Multi-oriented Symplectic Geometry and the Extension of Path Intersection Indices
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Växjö University Press

Series editors: Tommy Book and Kerstin Brodén
ISSN: 1404-4307
ISBN: 91-7636-477-1
Printed by: Intellecta Docusys, Gothenburg 2005
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I dedicate this work to my wife Theresie and my daughters Mathilda and Amanda. They have carried me through it all. I love you!
I thank my God and Lord Jesus Christ with all my heart for having blessed me with the family I have.
Preface

The aim of this doctoral thesis is to give a detailed account of symplectic geometry and of some of its applications to the theory of Lagrangian and symplectic path intersections.

Some of the results contained in this thesis have appeared in the following joint work:


The original material of last Section, devoted an extension of the Conley–Zehnder index,

4. Extension of the Conley–Zehnder index and calculation of the Maslov-Type Index intervening in Gutzwiller’s trace formula.

is available as a preprint on arXiv.

I take the opportunity of expressing many thanks to Andrei Khrennikov for having accepted me as a Ph.D. student at MSI (University of Växjö). Without his active and kind support I would never have been able to complete this thesis in reasonable time. It is also my duty to thank the following mathematicians for stimulating conversations from which I have learnt much: Bernhelm Booss-Bavnbek, Kenro Furutani and Maurice de Gosson.
Introduction

Symplectic geometry, whose origin can be traced back to the early work of Lagrange on Celestial Mechanics, has become, since the early 1970’s, an increasingly active field of research in mathematics. It has greatly benefited from modern work in Hamiltonian mechanics (of which it is the natural language). However, symplectic geometry is definitely not a new theory, even if it has recently experienced a drastic rejuvenation. It has been well-known for a very long time (especially among physicists) under its older name, namely the theory of canonical transformations. Historically we can trace back the birth of symplectic geometry to Lagrange’s (1736–1813) work on celestial mechanics. His work was furthered –among others– by Poisson (1781–1840), Jacobi (1804–1851), Hamilton (1805–1865) and Liouville (1809–1882). We are perhaps witnessing a “symplectization of science” as Gotay and Isenberg call it in [32]. It is argued in this paper that every theorem in Riemannian geometry has a symplectic counterpart.

The term Maslov index is a collective denomination for a whole class of integer (or sometimes half-integer) valued functions defined on the covering spaces of the Lagrangian Grassmannian and of the symplectic group and having characteristic topological and cohomological properties. Historically one can trace the history of the Maslov index back to J. B. Keller, who in his seminal work [42] showed the need for the use of an intersection index to express in a mathematically rigorous manner the Bohr–Sommerfeld quantization rules of the classical trajectories, and also to elucidate the problems due to the appearance of caustics, which are obstructions to the construction of global solutions in the WKB approximation method of quantum mechanics. A few years later, the Russian mathematician V. P. Maslov published his famous book [56] where he systematically developed Keller’s ideas and where he showed that the caustics are only apparent singularities which can be eliminated by the use of a convenient Fourier transform (also see Maslov and Fedoriuk [57]). To perform this task, Maslov used an index that described the phase jumps when one crosses the caustics. the definition of this index was then clarified by V. I. Arnold [3] (also see the appendix to the French translation of Maslov’s book [56]). Arnold’s study was eventually taken up by the French mathematician J. Leray [50, 51, 52] who showed the existence of a unique locally constant integer-valued function defined on transverse pairs of elements of the universal covering $\text{Lag}_\infty(n)$ of the Lagrangian Grassmannian and whose coboundary is the index of inertia of a triple of Lagrangian planes. We have chosen to call it the “Leray index” although Leray himself called it, with his customary modesty “Maslov index”. In addition to fairness, this choice has the advantage that it will allow us to use the name Maslov index for the restriction of Leray’s index to loops (as is customary in the literature devoted to quantization). We mention that Leray’s work has been taken up, extended, and pursued by several authors, for instance Cappell et al. [14], Dazord [17], and M. de Gosson [25, 26].
This thesis is solely devoted to problems in symplectic geometry on finite dimensional spaces. Of course, similar problems can be studied for infinite dimensional spaces. We just mention that infinite dimensional symplectic geometry was intensively used by Andrei Khrennikov in applications to the foundations of quantum mechanics, see [43]–[45]. See also his investigations on the infinite dimensional Liouville equation [46] and the correspondence principle [47] (in [48] super-symmetric models were considered while non-archimedean ones were considered in [49])
1 Symplectic Geometry

The purpose of this Chapter is to review the notions of symplectic geometry that will be used in the rest of the thesis. There are many good texts on symplectic geometry. To cite a few (in alphabetical order): Abraham and Marsden [1], Libermann and Marle [53], the first Chapter of McDuff and Salamon [55], Vaisman [71]. The latter contains an interesting study of characteristic classes intervening in symplectic geometry. A nicely written review of symplectic geometry and of its applications is Gotay and Isenberg’s paper [32] on the “symplectization of science”. Other interesting reviews of symplectic geometry can be found in Weinstein [76, 77].

1.1 Symplectic Vector Spaces

We will exclusively deal with finite-dimensional real symplectic spaces. We begin by discussing the notion of symplectic form on a vector space. Symplectic forms allow the definition of symplectic bases, which are the analogues of orthonormal bases in Euclidean geometry.

Generalities

Let $E$ be a real vector space. Its generic vector will be denoted by $z$. A symplectic form (or skew-product) on $E$ is a mapping $\omega : E \times E \rightarrow \mathbb{R}$ which is

- linear in each of its components:

  \[
  \omega(\alpha_1 z_1 + \alpha_2 z_2, z') = \alpha_1 \omega(z_1, z') + \alpha_2 \omega(z_2, z') \\
  \omega(z, \alpha_1 z_1' + \alpha_2 z_2') = \alpha_1 \omega(z, z_1') + \alpha_2 \omega(z, z_2')
  \]

  for all $z, z', z_1, z_1', z_2, z_2'$ in $E$ and $\alpha_1, \alpha_1', \alpha_2, \alpha_2'$ in $\mathbb{R}$;

- antisymmetric (one also says skew-symmetric):

  \[
  \omega(z, z') = -\omega(z', z) \quad \text{for all } z, z' \in E
  \]

  (equivalently, in view of the bilinearity of $\omega$: $\omega(z, z) = 0$ for all $z \in E$):

- non-degenerate:

  \[
  \omega(z, z') = 0 \quad \text{for all } z \in E \text{ if and only if } z' = 0.
  \]

**Definition 1** A real symplectic space is a pair $(E, \omega)$ where $E$ is a real vector space on $\mathbb{R}$ and $\omega$ a symplectic form. The dimension of $(E, \omega)$ is, by definition, the dimension of $E$.

The most basic –and important– example of a finite-dimensional symplectic space is the standard symplectic space $(\mathbb{R}^{2n}_z, \sigma)$ where $\sigma$ (the standard symplectic form) is defined by
\[
\sigma(z, z') = \sum_{j=1}^{n} p_j x'_j - p'_j x_j
\]  

(1.1)

when \( z = (x_1, \ldots, x_n; p_1, \ldots, p_n) \) and \( z' = (x'_1, \ldots, x'_n; p'_1, \ldots, p'_n) \). In particular, when \( n = 1 \),

\[
\sigma(z, z') = -\det(z, z').
\]

In the general case \( \sigma(z, z') \) is (up to the sign) the sum of the areas of the parallelograms spanned by the projections of \( z \) and \( z' \) on the coordinate planes \( x_j, p_j \).

Here is a coordinate-free variant of the standard symplectic space: set \( X = \mathbb{R}^n \) and define a mapping \( \xi : X \oplus X^* \to \mathbb{R} \) by

\[
\xi(z, z') = \langle p, x' \rangle - \langle p', x \rangle
\]

(1.2)

if \( z = (x, p) \), \( z' = (x', p') \). That mapping is then a symplectic form on \( X \oplus \ell_P \). Expressing \( z \) and \( z' \) in the canonical bases of \( X \) and \( \ell_P \) then identifies \( (\mathbb{R}^{2n}, \sigma) \) with \( (X \oplus X^*, \xi) \).

**Remark 2** Let \( \Phi \) be the mapping \( E \to E^* \) which to every \( z \in E \) associates the linear form \( \Phi_z \) defined by

\[
\Phi_z(z') = \omega(z, z').
\]

(1.3)

The non-degeneracy of the symplectic form can be restated as follows:

\( \omega \) is non-degenerate \( \iff \Phi \) is a monomorphism \( E \to E^* \).

