formulated already by Weyl in the 1930s when he associated operators to the phase space variables, and lies at the heart of quantum mechanics. The Wigner distribution enters into the Weyl quantization by the formula

$$\langle a_{1/2}(x, D)f, g \rangle_{L^2(\mathbb{R}^d)} = (2\pi)^{-d/2}\langle a, W_{g,f} \rangle_{L^2(\mathbb{R}^{2d})}.$$  

If the symbol $a$ is the indicator of a measurable subset of the phase space, this formula suggests that the operator $a_{1/2}(x, D)$ acts by cutting out the content of a function $f$ in the chosen subset. However, by the Uncertainty Principle, a function cannot have time-frequency content in arbitrary sets, so this interpretation must be taken with a grain of salt.

If $a \in S'(\mathbb{R}^{2d})$ then (3) defines a bounded operator $S(\mathbb{R}^d) \mapsto S'(\mathbb{R}^d)$ by the Schwartz kernel theorem and the continuity of partial Fourier transformation on $S'(\mathbb{R}^{2d})$. Thus pseudodifferential operators can be seen as a framework for this large space of operators. However, it is often useful to impose more restrictive conditions on symbols, in order to obtain a family of operators that possess more restrictive continuity properties and admits composition.

The basic symbol classes are denoted $S^m$ where $m$ is real, and defined by the requirement that $a$ is a smooth function on $\mathbb{R}^{2d}$ that satisfies

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{m-|\beta|}$$

for all multiindices $\alpha$ and $\beta$ [15, 19]. Thus $a$ has polynomial growth (or decrease if $m < 0$) in the frequency variables, with decay that improves with differentiation in the frequency variables. Obviously this definition extends the symbols of partial differential operators with smooth coefficients (4).

The spaces $S^m$ are Fréchet spaces. The corresponding operator classes are an extension of the partial differential operators that contain parametrices (inverses modulo smoothing operators) for elliptic linear partial differential equations. New symbol classes of increasing generality were gradually introduced into pseudodifferential calculus, culminating in Hörmander’s Weyl calculus [15] where symbols are defined in terms of slowly varying metrics on the phase space and weight functions.
A considerable part of the literature on pseudodifferential operators concerns boundedness results. They are often formulated as conditions on a symbol $a$ that are sufficient for $Op(a)$ to extend to a continuous operator between certain function spaces. This type of results constitute an essential part of the theory, useful also to prove lower bound results.

As a first example of boundedness results, let us indicate that a symbol $a \in S^m$, for any $m \in \mathbb{R}$, gives rise to a bounded linear operator on the space $C^\infty_b(\mathbb{R}^d)$ of smooth functions with uniformly bounded derivatives. For simplicity we restrict to the Kohn–Nirenberg quantization $t = 0$, but the results are independent of $t \in \mathbb{R}$. For a symbol $a \in S^m$, the definition (3) makes sense as an iterated integral when $f \in S$ and defines a continuous operator on $S$ that extends to a continuous operator on $S'$. For $f \in C^\infty_b(\mathbb{R}^d)$, one defines

$$a(x,D)f(x) = \lim_{\varepsilon \to 0^+} (2\pi)^{-d} \iint_{\mathbb{R}^{2d}} \psi(\varepsilon \xi) \psi(\varepsilon y) a(x,\xi) e^{i(x-y,\xi)} f(y) \, dy \, d\xi$$

where $\psi \in C^\infty_0$ equals one in a neighborhood of the origin. This definition, the formulas

$$(1 + |\xi|^2)^{-N} (1 - \Delta_y)^N e^{i(x-y,\xi)} = e^{i(x-y,\xi)},$$

$$(1 + |x-y|^2)^{-M} (1 - \Delta_\xi)^M e^{i(x-y,\xi)} = e^{i(x-y,\xi)},$$

where $\Delta$ denotes the Laplacian, and integration by parts leads quickly to a proof of boundedness of the operator on $C^\infty_b(\mathbb{R}^d)$. Pseudodifferential operators also preserves smoothness locally, which is known as the pseudolocal property.

A fundamental boundedness result of global character is that $a \in S^0$ implies that $Op(a)$ is bounded on $L^2(\mathbb{R}^d)$. The result has the consequence that if $a \in S^m$ then $Op(a) : H_s(\mathbb{R}^d) \mapsto H_{s-m}(\mathbb{R}^d)$ continuously for any $s \in \mathbb{R}$. Here $H_s(\mathbb{R}^d)$ denotes the scale of Sobolev spaces, that quantifies smoothness, of tempered distributions $f$ such that

$$\|f\|_{H_s}^2 = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s \, d\xi < \infty.$$  

Modulation spaces will be a recurrent topic in the present thesis [11]. This scale of spaces was introduced by H. G. Feichtinger 1983 [6]. Originally they were defined on a locally compact Abelian group, but we will restrict to the special case $\mathbb{R}^d$. For a fixed window function $g \in \mathcal{S}(\mathbb{R}^d)$, a positive weight function $\omega$ and exponents $p, q \in [1, \infty]$, the weighted modulation space $M^p_q(\mathbb{R}^d)$ is defined with aid of the STFT as the subspace of $\mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{M^p_q(\omega)} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \left| V_g f(x,\xi) \omega(x,\xi) \right|^p \, dx \right)^{q/p} \, d\xi \right)^{1/q} < \infty. \quad (6)$$

If the weight is trivial $\omega \equiv 1$ the modulation space is denoted $M^{p,q}(\mathbb{R}^d)$, and $M^{p,p} = M^p$. The requirement (6) means that the scale of modulation
spaces quantifies the asymptotic decay of \( f \) in the time and frequency variables simultaneously, in a global way. The smallest space \( M^1 \), also denoted \( S_0 \), is called Feichtinger’s algebra [5], and \( M^1 \subset L^1 \cap \mathcal{F}L^1 \). The modulation spaces are Banach spaces. They comprise several other families of spaces, for example \( L^2 = M^2, L^2(\omega) = M^2(\omega) \) if \( \omega(x,\xi) = \omega(x) \) and \( H_s = M^2(\omega) \) if \( \omega(x,\xi) = (1 + |\xi|^2)^s/2 \). Moreover, \( M^p \) is invariant under Fourier transformation for \( p \in [1,\infty] \), and

\[
\mathcal{F}(\mathbb{R}^d) \subseteq M_{\omega_1}^{p_1,q_1}(\mathbb{R}^d) \subseteq M_{\omega_2}^{p_2,q_2}(\mathbb{R}^d) \subseteq \mathcal{F}'(\mathbb{R}^d)
\]

provided \( p_1 \leq p_2, q_1 \leq q_2, \omega_2 \leq C\omega_1 \), and \( \omega_1, \omega_2 \) are polynomially bounded.

The theory of modulation spaces was generalized to coorbit spaces by Feichtinger and Gröchenig in the late 1980s [7, 8], which is a broad framework including several classes of spaces, defined using group representation theory. Moreover, Gabor frame theory has been generalized from \( L^2 \) to modulation spaces by Feichtinger, Gröchenig and Leinert [9, 13]. They proved that if \( \mathcal{G} = \{ M_n T_{ak}\varphi \}_{n,k \in \mathbb{Z}^d} \) is a frame for \( L^2(\mathbb{R}^d) \) and \( \varphi \in M_1^1 \) for a certain class of weight functions \( \nu \), then \( \mathcal{G} \) is automatically a frame for all modulation spaces \( M_{\omega}^{p,q}(\mathbb{R}^d) \) where \( p, q \in [1,\infty] \) and \( \omega \) is \( \nu \)-moderate [11].

This means that the condition (1) implies the norm equivalence

\[
\|f\|_{M_{\omega}^{p,q}} \approx \left( \sum_{k \in \mathbb{Z}^d} \left( \sum_{n \in \mathbb{Z}^d} |V_{\mathcal{F}} f(an, 2\pi bk) \omega(an, 2\pi bk)|^p \right)^q / q \right)^{1/q},
\]

and the Gabor expansion (2) holds. Furthermore, \( \tilde{\varphi} \in M_1^1 \) which means that the Gabor coefficient \( \langle f, M_{bn} T_{ak}\varphi \rangle \) represents the time-frequency content of \( f \) at the phase space point \( (an, bk) \) accurately.