We will say that two symplectic spaces \( (E, \omega) \) and \( (E', \omega') \) are isomorphic if there exists a vector space isomorphism \( s : E \to E' \) such that

\[
\omega'(s(z), s(z')) = \omega'(z, z')
\]

for all \( z, z' \) in \( E \). Two isomorphic symplectic spaces thus have same dimension. We will see below that, conversely, two finite-dimensional symplectic spaces are always isomorphic in the sense above if they have same dimension. The proof of this property requires the notion of symplectic basis, studied in next subsection.

Let \( (E_1, \omega_1) \) and \( (E_2, \omega_2) \) be two arbitrary symplectic spaces. The mapping

\[
\omega = \omega_1 \oplus \omega_2 : E_1 \oplus E_2 \to \mathbb{R}
\]

defined by

\[
\omega(z_1 \oplus z_2; z'_1 \oplus z'_2) = \omega_1(z_1, z'_1) + \omega_2(z_2, z'_2)
\]

(1.4)

for \( z_1 \oplus z_2, z'_1 \oplus z'_2 \in E_1 \oplus E_2 \) is obviously antisymmetric and bilinear. It is also non-degenerate: assume that

\[
\omega(z_1 \oplus z_2; z'_1 \oplus z'_2) = 0 \text{ for all } z'_1 \oplus z'_2 \in E_1 \oplus E_2;
\]
then, in particular, $\omega_1(z_1, z'_1) = \omega_2(z_2, z'_2) = 0$ for all $(z'_1, z'_2)$ and hence $z_1 = z_2 = 0$. The pair

$$(E, \omega) = (E_1 \oplus E_2, \omega_1 \oplus \omega_2)$$

is thus a symplectic space. It is called the \textit{direct sum} of $(E_1, \omega_1)$ and $(E_2, \omega_2)$.

**Example 3** Let $(\mathbb{R}^{2n}_z, \sigma)$ be the standard symplectic space. Then we can define on $\mathbb{R}^{2n}_z \oplus \mathbb{R}^{2n}_z$ two symplectic forms $\sigma^\oplus$ and $\sigma^\ominus$ by

$$\sigma^\oplus(z_1, z_2; z'_1, z'_2) = \sigma(z_1, z'_1) + \sigma(z_2, z'_2)$$
$$\sigma^\ominus(z_1, z_2; z'_1, z'_2) = \sigma(z_1, z'_1) - \sigma(z_2, z'_2).$$

The corresponding symplectic spaces are denoted $(\mathbb{R}^{2n}_z \oplus \mathbb{R}^{2n}_z, \sigma^\oplus)$ and $(\mathbb{R}^{2n}_z \oplus \mathbb{R}^{2n}_z, \sigma^\ominus)$.

Let us briefly discuss the notion of complex structure on a vector space. We refer to the literature, for instance Hofer–Zehnder [38] or McDuff–Salamon [55], where this notion is emphasized and studied in detail.

We begin by noting that the standard symplectic form $\sigma$ on $\mathbb{R}^{2n}_z$ can be expressed in matrix form as

$$\sigma(z, z') = (z')^T J z, \quad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \quad (1.5)$$

where $0$ and $I$ stand for the $n \times n$ zero and identity matrices. The matrix $J$ is called the standard symplectic matrix. Alternatively, we can view $\mathbb{R}^{2n}_z$ as the complex vector space $\mathbb{C}^n$ by identifying $(x, p)$ with $x + ip$. The standard symplectic form can with this convention be written as

$$\sigma(z, z') = \text{Im} \langle z, z' \rangle_{\mathbb{C}^n} \quad (1.6)$$

where $\langle \cdot, \cdot \rangle_{\mathbb{C}^n}$ is the usual (Hermitian) scalar product on $\mathbb{C}^n$. Notice that multiplication of $x + ip$ by $i$ then corresponds to multiplication of $(x, p)$ by $-J$. These considerations lead to the following definition:

**Definition 4** A “complex structure” on a vector space $E$ is any linear isomorphism $j : E \rightarrow E$ such that $j^2 = -I$.

Since $\det(j^2) = (-1)^{\dim E} > 0$ we must have $\dim E = 2n$ so that only even-dimensional vector spaces can have a complex structure.

**Symplectic bases**

We begin by observing that the dimension of a finite-dimensional symplectic vector is always even: choosing a scalar product $\langle \cdot, \cdot \rangle_E$ on $E$, there exists an endomorphism $j$ of $E$ such that $\omega(z, z') = \langle j(z), z' \rangle_E$ and the antisymmetry
of $\omega$ is then equivalent to $j^T = -j$ where $^T$ denotes here transposition with respect to $\langle \cdot, \cdot \rangle_E$. Hence

$$\det j = (-1)^{\dim E} \det j^T = (-1)^{\dim E} \det j.$$ 

The non-degeneracy of $\omega$ implies that $\det j \neq 0$ so that $(-1)^{\dim E} = 1$, hence $\dim E = 2n$ for some integer $n$, as claimed.

**Definition 5** A set $\mathcal{B}$ of vectors

$$\mathcal{B} = \{e_1, \ldots, e_n\} \cup \{f_1, \ldots, f_n\}$$

of $E$ is called a “symplectic basis” of $(E, \omega)$ if the conditions

$$\omega(e_i, e_j) = \omega(f_i, f_j) = 0, \quad \omega(f_i, e_j) = \delta_{ij} \quad \text{for} \quad 1 \leq i, j \leq n \quad (1.7)$$

hold ($\delta_{ij}$ is the Kronecker index: $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$).

It is clear that the conditions (1.7) automatically ensure the linear independence of the vectors $e_i, f_j$ for $1 \leq i, j \leq n$ (hence a symplectic basis really is a basis).

Here is a trivial example of a symplectic basis: define vectors $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ in $\mathbb{R}_2^{2n}$ by

$$e_i = (c_i, 0), \quad e_i = (0, c_i)$$

where $(c_i)$ is the canonical basis of $\mathbb{R}^n$. (For instance, if $n = 1$, $e_1 = (1, 0)$ and $f_1 = (0, 1)$). These vectors form the canonical basis

$$\mathcal{B} = \{e_1, \ldots, e_n\} \cup \{f_1, \ldots, f_n\}$$

of the standard symplectic space $(\mathbb{R}_2^{2n}, \sigma)$. One immediately checks that they satisfy the conditions $\sigma(e_i, e_j) = 0, \sigma(f_i, f_j) = 0$, and $\sigma(f_i, e_j) = \delta_{ij}$ for $1 \leq i, j \leq n$. This basis is called the canonical symplectic basis.

It is not immediately obvious that each symplectic space has a symplectic basis. That this is however true will be established in Section 1.2, where we will in addition prove the symplectic equivalent of the Gram–Schmidt orthonormalization process.

Taking for granted the existence of symplectic bases we can prove that all symplectic vector spaces of same finite dimension $2n$ are isomorphic: let $(E, \omega)$ and $(E', \omega')$ have symplectic bases $\{e_i, f_j; 1 \leq i, j \leq n\}$ and $\{e'_i, f'_j; 1 \leq i, j \leq n\}$ and consider the linear isomorphism $s : E \longrightarrow E'$ defined by the conditions $s(e_i) = e'_i$ and $s(f_i) = f'_i$ for $1 \leq i \leq n$. That $s$ is symplectic is clear since we have

$$\omega'(s(e_i), s(e_j)) = \omega'(e'_i, e'_j) = 0$$

$$\omega'(s(f_i), s(f_j)) = \omega'(f'_i, f'_j) = 0$$

$$\omega'(s(f_j), s(e_i)) = \omega'(f'_j, e'_i) = \delta_{ij}$$
1 Symplectic Geometry

for $1 \leq i, j \leq n$.

The set of all symplectic automorphisms $(E, \omega) \rightarrow (E, \omega)$ form a group $\text{Sp}(E, \omega)$ –the symplectic group of $(E, \omega)$– for the composition law. Indeed, the identity is obviously symplectic, and so is the compose of two symplectic transformations. If $\omega(s(z), s(z')) = \omega(z, z')$ then, replacing $z$ and $z'$ by $s^{-1}(z)$ and $s^{-1}(z')$, we have $\omega(z, z') = \omega(s^{-1}(z), s^{-1}(z'))$ so that $s^{-1}$ is symplectic as well.