Another way of looking at modulation spaces is to view them as Fourier transforms of Wiener amalgam spaces [4, 6], which allows discretization of the global norm. More precisely, Parseval’s formula gives

\[
|V_{\mathcal{F}} f(x,\xi)| = |V_{\mathcal{F}} \tilde{f}(\xi,-x)| = |\mathcal{F}(\hat{f} T_{\xi} g)(-x)|.
\]

Assume for simplicity that \( \omega \equiv 1 \). Thus the inner \( L^p \)-norm in (6) is an \( \mathcal{F}L^p \)-norm, and

\[
\|f\|_{M_{\omega}^{p,q}} = \left( \int_{\mathbb{R}^d} \left\| \hat{f} T_{\xi} g \right\|_{\mathcal{F}L^p}^q \, d\xi \right)^{1/q} = \left( \sum_{j \in J} \|\hat{f} \psi_j\|_{\mathcal{F}L^p}^q \right)^{1/q} .
\]

The idea of Wiener amalgam spaces is to quantify distributions with two norms that controls the local and global behavior, respectively. In (8) \( \hat{f} \) is localized with a translated window function \( T_{\xi} g \) and measured locally in the norm \( \mathcal{F}L^p(\mathbb{R}^d) \), and then globally normed with respect to \( \xi \) in the \( L^1(\mathbb{R}^d) \) norm. The norm equivalence (8) holds provided \( \{\psi_j\}_{j \in J} \) is a countable set of functions that constitutes a bounded uniform partition of unity. This
Introduction

means that \( \{ \psi_j \}_{j \in J} \) is a set of of nonnegative functions that satisfy, for a
 discrete set \( \{ x_j \}_{j \in J} \) and a relatively compact set \( U \) with nonempty interior,

\[
\text{(i) supp} \psi_j \subset x_j + U \quad \forall j \in J, \quad \text{(ii) sup} \| \psi_j \|_{L^1(\mathbb{R}^d)} < \infty,\]

\[
\text{(iii) sup} \{ j : (x_i + U) \cap (x_j + U) \neq \emptyset \} < \infty, \quad \text{(iv) } \sum_{j \in J} \psi_j(x) = 1.\]

This form of the modulation space norm makes the connection to Besov
spaces visible. In fact, Besov spaces are defined by a similar decomposition
in the frequency domain, but instead of a uniform decomposition (8) one
has for Besov spaces a dyadic decomposition defined by functions of the
form \( \psi_j(\xi) = \psi(2^{-j} \xi) \).

The classical boundedness results for pseudodifferential operators concern
symbols that are smooth [15, 19]. Nevertheless, the fact that the smooth-
ness may be relaxed while maintaining operator boundedness properties has
been known for a long time. A basic example of nonsmooth symbols occurs
already in the theory of Fourier multiplier operators, that is, pseudodia-

terential operators where the symbol \( a(x, \xi) = a(\xi) \) does not depend on \( x \). In
fact, any symbol \( a \in L^\infty(\mathbb{R}^d) \) gives rise to an operator that is \( L^2 \)-bounded.

Another well-known instance of symbols of limited smoothness appears in
the Calderón–Vaillancourt theorem [10]. This result says that if all symbol
derivatives \( \partial^{\alpha}x\partial^{\beta}\xi a \) of order \( |\alpha| + |\beta| \leq 2d + 1 \) are uniformly bounded, then
\( \text{Op}(a) \) is bounded on \( L^2 \).

Apart from investigations of continuity properties, an important area in
pseudodifferential calculus is the study of composition of operators [10, 15,
19]. In the basic classical calculus with symbols \( S^m \) the fundamental result
reads: if \( a \in S^m \) and \( b \in S^{m'} \) then \( \text{Op}(a) \circ \text{Op}(b) = \text{Op}(c) \) where \( c \in S^{m+m'} \).
Moreover, there is an asymptotic expansion, according to
\[
c(x, \xi) \sim \sum_{\alpha} i^{-|\alpha|} \partial^{\alpha}_\xi a(x, \xi) \partial^{\alpha}_x b(x, \xi) / \alpha!,
\]
whose leading term is the product of symbols. The product of symbols
corresponding to composition of operators in the Weyl calculus is called
Weyl product (or sometimes twisted product) and denoted \( a \# b \), which means
that \( a_{1/2}(x, D) \circ b_{1/2}(x, D) = (a \# b)_{1/2}(x, D) \). Hence we have \( S^0 \# S^0 \subset S^0 \).

In the smooth calculus the basic composition result has been extended to
greater generality in the Weyl–Hörmander calculus [15].

During the 1990s it was discovered that time-frequency analysis may be
applied in pseudodifferential calculus [11]. Starting with Sjöstrand’s seminal
papers [17, 18], modulation spaces have been used as symbol classes.
Sjöstrand used the modulation space \( M^{\infty,1}(\mathbb{R}^{2d}) \) that consists of continuous
but not necessarily differentiable functions. He proved that \( a \in M^{\infty,1} \)
implies that \( \text{Op}(a) \) is \( L^2 \)-bounded, and he also proved that composition of
two operators with symbols in \( M^{\infty,1} \) gives a new operator with symbol
in \( M^{\infty,1} \), that is, \( M^{\infty,1} \# M^{\infty,1} \subset M^{\infty,1} \) (the algebra property). Finally,
he proved the Wiener property for $M^{\infty,1}$: if $a \in M^{\infty,1}$ and $Op(a)$ is $L^2$-invertible, then $Op(a)^{-1} = Op(b)$ where $b \in M^{\infty,1}$. The symbol space $M^{\infty,1}$ contains the classical symbol class $S_{0,0}^0$, for which the Wiener property was proved by Beals. Unaware of Sjöstrand’s result on $L^2$-continuity, Gröchenig and Heil [12] generalized the result to say that if $a \in M^{\infty,1}$ then $Op(a)$ is continuous on $M^{p,q}(\mathbb{R}^d)$ for all $p, q \in [1, \infty]$, which contains one of Sjöstrand’s results because $M^{2,2} = L^2$. The symbol class $M^{\infty,1}$ thus preserves global phase space concentration measured by the scale of modulation spaces. This continuity result has been extended to weighted modulation spaces by Gröchenig [11] and Toft [22]. Sjöstrand’s algebraic result $M^{\infty,1} \subseteq M^{\infty,1}$ has been elaborated by Gröchenig [14] who extended it to modulation spaces with nontrivial weights. He also extended the Wiener property to weighted modulation spaces $M^{p,q}_\omega$. Toft showed that $M^{\infty,1}$ contains a family $M^{p,1}$, $p \in [1, \infty]$, of subalgebras with respect to the Weyl product [20], and Labate [16] proved that $M^{p}_\omega$ are algebras under the Weyl product for $p \in [1, 2]$ and certain weights $\omega$.

During the last ten years, a growing community has studied weighted modulation spaces $M^{p,q}_\omega$ as symbol classes for pseudodifferential operators. A new direction of mathematical time-frequency analysis has emerged. Broadly speaking, this direction seems to be defined as a part of pseudodifferential calculus where Banach phase space techniques is often used instead of the more classical Fréchet spaces, and the use of differentiation is minimal.

As mentioned, the formula (3) is a rather general way to perform quantization, since it can be used for any continuous linear operator from the Schwartz functions to the tempered distributions. A less general framework is localization operators, which are also known under the names Anti-Wick and Toeplitz operators [3, 10, 19]. Localization operators are based on the fact that the STFT admits a reconstruction formula

$$f = \iint_{\mathbb{R}^{2d}} V_g f(t, 2\pi \xi) M_{\xi} T_{\xi} \xi g(t, 2\pi \xi) M_{\xi} T_{\xi} g dt d\xi, \quad \|g\|_{L^2} = (2\pi)^{d/4}. $$