It turns out that all symplectic groups corresponding to symplectic spaces of same dimension are isomorphic:

**Proposition 6** Let $(E, \omega)$ and $(E', \omega')$ be two symplectic spaces of same dimension $2n$. The symplectic groups $\text{Sp}(E, \omega)$ and $\text{Sp}(E', \omega')$ are isomorphic.

**Proof.** Let $\Phi$ be a symplectic isomorphism $(E, \omega) \rightarrow (E', \omega')$ and define a mapping $f_\Phi : \text{Sp}(E, \omega) \rightarrow \text{Sp}(E', \omega')$ by $f_\Phi(s) = f \circ s \circ f^{-1}$. Clearly $f_\Phi(ss') = f_\Phi(s)\Phi(s')$ hence $f_\Phi$ is a group monomorphism. The condition $f_\Phi(S) = I$ (the identity in $\text{Sp}(E', \omega')$) is equivalent to $f \circ s = f$ and hence to $s = I$ (the identity in $\text{Sp}(E, \omega)$); $f_\Phi$ is thus injective. It is also surjective because $s = f^{-1} \circ s' \circ f$ is a solution of the equation $f \circ s \circ f^{-1} = s'$.

These results show that it is no restriction to study finite-dimensional symplectic geometry by singling out one particular symplectic space, for instance the standard symplectic space, or its variants. This will be done in next section.

Note that if $B_1 = \{e_1, f_1; 1 \leq i, j \leq n_1\}$ and $B_2 = \{e_2, f_2; 1 \leq k, \ell \leq n_2\}$ are symplectic bases of $(E_1, \omega_1)$ and $(E_2, \omega_2)$ then

$$B = \{e_{1i} \oplus e_{2k}, f_{1j} \oplus f_{2\ell}; 1 \leq i, j \leq n_1 + n_2\}$$

is a symplectic basis of $(E_1 \oplus E_2, \omega_1 \oplus \omega_2)$.

**Differential interpretation of $\sigma$**

A differential two-form on a vector space $\mathbb{R}^m$ is the assignment to every $x \in \mathbb{R}^m$ of a linear combination

$$\beta_x = \sum_{i<j \leq m} b_{ij}(x)dx_i \wedge dx_j$$

where the $b_{ij}$ are (usually) chosen to be $C^\infty$ functions, and the wedge product $dx_i \wedge dx_j$ is defined by

$$dx_i \wedge dx_j = dx_i \otimes dx_j - dx_j \otimes dx_i$$

where $dx_i : \mathbb{R}^m \rightarrow \mathbb{R}$ is the projection on the $i$-th coordinate. Returning to $\mathbb{R}^{2n}_z$, we have

$$dp_j \wedge dx_j(z, z') = p_j x'_j - p'_j x_j$$
hence we can identify the standard symplectic form $\sigma$ with the differential 2-form
\[
dp \wedge dx = \sum_{j=1}^{n} dp_j \wedge dx_j = d\left(\sum_{j=1}^{n} p_j dx_j\right);
\]
the differential one-form
\[
pdx = \sum_{j=1}^{n} p_j dx_j
\]
plays a fundamental role in both classical and quantum mechanics. It is sometimes called the (reduced) action form in physics and the Liouville form in mathematics.

Since we are in the business of differential form, let us make the following remark: the exterior derivative of $dp_j \wedge dx_j$ is
\[
d(dp_j \wedge dx_j) = d(dp_j) \wedge dx_j + dp_j \wedge d(dx_j) = 0
\]
so that we have
\[
d\sigma = d(dp \wedge dx) = 0.
\]

The standard symplectic form is thus a closed non-degenerate 2-form on $\mathbb{R}^{2n}$. This remark is the starting point of the generalization of the notion of symplectic form to a class of manifolds: a symplectic manifold is a pair $(M, \omega)$ where $M$ is a differential manifold $M$ and $\omega$ a non-degenerate closed 2-form on $M$. This means that every tangent plane $T_z M$ carries a symplectic form $\omega_z$ varying smoothly with $z \in M$. As a consequence, a symplectic manifold always has even dimension (we will not discuss the infinite-dimensional case).

One basic example of a symplectic manifold is the cotangent bundle $T^*\mathbb{V}^n$ of a manifold $\mathbb{V}^n$. The symplectic form is here the “canonical 2-form” on $T^*\mathbb{V}^n$, defined as follows: let $\pi : T^*\mathbb{V}^n \longrightarrow \mathbb{V}^n$ be the projection to the base and define a 1-form $\lambda$ on $T^*\mathbb{V}^n$ by $\lambda_z(X) = p(\pi_*(X))$ for a tangent vector $\mathbb{V}^n$ to $T^*\mathbb{V}^n$ at $z = (z, p)$. The form $\lambda$ is called the “canonical 1-form” on $T^*\mathbb{V}^n$. Its exterior derivative $\omega = d\lambda$ is called the “canonical 2-form” on $T^*\mathbb{V}^n$ and one easily checks that it indeed is a symplectic form (in local coordinates $\lambda = pdx$ and $\sigma = dp \wedge dx$). The symplectic manifold $(T^*\mathbb{V}^n, \omega)$ is in a sense the most straightforward non-linear version of the standard symplectic space (to which it reduces when $\mathbb{V}^n = \mathbb{R}^n_\times$ since $T^*\mathbb{R}^n_\times$ is just $\mathbb{R}^n_\times \times (\mathbb{R}^n_\times)^* \equiv \mathbb{R}^{2n}$).

A symplectic manifold is always orientable: the non-degeneracy of $\omega$ namely implies that the $2n$-form
\[
\omega^{\wedge n} = \underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ factors}}
\]
ever vanishes on $M$ and is thus a volume form on $M$. We will call the exterior power $\omega^{\wedge n}$ the symplectic volume form. When $M$ is the standard symplectic space then the usual volume form on $\mathbb{R}^{2n}$
\[
\text{Vol}_{2n} = (dp_1 \wedge \cdots \wedge dp_n) \wedge (dx_1 \wedge \cdots \wedge dx_n)
\]
is related to the symplectic volume form by

\[ \text{Vol}_{2n} = (-1)^{n(n-1)/2} \frac{1}{n!} \sigma \wedge^n. \tag{1.8} \]

Notice that, as a consequence, every cotangent bundle \( T^* \mathbb{R}^n \) is an oriented manifold!

### 1.2 Skew-Orthogonality

All vectors in a symplectic space \((E, \omega)\) are skew–orthogonal (one also says “isotropic”) in view of the antisymmetry of a symplectic form: \( \omega(z, z') = 0 \) for all \( z \in E \). The notion of length therefore does not make sense in symplectic geometry (whereas the notion of area does). The notion of “skew orthogonality” is extremely interesting in the sense that it allows the definition of subspaces of a symplectic space having special properties.

#### Isotropic and Lagrangian subspaces

Let \( M \) be an arbitrary subset of a symplectic space \((E, \omega)\). The skew-orthogonal set to \( M \) (one also says annihilator) is by definition the set

\[ M^\omega = \{ z \in E : \omega(z, z') = 0, \ \forall z' \in M \}. \]

Notice that we always have

\[ M \subset N \implies N^\omega \subset M^\omega \text{ and } (M^\omega)^\omega \subset M. \]

It is traditional to classify subsets \( M \) of a symplectic space \((E, \omega)\) as follows: \( M \subset E \) is said to be:

- **Isotropic** if \( M^\omega \supset M : \omega(z, z') = 0 \) for all \( z, z' \in M \);
- **Coisotropic** (or: *involutive*) if \( M^\omega \subset M \);
- **Lagrangian** if \( M \) is both isotropic and co-isotropic: \( M^\omega = M \);
- **Symplectic** if \( M \cap M^\omega = 0 \).

Notice that the non-degeneracy of a symplectic form is equivalent to saying that the only vector of a symplectic space which is skew-orthogonal to all other vectors is 0.