Instead of reconstructing the STFT of a function one makes a multiplicative modification over the phase space first. That is, an operator is defined by

$$(A_a f)(x) = \iint_{\mathbb{R}^{2d}} a(t, 2\pi \xi) V_g f(t, 2\pi \xi) M_{\xi} T_{\xi} \xi g(t, 2\pi \xi) M_{\xi} T_{\xi} g(x) dt d\xi$$

for a localization symbol $a$. This class of operators originates from quantum field theory on the one hand, and signal theory on the other hand, where they are used as time-frequency filters. A localization operator can be written as a Weyl operator with symbol $(2\pi)^{-d} a * W_{\xi} T_{\xi}$. Since not all Weyl symbols are convolutions of this kind, this explains why not all Weyl operators are localization operators. Localization operators are currently an active area [2, 21, 22]. Cordero and Gröchenig [2] have proved that if $a \in M^{\infty}(\mathbb{R}^{2d})$ then the localization operator $A_a$ is continuous on all modulation spaces $M^{p,q}(\mathbb{R}^d)$, $p, q \in [1, \infty]$. 
Summary of included papers

Paper I: The Wigner distribution of Gaussian weakly harmonizable stochastic processes

Summary

This paper concerns the Wigner distribution of a complex-valued stochastic process \( X \) defined on \( \mathbb{R}^d \). We impose the restrictions that the stochastic process is Gaussian and weakly harmonizable. The latter condition means that the covariance function \( r(t, u) = \langle X(t), X(u) \rangle_H \), where \( H \) denotes a Hilbert space of second-order stochastic variables, can be represented as a Fourier transform of a spectral bimeasure. A bimeasure is a function \( \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d) \mapsto \mathbb{C} \), where \( \mathcal{B}(\mathbb{R}^d) \) denotes the Borel \( \sigma \)-algebra, that is a bounded measure in each variable, but not necessarily a bounded measure on \( \mathcal{B}(\mathbb{R}^{2d}) \). Weakly harmonizable stochastic processes are a generalization of weakly stationary processes, which means that \( r(t, u) = r^s(t-u) \) for a continuous function \( r^s \). In fact, these assumptions and Bochner’s theorem implies that \( r^s \) is the Fourier transform of a bounded nonnegative measure, since \( r^s \) is positive definite. The condition weakly harmonizable implies that the stochastic process \( X \) itself has a representation as a Fourier integral of a stochastic measure \( Z \), called spectral vector measure. By a stochastic measure we mean here a vector measure \( \mathcal{B}(\mathbb{R}^d) \mapsto H \). In the weakly stationary case the stochastic measure \( Z \) has the property \( (Z(A), Z(B))_H = 0 \) if \( A \cap B = \emptyset \).

The main goal of the paper is to prove that the assumptions Gaussianity and weak harmonizability are sufficient for a stochastic Wigner distribution to exist. The stochastic Wigner distribution is a map \( W : \mathbb{R}^d \times \mathcal{B}(\mathbb{R}^d) \mapsto H \), and may be interpreted as a time-frequency version of the stochastic process \( X \). The stochastic process \( W \) is continuous in the first variable (“time”), a property it has in common with \( X \), and is a \( \sigma \)-additive set function in the second argument (“frequency”), a property shared with \( Z \).

In the proof we use the theory of vector-valued measures and a result by R. C. Blei, which implies that a product of two bimeasures of the form \( m(A \times A', B \times B') := m_1(A \times B)m_2(A' \times B') \) may be extended to a bimeasure on \( \mathcal{B}(\mathbb{R}^{2d}) \times \mathcal{B}(\mathbb{R}^{2d}) \). Blei’s result is based on the Grothendieck inequality.

Furthermore, we specialize to the case of weakly stationary Gaussian stochastic processes, which implies that the probabilistic structure is translation-invariant. We present the following results for the Wigner distribution of a stationary Gaussian stochastic process: (i) \( (W(t, A), W(u, B))_H = 0 \) if \( A \cap B = \emptyset \) for all \( t, u \in \mathbb{R}^d \), (ii) \( W(\cdot, A) \) is weakly stationary for each \( A \in \mathcal{B}(\mathbb{R}^d) \), and (iii) \( (W(t + t_0, A), W(t_0, A))_H = (2\pi)^d W(r^s)(t, A) \) for all \( t_0 \in \mathbb{R}^d \), where \( W(r^s) \) is the Wigner distribution of the covariance function \( r^s \). These results are rather natural to anticipate from the interpretation of \( W \) as a time-frequency stochastic process corresponding to \( X \) and \( Z \), since \( X \) is stationary and \( Z \) has the property \( (Z(A), Z(B))_H = 0 \) if \( A \cap B = \emptyset \).
Paper II: Vector-valued modulation spaces and localization operators with operator-valued symbols

Summary

In this paper we generalize some continuity results for pseudodifferential operators with symbols in modulation spaces to the situation where the function spaces consists of vector-valued functions and symbols are operator-valued. The vector space $X$ is either a Hilbert space or a Banach space.

First we survey the basic theory of the short-time Fourier transform and weighted modulation spaces $M_{p,q}^m(\mathbb{R}^d, X)$ for vector-valued tempered distributions. When the vector space is a Hilbert space, modulation spaces have basic properties virtually identical to Feichtinger’s theory for complex-valued spaces. This is due to Kwapień’s result that Parseval’s formula holds for $X$-valued $L^2$-spaces if and only if $X$ is isomorphic to a Hilbert space. When the vector space is a Banach space, some modifications of the scalar theory occur. For example, the identity $L^2 = M^2$ for complex-valued modulation spaces is weakened to the embedding $L^p \cap \mathcal{F} L^p \subseteq M^p$ for all $p \in [1, r]$, where $r$ denotes the Fourier type of the vector (Banach) space and $1/p + 1/p' = 1$. The Fourier type of a Banach space is a number in the interval $[1, 2]$ defined as the maximum exponent such that a restricted form of the Hausdorff–Young inequality is valid, that is $\mathcal{F} : L^p \to L^{p'}$ for $p \in [1, r]$. We prove the duality result $(M_{p,q}^m(\mathbb{R}^d, X))' = M_{p',q'}^{1/m}(\mathbb{R}^d, X')$ provided $X$ is a reflexive Banach space and $(p, q) \in \{(1, 1)\} \cup (1, \infty) \times [1, \infty)$. As a connection to Paper I, we prove that each Hilbert space valued weakly stationary tempered distribution is a member of a modulation space $M_{m}^{\infty,1}$ where the weight $m(x, \xi) = (1 + |\xi|^2)^{-s/2}$ for some $s \geq 0$. We also prove that Hilbert space valued modulation spaces may be discretized in the same way as the complex-valued spaces.

In the latter part of the paper, we generalize some recent continuity results for localization and Weyl operators acting on modulation spaces to vector-valued distributions and operator-valued symbols. First we observe that a result by Boggiatto on modulation space continuity for localization operators with symbols in $L^{p,q}$ extends immediately to $X$-valued function spaces and operator-valued symbols, for any Banach space $X$. This result implies that a symbol in $L^\infty$ gives rise to a localization operator that is continuous on $M^{p,q}$ for all $p, q \in [1, \infty]$, in particular $M^2 = L^2$ (which holds in the scalar case). By means of a counterexample we show that the $L^2$-continuity is not always true in the vector-valued case.

Secondly, we give a version of Cordero and Gröchenig’s result that a localization operator with symbol in $M^\infty$ is bounded on $M^{p,q}$ for all $p, q \in [1, \infty]$. The result extends to Hilbert space-valued modulation spaces. In the procedure of extending the proof, we also obtain an extension of the results by Gröchenig, Heil, and Sjöstrand which say that a Weyl operator with symbol in $M_{m}^{\infty,1}$ is continuous on $M^{p,q}$ for all $p, q \in [1, \infty]$. 

Introduction

**Paper III: Weyl product algebras and modulation spaces**

Joint work with A. Holst and J. Toft.

**Summary**

In this paper we study algebraic properties of the Weyl product acting on modulation spaces defined on $\mathbb{R}^d$. The Weyl (or twisted) product is the symbol product corresponding to operator composition in the Weyl quantization of pseudodifferential operators.

A basic ingredient in our investigation is a formula by Toft for the short-time Fourier transform (STFT) of the Weyl product of two symbols, expressed in terms of the STFT of the involved symbols. We introduce a twisted domination condition on weight function triples that is adapted to the formula and reduces to a submultiplicativity property when all weights depend on the second (frequency) variable only. Using the STFT formula we prove continuity results for the Weyl product acting on modulation spaces of the form

$$\|a \# b\|_{M_{\omega_0}^p,q_0} \leq C \|a\|_{M_{\omega_1}^{p_1,q_1}} \|b\|_{M_{\omega_2}^{p_2,q_2}}.$$  

In the results we assume that the weight triples $(\omega_0, \omega_1, \omega_2)$ satisfy the twisted domination condition. Our continuity results are conditions on the exponents $p_j, q_j \in [1, \infty], j = 0, 1, 2$.