Following proposition describes some straightforward but useful properties of the skew-orthogonal of a linear subspace of a symplectic space:

**Proposition 7**

(i) If \( M \) is a linear subspace of \( E \), then so is \( M^\omega \) and

\[ \dim M + \dim M^\omega = \dim E \text{ and } (M^\omega)^\omega = M. \tag{1.9} \]

(ii) If \( M_1, M_2 \) are linear subspaces of a symplectic space \((E, \omega)\), then

\[ (M_1 + M_2)^\omega = M_1^\omega \cap M_2^\omega, \quad (M_1 \cap M_2)^\omega = M_1^\omega + M_2^\omega. \tag{1.10} \]
Proof. Proof of (i). That $M^\omega$ is a linear subspace of $E$ is clear. Let $\Phi : E \to E^*$ be the linear mapping (1.3). Since the dimension of $E$ is finite the non-degeneracy of $\omega$ implies that $\Phi$ is an isomorphism. Let $\{e_1, \ldots, e_k\}$ be a basis of $M$. We have

$$M^\omega = \bigcap_{j=1}^{k} \ker(\Phi(e_j))$$

so that $M^\omega$ is defined by $k$ independent linear equations, hence

$$\dim M^\omega = \dim E - k = \dim E - \dim M$$

which proves the first formula (1.9). Applying that formula to the subspace $(M^\omega)^\omega$ we get

$$\dim (M^\omega)^\omega = \dim E - \dim M^\omega = \dim M$$

and hence $M = (M^\omega)^\omega$ since $(M^\omega)^\omega \subset M$ whether $M$ is linear or not.

Proof of (ii). It is sufficient to prove the first equality (1.10) since the second follows by duality, replacing $M_1$ by $M_1^\omega$ and $M_2$ by $M_2^\omega$ and using the first formula (1.9). Assume that $z \in (M_1 + M_2)^\omega$. Then $\omega(z, z_1 + z_2) = 0$ for all $z_1 \in M_1, z_2 \in M_2$. In particular $\omega(z, z_1) = \omega(z, z_2) = 0$ so that we have both $z \in M_1^\omega$ and $z \in M_2^\omega$, proving that $(M_1 + M_2)^\omega \subset M_1^\omega \cap M_2^\omega$. If conversely $z \in M_1^\omega \cap M_2^\omega$ then $\omega(z, z_1) = \omega(z, z_2) = 0$ for all $z_1 \in M_1, z_2 \in M_2$ and hence $\omega(z, z') = 0$ for all $z' \in M_1 + M_2$. Thus $z \in (M_1 + M_2)^\omega$ and $M_1^\omega \cap M_2^\omega \subset (M_1 + M_2)^\omega$. □

The symplectic Gram–Schmidt theorem

The following result is a symplectic version of the Gram–Schmidt orthonormalization process of Euclidean geometry. Because of its importance and its many applications we give it the status of a theorem:

**Theorem 8** Let $A$ and $B$ be two (possibly empty) subsets of $\{1, \ldots, n\}$. For any two subsets $E = \{e_i : i \in A\}$, $F = \{f_j : j \in B\}$ of the symplectic space $(E, \omega)$ ($\dim E = 2n$), such that the elements of $E$ and $F$ satisfy the relations

$$\omega(e_i, e_j) = \omega(f_i, f_j) = 0, \omega(f_i, e_j) = \delta_{ij} \text{ for } (i, j) \in A \times B \quad (1.11)$$

there exists a symplectic basis $B$ of $(E, \omega)$ containing $E \cup F$.

**Proof.** We will distinguish three cases. (i) The case $A = B = \emptyset$. Choose a vector $e_1 \neq 0$ in $E$ and let $f_1$ be another vector with $\omega(f_1, e_1) \neq 0$ (the existence of $f_1$ follows from the non-degeneracy of $\omega$). These vectors are linearly independent, which proves the theorem in the considered case when $n = 1$. Suppose $n > 1$ and let $M$ be the subspace of $E$ spanned by $\{e_1, f_1\}$ and set $E_1 = M^\omega$. In view of the first formula (1.9) we have $\dim M + \dim E_1 = 2n$. Since $\omega(f_1, e_1) \neq 0$ we have $E_1 \cap M = 0$ hence
$E = E_1 \oplus M$, and the restriction $\omega_1$ of $\omega$ to $E_1$ is non-degenerate (because if $z_1 \in E_1$ is such that $\omega_1(z_1, z) = 0$ for all $z \in E_1$ then $z_1 \in E_1^\omega = M$ and hence $z_1 = 0$); $(E_1, \omega_1)$ is thus a symplectic space of dimension $2(n - 1)$. Repeating the construction above $n - 1$ times we obtain a strictly decreasing sequence

$$\ldots \supset (E_{n-1}, \omega_{n-1}) \supset (E_n, \omega_n) \supset (E, \omega)$$

of symplectic spaces with $\dim E_k = 2(n - k)$ and also an increasing sequence

$$\{e_1, f_1\} \subset \{e_1, e_2; f_1; f_2\} \subset \ldots \subset \{e_1, \ldots, e_n; f_1, \ldots, f_n\}$$

of sets of linearly independent vectors in $E$, each set satisfying the relations (1.11). (ii) The case $A = B \neq \emptyset$. We may assume without restricting the argument that $A = B = \{1, 2, \ldots, k\}$. Let $M$ be the subspace spanned by $\{e_1, \ldots, e_k; f_1, \ldots, f_k\}$. As in the first case we find that $E = M \oplus M'$ and that the restrictions $\omega_M$ and $\omega_{M'}$ of $\omega$ to $M$ and $M'$, respectively, are symplectic forms. Let $\{e_{k+1}, \ldots, e_n; f_{k+1}, \ldots, f_n\}$ be a symplectic basis of $M'$; then

$$B = \{e_1, \ldots, e_n; f_1, \ldots, f_n\}$$

is a symplectic basis of $E$. (iii) The case $B \setminus A \neq \emptyset$ (or $B \setminus A \neq \emptyset$). Suppose for instance $k \in B \setminus A$ and choose $e_k \in E$ such that $\omega(e_i, e_k) = 0$ for $i \in A$ and $\omega(f_j, e_k) = \delta_{jk}$ for $j \in B$. Then $E \cup F \cup \{e_k\}$ is a system of linearly independent vectors: the equality

$$\lambda_k e_k + \sum_{i \in A} \lambda_i e_i + \sum_{j \in B} \mu_j e_j = 0$$

implies that we have

$$\lambda_k \omega(f_k, e_k) + \sum_{i \in A} \lambda_i \omega(f_k, e_i) + \sum_{j \in B} \mu_j \omega(f_k, e_j) = \lambda_k = 0$$

and hence also $\lambda_i = \mu_j = 0$. Repeating this procedure as many times as necessary, we are led back to the case $A = B \neq \emptyset$. ■

**Remark 9** The proof above shows that we can construct symplectic subspaces of $(E, \omega)$ having any given even dimension $2m < \dim E$ containing any pair of vectors $e, f$ such that $\omega(f, e) = 1$. In fact, $M = \text{Span}\{e, f\}$ is a two-dimensional symplectic subspace ("symplectic plane") of $(E, \omega)$. In the standard symplectic space $(\mathbb{R}^{2n}, \sigma)$ every plane $x_j, p_j$ of "conjugate coordinates" is a symplectic plane.

It follows from the theorem above that if $(E, \omega)$ and $(E', \omega')$ are two symplectic spaces with same dimension $2n$ there always exists a symplectic isomorphism $\Phi : (E, \omega) \longrightarrow (E', \omega')$. Let in fact

$$B = \{e_1, \ldots, e_n\} \cup \{f_1, \ldots, f_n\} \quad B' = \{e'_1, \ldots, e'_n\} \cup \{f'_1, \ldots, f'_n\}$$
be symplectic bases of \((E, \omega)\) and \((E', \omega')\), respectively. The linear mapping \(\Phi : E \rightarrow E'\) defined by \(\Phi(e_j) = e'_j\) and \(\Phi(f_j) = f'_j\) \((1 \leq j \leq n)\) is a symplectic isomorphism.

This result, together with the fact that any skew-product takes the standard form in a symplectic basis shows why it is no restriction to develop symplectic geometry from the standard symplectic space: all symplectic spaces of a given dimension are just isomorphic copies of \((\mathbb{R}^{2n}_z, \sigma)\).

We end this subsection by briefly discussing the restrictions of symplectic transformations to subspaces:

**Proposition 10** Let \((F, \omega|_F)\) and \((F', \omega|_{F'})\) be two symplectic subspaces of \((E, \omega)\). If \(\dim F = \dim F'\) there exists a symplectic automorphism of \((E, \omega)\) whose restriction \(\varphi|_F\) is a symplectic isomorphism \(\varphi|_F : (F, \omega|_F) \rightarrow (F', \omega|_{F'})\).