Using Minkowski’s, Hölder’s and Young’s inequalities in two different combinations we prove two continuity results that extend earlier continuity results for the Weyl product on modulation spaces. By means of multilinear complex interpolation theory, we extend the two sets of conditions on the exponents to

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{q_1} + \frac{1}{q_2} = 1 + \frac{1}{p_0} + \frac{1}{q_0}, \quad q_1, q_2 \leq q_0,$$

$$0 \leq \frac{1}{p_1} + \frac{1}{p_2} \leq \frac{1}{p_j} \quad \frac{1}{q_1} \leq \frac{1}{q_2} \leq \frac{1}{q_0}, \quad j = 0, 1, 2.$$

An immediate consequence of this result is that $M_{\omega}^{p,q}$ is an algebra under the Weyl product provided $p \in [1, \infty], q \in [1, \min(p, p')]$, and $(\omega, \omega, \omega)$ satisfies the twisted domination condition.

In the second part of the paper we treat necessary algebraic conditions for the Weyl product acting on modulation spaces. Here we restrict to the unweighted modulation spaces. Our results imply that $M_{\omega}^{p,q}$ is not an algebra when $p \in [1, \infty]$ and $q \in (\min(p, p'), \infty)$, which together with the above mentioned positive results gives a classification of all modulation spaces $M_{\omega}^{p,q}$, $p, q \in [1, \infty]$, regarding the algebra property. We also discuss conditions on the exponents $p, q$ that are necessary and sufficient for the module property $M_{\omega}^{p,q} \# M_{\omega}^{r,s} \subseteq M_{\omega}^{p,q}$. For $r \in [1, \infty]$ and $s \in [1, \min(r, r')]$ fixed, we obtain a region $P \subset [0, 1] \times [0, 1]$ such that $(1/p, 1/q) \in P$ implies the module property, and a region $Q \subset [0, 1] \times [0, 1]$ such that $(1/p, 1/q) \in Q$
implies that the module property is false. There is also a small area outside $P$ and $Q$ where we have no information. Finally we study some implications of our results for the Wiener property, that is, statements about the symbol of the inverse of an operator that is assumed to be $L^2$-invertible.

**Paper IV: Gabor discretization of the Weyl product for modulation spaces and filtering of nonstationary stochastic processes**

Joint work with P. Schreier.

**Summary**

The paper may be looked upon as a sequel to Paper III. Using Feichtinger, Gröchenig and Leinert’s result on Gabor frame expansions for weighted modulation spaces, we discretize the Weyl product

$$M^{p_1,q_1}_{(\omega_1)}(\mathbb{R}^{2d}) \times M^{p_2,q_2}_{(\omega_2)}(\mathbb{R}^{2d}) \mapsto M^{p_0,q_0}_{(\omega_0)}(\mathbb{R}^{2d}),$$

where the weights $\omega_j$ and exponents $p_j, q_j$ satisfy the criteria of Paper III.

We choose to work with a Gabor frame defined by a Gauss function which admits explicit formulas. The result is a bilinear map

$$l^{p_1,q_1}_{(\omega_1)}(\mathbb{Z}^{4d}) \times l^{p_2,q_2}_{(\omega_2)}(\mathbb{Z}^{4d}) \mapsto l^{p_0,q_0}_{(\omega_0)}(\mathbb{Z}^{4d}),$$

where $l^{p_j,q_j}_{(\omega_j)}$ denotes weighted mixed-norm sequence spaces. If one factor is considered fixed, this map may be considered a matrix multiplication where the matrix depends linearly on the fixed factor. This is denoted $c_0 = M(c_2) \cdot c_1 = \tilde{M}(c_1) \cdot c_2$, where $c_j$ are the Gabor coefficient vectors for the symbols $a_j$, $j = 0, 1, 2$, $a_0 = a_1 \# a_2$, and $M(c_2), \tilde{M}(c_1)$ are matrices depending on $c_2$ and $c_1$, corresponding to right and left composition of Weyl operators, respectively.