**Proof.** Assume that the common dimension of \(F\) and \(F'\) is \(2k\) and let

\[
B_{(k)} = \{e_1, \ldots, e_k\} \cup \{f_1, \ldots, f_k\} \\
B'_{(k)} = \{e'_1, \ldots, e'_k\} \cup \{f'_1, \ldots, f'_k\}
\]

be symplectic bases of \(F\) and \(F'\), respectively. In view of Theorem 8 we may complete \(B_{(k)}\) and \(B'_{(k')}\) into full symplectic bases \(B\) and \(B'\) of \((E, \omega)\). Define a symplectic automorphism \(\Phi\) of \(E\) by requiring that \(\Phi(e_i) = e'_i\) and \(\Phi(f_j) = f'_j\). The restriction \(\varphi = \Phi|_F\) is a symplectic isomorphism \(F \rightarrow F'\).

Let us now work in the standard symplectic space \((\mathbb{R}^{2n}_z, \sigma)\). Everything can however be generalized to vector spaces with a symplectic form associated to a complex structure.

**Definition 11** A basis of \((\mathbb{R}^{2n}_z, \sigma)\) which is both symplectic and orthogonal (with respect to the scalar product \(\langle z, z' \rangle = \sigma(Jz, z')\)) is called an orthosymplectic basis.

The canonical basis is trivially an orthosymplectic basis. It is easy to construct orthosymplectic bases starting from an arbitrary set of vectors \(\{e'_1, \ldots, e'_n\}\) satisfying the conditions \(\sigma(e'_i, e'_j) = 0\): let \(\ell\) be the vector space (Lagrangian plane) spanned by these vectors. Using the classical Gram–Schmidt orthonormalization process we can construct an orthonormal basis \(\{e_1, \ldots, e_n\}\) of \(\ell\). Define now \(f_1 = -Je_1, \ldots, f_n = -Je_n\). The vectors \(f_i\) are orthogonal to the vectors \(e_j\) and are mutually orthogonal because \(J\) is a rotation. In addition,

\[\sigma(f_i, f_j) = \sigma(e_i, e_j) = 0, \quad \sigma(f_i, e_j) = \delta_{ij}\]

hence the basis

\[B = \{e_1, \ldots, e_n\} \cup \{f_1, \ldots, f_n\}\]
is both orthogonal and symplectic.

This construction generalizes to any set

\[ \{e_1, \ldots, e_k\} \cup \{f_1, \ldots, f_m\} \]

of normed pairwise orthogonal vectors satisfying in addition the symplectic conditions \( \sigma(f_i, f_j) = \sigma(e_i, e_j) = 0 \) and \( \sigma(f_i, e_j) = \delta_{ij} \).

1.3 The Lagrangian Grassmannian

Recall that a subset of \((E, \omega)\) is isotropic if \( \omega \) vanishes identically on it. An isotropic subspace \( \ell \) of \((E, \omega)\) having dimension \( n = \frac{1}{2} \dim E \) is called a Lagrangian plane. Equivalently, a Lagrangian plane in \((E, \omega)\) is a linear subspace of \( E \) which is both isotropic and co-isotropic.

Lagrangian planes

It follows from Theorem 8 that there always exists a Lagrangian plane containing a given isotropic subspace: let \( \{e_1, \ldots, e_k\} \) be a basis of such a subspace and complete that basis into a full symplectic basis

\[ B = \{e_1, \ldots, e_n\} \cup \{f_1, \ldots, f_n\} \]

of \((E, \omega)\). The space spanned by \( \{e_1, \ldots, e_n\} \) is then a Lagrangian plane. Notice that we have actually constructed in this way a pair \((\ell, \ell')\) of Lagrangian planes such that \( \ell \cap \ell' = 0 \), namely

\[ \ell = \text{Span} \{e_1, \ldots, e_n\} \quad \ell' = \text{Span} \{f_1, \ldots, f_n\} . \]

Since Lagrangian planes will play a recurring role in the rest of this thesis it is perhaps appropriate to summarize some terminology and notation:

**Definition 12** The set of all Lagrangian planes in a symplectic space \((E, \omega)\) is denoted by \( \text{Lag}(E, \omega) \) and is called the \( \text{“Lagrangian Grassmannian of} \ (E, \omega) \text{”} \). When \((E, \omega)\) is the standard symplectic space \((\mathbb{R}^{2n}_x, \sigma)\) the Lagrangian Grassmannian is denoted by \( \text{Lag}(n) \), and we will use the notations

\[ \ell_X = \mathbb{R}^n_x \times 0 \quad \text{and} \quad \ell_P = 0 \times \mathbb{R}^n_p . \]

\( \ell_X \) and \( \ell_P \) are called the \( \text{“horizontal”} \) and \( \text{“vertical”} \) Lagrangian planes in \((\mathbb{R}^{2n}_x, \sigma)\).

Other common notations for the Lagrangian Grassmannian are \( \Lambda(E, \omega) \), \( \Lambda(n, \mathbb{R}) \) or \( \Lambda(2n, \mathbb{R}) \).

**Example 13** Suppose \( n = 1 \). \( \text{Lag}(1) \) consists of all straight lines passing through the origin in the symplectic plane \((\mathbb{R}^2_x, -\det)\).
When $n > 1$ the Lagrangian Grassmannian is a proper subset of the set of all $n$-dimensional planes of $(\mathbb{R}^{2n}_z, \sigma)$.

Let us study the equation of a Lagrangian plane in the standard symplectic space.

In what follows we work in an arbitrary symplectic basis

$$B = \{e_1, ..., e_n\} \cup \{f_1, ..., f_n\}$$

of the standard symplectic space. The corresponding coordinates are denoted by $x$ and $p$.

**Proposition 14** Let $\ell$ be $n$-dimensional linear subspace $\ell$ of the standard symplectic space $(\mathbb{R}^{2n}_z, \sigma)$.

(i) $\ell$ is a Lagrangian plane if and only if it can be represented by an equation

$$Xx + Pp = 0 \quad \text{with} \quad \text{rank}(X, P) = n \quad \text{and} \quad XP^T = PX^T. \quad (1.12)$$

(ii) Let $B = \{e_1, ..., e_n\} \cup \{f_1, ..., f_n\}$ be a symplectic basis and assume that $\ell = \text{Span}\{f_1, ..., f_n\}$. Then there exists a symmetric matrix $M \in M(n, \mathbb{R})$ such that the Lagrangian plane $\ell$ is represented by the equation $p = Mx$ in the coordinates defined by $B$.

**Proof.** Proof of (i). We first remark that $Xx + Pp = 0$ represents a $n$-dimensional space if and only if

$$\text{rank}(X, P) = \text{rank}(X^T, P^T) = n. \quad (1.13)$$

Assume that in addition $X^T P = P^T X$ and parametrize $\ell$ by setting $x = P^T u, p = -X^T u$. It follows that if $z, z'$ are two vectors of $\ell$ then

$$\sigma(z, z') = \langle -X^T u, P^T u' \rangle - \langle -X^T u', P^T u \rangle = 0$$

so that (1.12) indeed is the equation of a Lagrangian plane. Reversing the argument shows that if $Xx + Pp = 0$ represents a $n$-dimensional space then the condition $\sigma(z, z') = 0$ for all vectors $z, z'$ of that space implies that we must have $XP^T = PX^T$.

Proof of (ii). It is clear from (i) that $p = Mx$ represents a Lagrangian plane $\ell$. It is also clear that this plane $\ell$ is transversal to $\text{Span}\{f_1, ..., f_n\}$. The converse follows from the observation that if $\ell : Xx + Pp = 0$ is transversal to $\text{Span}\{f_1, ..., f_n\}$ then $P$ must invertible. The property follows taking $M = -P^{-1}X$ which is symmetric since $XP^T = PX^T$.

Two Lagrangian planes are said to be transversal if $\ell \cap \ell' = 0$. Since $\dim \ell = \dim \ell' = \frac{1}{2} \dim E$ this is equivalent to saying that $E = \ell \oplus \ell'$. For instance the horizontal and vertical Lagrangian planes $\ell_X = \mathbb{R}_x^n \times 0$ and $\ell_P = 0 \times \mathbb{R}_p^n$ are obviously transversal in $(\mathbb{R}^{2n}_z, \sigma)$.

Part (ii) of Proposition 14 above implies:
Corollary 15  
(i) A $n$-plane $\ell$ in $(\mathbb{R}^{2n}, \sigma)$ is a Lagrangian plane transversal to $\ell_P$ if and only if there exists a symmetric matrix $M \in M(n, \mathbb{R})$ such that $\ell : p = Mx$.

(ii) For any $n$-plane $\ell : Xx + Pp = 0$ in $\mathbb{R}^{2n}$ we have

$$\dim(\ell \cap \ell_P) = n - \text{rank}(P).$$

(iii) For any symplectic matrix

$$s = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

the rank of $B$ is given by the formula

$$\text{rank}(B) = n - \dim(S\ell_P \cap \ell_P).$$

Proof. Proof of (i). The condition is necessary taking for $B$ the canonical symplectic bases. If conversely $\ell$ is the graph of a symmetric matrix $M$ then it is immediate to check that $\sigma(z; z') = 0$ for all $z \in \ell$. (ii) The intersection $\ell \cap \ell_P$ consists of all $(x, p)$ which satisfy both conditions $Xx + Pp = 0$ and $x = 0$. It follows that

$$(x, p) \in \ell \cap \ell_P \iff Pp = 0$$

and hence (1.14).

Proof of (iii). Formula (1.15) follows from the trivial equivalence

$$(x, p) \in S\ell_P \cap \ell_P \iff Bp = 0.$$
The action of $\text{Sp}(n)$ on $\text{Lag}(n)$

Let us prove the following important result on the action of $\text{Sp}(n)$ and its subgroup $U(n)$ on the Lagrangian Grassmannian $\text{Lag}(n)$.

**Theorem 17** The action of $U(n)$ and $\text{Sp}(n)$ on $\text{Lag}(n)$ has the following properties:

(i) $U(n)$ (and hence $\text{Sp}(n)$) acts transitively on $\text{Lag}(n)$: for every pair $(\ell, \ell')$ of Lagrangian planes there exists $U \in U(n)$ such that $\ell' = U\ell$.

(ii) The group $\text{Sp}(n)$ acts transitively on the set of all pairs of transverse Lagrangian planes: if $(\ell_1, \ell_1')$ and $(\ell_2, \ell_2')$ are such that $\ell_1 \cap \ell_1' = \ell_1 \cap \ell_2' = 0$ then there exits $S \in \text{Sp}(n)$ such that $(\ell_2, \ell_2') = (S\ell_1, S\ell_1')$.

**Proof.** Proof of (i). Let $O = \{e_1, \ldots, e_n\}$ and $O' = \{e_1', \ldots, e_n'\}$ be orthonormal bases of $\ell$ and $\ell'$, respectively. Then $B = O \cup JO$ and $B' = O' \cup JO'$ are orthosymplectic bases of $(\mathbb{R}^{2n}, \sigma)$. There exits $U \in O(2n)$ such that $U(e_i) = e_i'$ and $U(f_i) = f_i'$ where $f_i = Je_i$, $f_i' = Je_i'$. We have $U \in \text{Sp}(n)$ hence

$$U \in O(2n) \cap \text{Sp}(n) = U(n)$$

((1.26) in Proposition 23).

**Proof of (ii).** Choose a basis $\{e_{11}, \ldots, e_{1n}\}$ of $\ell_1$ and a basis $\{f_{11}, \ldots, f_{1n}\}$ of $\ell_1'$ such that $\{e_{1i}, f_{1j}\}_{1 \leq i,j \leq n}$ is a symplectic basis of $(\mathbb{R}^{2n}_2, \sigma)$. Similarly choose bases of $\ell_2$ and $\ell_2'$ whose union $\{e_{2i}, f_{2j}\}_{1 \leq i,j \leq n}$ is also a symplectic basis. Define a linear mapping $S : \mathbb{R}^{2n}_2 \to \mathbb{R}^{2n}_2$ by $S(e_{1i}) = e_{2i}$ and $S(f_{1i}) = f_{2i}$ for $1 \leq i \leq n$. We have $S \in \text{Sp}(n)$ and $(\ell_2, \ell_2') = (S\ell_1, S\ell_1')$. ■

We will see in the next section that the existence of an integer measuring the relative position of triples of Lagrangian planes implies that $\text{Sp}(n)$ cannot act transitively on triples (or, more generally, of $k$-uples, $k \geq 3$) of Lagrangian planes.

For two integers $n_1, n_2 > 0$ consider the group

$$\text{Sp}(n_1) \oplus \text{Sp}(n_2) = \{(S_1, S_2) : S_1 \in \text{Sp}(n_1), S_2 \in \text{Sp}(n_2)\}$$

equipped with the composition law

$$(S_1 \oplus S_2)(S_1' \oplus S_2') = S_1 S_1' \oplus S_2 S_2'.$$

Setting $n = n_1 + n_2$ then $\text{Sp}(n_1) \oplus \text{Sp}(n_2)$ acts on the Lagrangian Grassmannian $\text{Lag}(n)$. We have in particular a natural action

$$\text{Sp}(n_1) \oplus \text{Sp}(n_2) : \text{Lag}(n_1) \oplus \text{Lag}(n_2) \to \text{Lag}(n_1) \oplus \text{Lag}(n_2)$$

where $\text{Lag}(n_1) \oplus \text{Lag}(n_2)$ is the set of all direct sums $\ell_1 \oplus \ell_2$ with $\ell_1 \in \text{Lag}(n_1)$, $\ell_2 \in \text{Lag}(n_2)$. This action is defined by the obvious formula

$$(S_1 \oplus S_2)(\ell_1 \oplus \ell_2) = S_1 \ell_1 \oplus S_2 \ell_2.$$

Observe that $\text{Lag}(n_1) \oplus \text{Lag}(n_2)$ is a subset of $\text{Lag}(n)$:

$$\text{Lag}(n_1) \oplus \text{Lag}(n_2) \subset \text{Lag}(n)$$
since $\sigma_1 \oplus \sigma_2$ vanishes on each $\ell_1 \oplus \ell_2$. 

14
1.4 The Symplectic Group

In this section we study in some detail the symplectic group of a symplectic space \((E, \omega)\), with a special emphasis on the standard symplectic group \(\text{Sp}(n)\), corresponding to the case \((E, \omega) = (\mathbb{R}^{2n}, \sigma)\).

There exists an immense literature devoted to the symplectic group. A few classical references are Libermann and Marle [53], Guillemin and Sternberg [33, 34] and Abraham and Marsden [1].

The group \(\text{Sp}(n)\)

Let us begin by working in the standard symplectic space \((\mathbb{R}^{2n}, \sigma)\).

**Definition 18** The group of all automorphisms \(s\) of \((\mathbb{R}^{2n}, \sigma)\) such that

\[\sigma(sz, sz') = \sigma(z, z')\]

for all \(z, z' \in \mathbb{R}^{2n}\) is denoted by \(\text{Sp}(n)\) and called the “standard symplectic group”.

It follows from Proposition 6 that \(\text{Sp}(n)\) is isomorphic to the symplectic group \(\text{Sp}(E, \omega)\) of any \(2n\)-dimensional symplectic space.

The notion of linear symplectic transformation can be extended to diffeomorphisms:

**Definition 19** Let \((E, \omega)\), \((E', \omega')\) be two symplectic spaces. A diffeomorphism \(f : (E, \omega) \rightarrow (E', \omega')\) is called a “symplectomorphism” if the differential \(d_z f\) is a linear symplectic mapping \(E \rightarrow E'\) for every \(z \in E\). [In the physical literature one often says “canonical transformation” in place of “symplectomorphism”].

It follows from the chain rule that the compose \(g \circ f\) of two symplectomorphisms \(f : (E, \omega) \rightarrow (E', \omega')\) and \(g : (E', \omega') \rightarrow (E'', \omega'')\) is a symplectomorphism \((E, \omega) \rightarrow (E'', \omega'')\). When

\[(E, \omega) = (E', \omega') = (\mathbb{R}^{2n}, \sigma)\]

a diffeomorphism \(f\) of \((\mathbb{R}^{2n}, \sigma)\) is a symplectomorphism if and only if its Jacobian matrix (calculated in any symplectic basis) is in \(\text{Sp}(n)\). Summarizing:

\[f\] is a symplectomorphism of \((\mathbb{R}^{2n}, \sigma)\)

\[\iff\]

\[Df(z) \in \text{Sp}(n) \text{ for every } z \in (\mathbb{R}^{2n}, \sigma).\]

It follows directly from the chain rule \(D(g \circ f)(z) = Dg(f(z))Df(z)\) that the symplectomorphisms of the standard symplectic space \((\mathbb{R}^{2n}, \sigma)\) form a group. That group is denoted by \(\text{Symp}(n)\).

\[^1\]The word was reputedly coined by J.-M. Souriau.
Remark 20 The notion of symplectomorphism extends in the obvious way to symplectic manifold: if \((M, \omega)\) and \((M', \omega')\) are two such manifolds, then a diffeomorphism \(f : M \rightarrow M'\) is called a symplectomorphism if it preserves the symplectic structures on \(M\) and \(M'\), that is if \(f^* \omega' = \omega\) where \(f^*\omega'\) (the “pull-back of \(\omega'\) by \(f\)) is defined by

\[
f^*\omega'(z_0)(Z, Z') = \omega'(f(z_0))((d_{z_0}f)Z, (d_{z_0}f)Z')
\]

for every \(z_0 \in M\) and \(Z, Z' \in T_{z_0}M\). If \(f\) and \(g\) are symplectomorphisms \((M, \omega) \rightarrow (M', \omega')\) and \((M', \omega') \rightarrow (M'', \omega'')\) then \(g \circ f\) is a symplectomorphism \((M, \omega) \rightarrow (M'', \omega'')\).