In the case $(p_2, q_2) = (\infty, 1)$ and $\omega_0 = \omega_1$, the matrix $M(c_2)$ is bounded on $l^{p,q}_{(\omega_0)}$ for all $p, q \in [1, \infty]$. Moreover, using results by Sjöstrand and Gröchenig we prove that if the Weyl operator $a_2^w$ corresponding to the symbol $a_2$ is $L^2$-invertible and $\omega_2(X, Y) = \langle Y \rangle^s$ where $s \geq 0$, then $M(c_2)$ is invertible on $l^{p,q}_{(\omega_0)}$ for all $p, q \in [1, \infty]$. Analogous properties hold for the matrix $\tilde{M}(c_1)$.

If $(p_2, q_2) = (\infty, 1)$ and $\omega_2(X, Y) = \omega_2(Y)$ is sufficiently rapidly growing, we prove that the matrix $M(c_2)$ decays rapidly off the diagonal. More precisely, if $\omega_2(Y) = \langle Y \rangle^s$ where $s > 2d$, then the off-diagonal decay is polynomial of a positive order that increases with $s$. If $\omega_2(Y) = \exp(2\alpha |Y|)$, where $\alpha > 0$ is a certain constant that depends on the frame lattice parameters, then the off-diagonal decay is exponential. We reach these results using results by Janssen on the exponential decay of canonical dual Gabor windows for exponentially decaying windows. The assumption that $a_2^w$ is $L^2$-invertible implies that also the inverse matrix $M(c_2)^{-1}$ has similar off-diagonal decay properties.
As an application of the results, we discuss the problem of statistical estimation of a noise-corrupted second-order stochastic process. The equation for the linear operator ("filter") that gives minimum mean square error has the form \( a_0 w + a_1 a_2 = a_0 w + a_2 \) where \( a_0, a_2 \) are known symbols and \( a_1 \) is the unknown filter operator symbol. We obtain a time-frequency formula for the Gabor coefficients of the Weyl symbol for the optimal filter.

**Paper V: A transformation of almost periodic pseudodifferential operators to Fourier multiplier operators with operator-valued symbols**

**Summary**

We treat pseudodifferential operators on \( \mathbb{R}^d \) in the Kohn–Nirenberg quantization, where the symbol \( a(\cdot, \xi) \) is almost periodic (a.p.) for each \( \xi \in \mathbb{R}^d \), and belongs to a Hörmander class \( S^{\rho, \delta}_m \). We study the symbol transformation \( a \mapsto U(a) \) defined by

\[
U(a)(\xi) = M_x(a(x, \xi - \lambda))e^{-2\pi i x \cdot (\lambda - \lambda')},
\]

where \( M_x \) denotes the mean value for a.p. functions, which was introduced, for operator kernels rather than symbols, by E. Gladyshev. \( U(a)(\xi) \) can be considered a matrix indexed by \( (\lambda, \lambda') \in \Lambda \times \Lambda \) where \( \Lambda \) is the set of frequencies that occur in \( \{a(\cdot, \xi)\}_{\xi \in \mathbb{R}^d} \). Thus \( U(a) \) may be considered the symbol of an operator-valued Fourier multiplier operator \( U(a)(D) \).

Using results by M. A. Shubin, we show that the transformation respects operator composition,

\[
U(a\#_b)(\xi) = U(a)(\xi) \cdot U(b)(\xi),
\]

where \( a(D) \circ b(x, D) = (a\#_b)(x, D) \). Moreover, \( a(x, D) \geq 0 \) if and only if \( U(a)(D) \geq 0 \), which is similar to Gladyshev’s result. Positivity and boundedness on Sobolev–Besicovitch spaces of \( a(x, D) \) are encoded in the matrix \( U(a)(0) \). Finally we show that the condition that \( a(\cdot, \xi) \) is an a.p. function for all \( \xi \in \mathbb{R}^d \) is invariant under the parameter \( t \in \mathbb{R} \) in a standard family of quantizations that includes the Weyl quantization.

**Bibliography**


Bibliography