The symplectomorphisms \((M, \omega) \rightarrow (M', \omega')\) obviously form a group, denoted by \(\text{Symp}(M, \omega)\), and whose study is very active and far from being achieved.

For practical purposes it is often advantageous to work in coordinates and to represent the elements of \(\text{Sp}(n)\) by matrices.

Recall that definition (1.1) of the standard symplectic form can be rewritten in matrix form as

\[
\sigma(z, z') = (z')^T J z = \langle Jz, z' \rangle
\]

where \(J\) is the standard symplectic matrix:

\[
J = \begin{bmatrix}
0 & I \\
-I & 0
\end{bmatrix}.
\]

Notice that \(J^T = -J\) and \(J^2 = -I\).

Choose a symplectic basis in \((\mathbb{R}^{2n}_z, \sigma)\) we will identify a linear mapping \(s : \mathbb{R}^{2n}_z \rightarrow \mathbb{R}^{2n}_z\) with its matrix \(S\) in that basis. In view of (1.16) we have

\[
S \in \text{Sp}(n) \iff S^T J S = J
\]

where \(S^T\) is the transpose of \(S\). Since

\[
\det S^T J S = \det S^2 \det J = \det J
\]

it follows that \(\det S\) can, a priori, take any of the two values \(\pm 1\). It turns out, however, that

\[
S \in \text{Sp}(n) \implies \det S = 1.
\]

There are many ways of showing this. None of them is really totally trivial. Here is an algebraic proof making use of the notion of Pfaffian (we will give an alternative proof later on). Recall that to every antisymmetric matrix \(A\) one associates a polynomial \(\text{Pf}(A)\) (“the Pfaffian of \(A\)”) in the entries of \(A\), it has the following properties:

\[
\text{Pf}(S^T A S) = (\det S) \text{Pf}(A), \quad \text{Pf}(J) = 1.
\]
Choose now $A = J$ and $S \in \text{Sp}(n)$. Since $S^TJS = J$ we have

$$\text{Pf}(S^TJS) = \det S = 1$$

which was to be proven.

**Remark 21** The group $\text{Sp}(n)$ is stable under transposition: the condition $S \in \text{Sp}(n)$ is equivalent to $S^TJS = J$. Since $S^{-1}$ also is in $\text{Sp}(n)$ we have $(S^{-1})^TJS^{-1} = J$. Taking the inverses of both sides of this equality we get $SJ^{-1}S^T = J^{-1}$ that is $SJS^T = J$, so that $S^T \in \text{Sp}(n)$. It follows that we have the equivalences

$$S \in \text{Sp}(n) \iff S^TJS = J \iff SJS^T = J.$$  \hspace{1cm} (1.18)

A symplectic basis of $(\mathbb{R}^{2n}_z, \sigma)$ being chosen, we can always write $S \in \text{Sp}(n)$ in block-matrix form

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$ \hspace{1cm} (1.19)

where the entries $A, B, C, D$ are $n \times n$ matrices. The conditions (1.18) are then easily seen, by a direct calculation, equivalent to the two following sets of equivalent conditions:

$$A^T C, B^TD \text{ symmetric, and } A^T D - C^T B = I$$ \hspace{1cm} (1.20)

$$AB^T, CD^T \text{ symmetric, and } AD^T - BC^T = I.$$ \hspace{1cm} (1.21)

It follows from the second of these sets of conditions that the inverse of $S$ is

$$S^{-1} = \begin{bmatrix} D^T & -B^T \\ -C^T & A^T \end{bmatrix}.$$ \hspace{1cm} (1.22)

Here are three classes of symplectic matrices which are useful: if $P$ and $L$ are, respectively, a symmetric and an invertible $n \times n$ matrix we set

$$V_P = \begin{bmatrix} I & 0 \\ -P & I \end{bmatrix}, U_P = \begin{bmatrix} -P & I \\ -I & 0 \end{bmatrix}, M_L = \begin{bmatrix} L^{-1} & 0 \\ 0 & L^T \end{bmatrix}.$$ \hspace{1cm} (1.23)

The matrices $V_P$ are sometimes called “symplectic shears”.

It turns out –as we shall prove later on– that both sets

$$\mathcal{G} = \{ J \} \cup \{ V_P : P \in \text{Sym}(n, \mathbb{R}) \} \cup \{ M_L : L \in \text{GL}(n, \mathbb{R}) \}$$

and

$$\mathcal{G'} = \{ J \} \cup \{ U_P : P \in \text{Sym}(n, \mathbb{R}) \} \cup \{ M_L : L \in \text{GL}(n, \mathbb{R}) \}$$

generate the symplectic group $\text{Sp}(n)$.

We can also form direct sums of symplectic groups. Consider for instance $(\mathbb{R}^{2n_1}, \sigma_1)$ and $(\mathbb{R}^{2n_2}, \sigma_2)$, the standard symplectic spaces of dimension $2n_1$.
and $2n_2$. Let $\text{Sp}(n_1)$ and $\text{Sp}(n_2)$ be the respective symplectic groups. The direct sum $\text{Sp}(n_1) \oplus \text{Sp}(n_2)$ is the group of automorphisms of 

$$(\mathbb{R}^{2n}, \sigma) = (\mathbb{R}^{2n_1} \oplus \mathbb{R}^{2n_2}, \sigma_1 \oplus \sigma_2)$$

defined, for $z_1 \in \mathbb{R}^{2n_1}$ and $z_2 \in \mathbb{R}^{2n_2}$, by

$$(s_1 \oplus s_2)(z_1 \oplus z_2) = s_1 z_1 \oplus s_2 z_2.$$ 

It is evidently a subgroup of $\text{Sp}(n)$:

$$\text{Sp}(n_1) \oplus \text{Sp}(n_2) \subset \text{Sp}(n)$$

which can be expressed in terms of block-matrices as follows: let

$$S_1 = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \quad \text{and} \quad S_2 = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}$$

be elements of $\text{Sp}(n_1)$ and $\text{Sp}(n_2)$, respectively. Then

$$S_1 \oplus S_2 = \begin{bmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{bmatrix} \in \text{Sp}(n_1 + n_2). \quad (1.24)$$

The mapping $(S_1, S_2) \mapsto S_1 \oplus S_2$ thus defined is a group monomorphism

$$\text{Sp}(n_1) \oplus \text{Sp}(n_2) \longrightarrow \text{Sp}(n).$$

Let us now briefly discuss the eigenvalues of a symplectic matrix. It has been known for a long time that the eigenvalues of symplectic matrices play an fundamental role in the study of Hamiltonian periodic orbits. This is because the stability of these orbits depend in a crucial way on the structure of the associated linearized system. It turns out that these eigenvalues also play an essential role in the understanding of symplectic squeezing theorems, which we study later in this book.

Let us first prove the following result:

**Proposition 22** Let $S \in \text{Sp}(n)$.

(i) If $\lambda$ is an eigenvalue of $S$ then so are $\bar{\lambda}$ and $1/\lambda$ (and hence also $1/\bar{\lambda}$);  
(ii) if the eigenvalue $\lambda$ of $S$ has multiplicity $k$ then so has $1/\lambda$.  
(iii) $S$ and $S^{-1}$ have the same eigenvalues.

**Proof.** Proof of (i). We are going to show that the characteristic polynomial $P_S(\lambda) = \det(S - \lambda I)$ of $S$ satisfies the reflexivity relation

$$P_S(\lambda) = \lambda^{2n} P_S(1/\lambda); \quad (1.25)$$

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(i) will follow since for real matrices eigenvalues appear in conjugate pairs. Since $S^TJS = J$ we have $S = -J(S^T)^{-1}J$ and hence
\[
P_S(\lambda) = \det(-J(S^T)^{-1}J - \lambda I) \\
= \det(-(S^T)^{-1}J + \lambda I) \\
= \det(-J + \lambda S) \\
= \lambda^{2n} \det(S - \lambda^{-1}I)
\]
which is precisely (1.25).

Proof of (ii). Let $P_S^{(j)}$ be the $j$-th derivative of the polynomial $P_S$. If $\lambda_0$ has multiplicity $k$. Then $P_S^{(j)}(\lambda_0) = 0$ for $0 \leq j \leq k-1$ and $P_S^{(k)}(\lambda) \neq 0$. In view of (1.25) we also have $P_S^{(j)}(1/\lambda) = 0$ for $0 \leq j \leq k-1$ and $P_S^{(k)}(1/\lambda) \neq 0$. Property (iii) immediately follows from (ii). \[\square\]

Notice that as an immediate consequence of this result is that if $\pm 1$ is an eigenvalue of $S \in \text{Sp}(n)$ then its multiplicity is necessarily even.

We will see in next subsection (Proposition 24) that any positive-definite symmetric symplectic matrix can be diagonalized using an orthogonal transformation which is at the same time symplectic.

The unitary group $\text{U}(n)$

The complex structure associated to the standard symplectic matrix $J$ is very simple: it is defined by
\[
(\alpha + i\beta)z = \alpha + \beta Jz
\]
and corresponds to the trivial identification $z = (x, p) \equiv x + ip$. The unitary group $\text{U}(n, \mathbb{C})$ acts in a natural way on $(\mathbb{R}^{2n}_x, \sigma)$ and that action preserves the symplectic structure. Let us make this statement somewhat more explicit:

**Proposition 23** The monomorphism $\mu : M(n, \mathbb{C}) \rightarrow M(2n, \mathbb{R})$ defined by $u = A + iB \mapsto \mu(u)$ with

\[
\mu(u) = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}
\]

$(A$ and $B$ real) identifies the unitary group $\text{U}(n, \mathbb{C})$ with the subgroup $\text{U}(n) = \text{Sp}(n) \cap \text{O}(2n, \mathbb{R})$. (1.26)

Proof. In view of (1.22) the inverse of $U = \mu(u), u \in \text{U}(n, \mathbb{C})$, is

\[
U^{-1} = \begin{bmatrix} A^T & B^T \\ -B^T & A^T \end{bmatrix} = U^T
\]
hence $U \in O(2n, \mathbb{R})$ which proves the inclusion $U(n) \subset \text{Sp}(n) \cap O(2n, \mathbb{R})$. Suppose conversely that $U \in \text{Sp}(n) \cap O(2n, \mathbb{R})$. Then

$$JU = (U^T)^{-1}J = UJ$$

which implies that $U \in U(n)$ so that $\text{Sp}(n) \cap O(2n, \mathbb{R}) \subset U(n)$. ■

We will loosely talk about $U(n)$ as of the “unitary group” when there is no risk of confusion. Notice that it immediately follows from conditions (1.20), (1.21) that we have the equivalences:

$$A + iB \in U(n) \quad \iff \quad A^T B \text{ symmetric and } A^T A + B^T B = I \quad (1.28)$$

$$\iff \quad AB^T \text{ symmetric and } AA^T + BB^T = I; \quad (1.29)$$

of course these conditions are just the same thing as the conditions

$$(A + iB)^*(A + iB) = (A + iB)(A + iB)^* = I$$

for the matrix $A + iB$ to be unitary.

In particular, taking $B = 0$ we see the matrices

$$R = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \quad \text{with} \quad AA^T = A^T A = I \quad (1.30)$$

also are symplectic, and form a subgroup $O(n)$ of $U(n)$ which we identify with the rotation group $O(n, \mathbb{R})$. We thus have the chain of inclusions

$$O(n) \subset U(n) \subset \text{Sp}(n).$$

Let us end this subsection by mentioning that it is sometimes useful to identify elements of $\text{Sp}(n)$ with complex symplectic matrices. The group $\text{Sp}(n, \mathbb{C})$ is defined, in analogy with $\text{Sp}(n)$, by the condition

$$\text{Sp}(n, \mathbb{C}) = \{ M \in M(2n, \mathbb{C}) : M^T J M = J \}.$$

Let now $K$ be the complex matrix

$$K = \frac{1}{\sqrt{2}} \begin{bmatrix} I & iI \\ iI & I \end{bmatrix} \in U(2n, \mathbb{C})$$

and consider the mapping

$$\text{Sp}(n) \rightarrow \text{Sp}(n, \mathbb{C}) , \quad S \mapsto S_c = K^{-1} SK.$$
One verifies by a straightforward calculation that $S_c \in \text{Sp}(n, \mathbb{C})$. Notice that if $U \in U(n)$ then

$$U_c = \begin{bmatrix} U & 0 \\ 0 & U^* \end{bmatrix}.$$

We know from elementary linear algebra that one can diagonalize a symmetric matrix using orthogonal transformations. From the properties of the eigenvalues of a symplectic matrix follows that when this matrix is in addition symplectic and positive definite this diagonalization can be achieved using a symplectic rotation:

**Proposition 24** Let $S$ be a positive definite and symmetric symplectic matrix. Let $\lambda_1 \leq \cdots \leq \lambda_n \leq 1$ be the $n$ smallest eigenvalues of $S$ and set

$$\Lambda = \text{diag}[\lambda_1, \ldots, \lambda_n; 1/\lambda_1, \ldots, 1/\lambda_n]. \quad (1.31)$$

There exists $U \in U(n)$ such that $S = U^T \Lambda U$.

**Proof.** Since $S > 0$ its eigenvalues occur in pairs $(\lambda, 1/\lambda)$ of positive numbers (Proposition 22). If $\lambda_1 \leq \cdots \leq \lambda_n$ are $n$ eigenvalues then $1/\lambda_1, \ldots, 1/\lambda_n$ are the other $n$ eigenvalues. Let now $U$ be an orthogonal matrix such that $S = U^T \Lambda U$ with, $\Lambda$ being given by (1.31). We claim that $U \in U(n)$. It suffices to show that we can write $U$ in the form

$$U = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$$

with

$$AB^T = B^T A \quad \text{and} \quad AA^T + BB^T = I. \quad (1.32)$$

Let $e_1, \ldots, e_n$ be $n$ orthonormal eigenvectors of $U$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$. Since $SJ = JS^{-1}$ (because $S$ is both symplectic and symmetric) we have, for $1 \leq k \leq n$,

$$SJ e_k = JS^{-1} e_k = \frac{1}{\lambda_j} J e_k$$

hence $\pm Je_1, \ldots, \pm Je_n$ are the orthonormal eigenvectors of $U$ corresponding to the remaining $n$ eigenvalues $1/\lambda_1, \ldots, 1/\lambda_n$. Write now the $2n \times n$ matrix $(e_1, \ldots, e_n)$ as

$$[e_1, \ldots, e_n] = \begin{bmatrix} A \\ B \end{bmatrix}$$

where $A$ and $B$ are $n \times n$ matrices. We have

$$[-J e_1, \ldots, -J e_n] = -J \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -B \\ A \end{bmatrix}$$

hence $U$ is indeed of the type

$$U = [e_1, \ldots, e_n; -J e_1, \ldots, -J e_n] = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}.$$
The symplectic conditions (1.32) are automatically satisfied since $U^T U = I$.

An immediate consequence of Proposition 24 is that the square root of a positive-definite symmetric symplectic matrix is also symplectic. More generally:

**Corollary 25** (i) For every $\alpha \in \mathbb{R}$ there exists a unique $R \in \text{Sp}(n)$, $R > 0$, $R = R^T$, such that $S = R^\alpha$.

(ii) Conversely, if $R \in \text{Sp}(n)$ is positive definite, then $R^\alpha \in \text{Sp}(n)$ for every $\alpha \in \mathbb{R}$.

**Proof.** Proof of (i). Set $R = U^T \Lambda^{1/\alpha} U$. Then $R^\alpha = U^T \Lambda U = S$.

Proof of (ii). It suffices to note that we have $R^\alpha = (U^T \Lambda U)^\alpha = U^T \Lambda^\alpha U \in \text{Sp}(n)$.

The symplectic algebra

$\text{Sp}(n)$ is a Lie group. We will call its Lie algebra the “symplectic algebra”, and denote it by $\mathfrak{sp}(n)$. There is one-to-one correspondence between the elements of $\mathfrak{sp}(n)$ and the one-parameter groups in $\text{Sp}(n)$. This correspondence is the starting point of linear Hamiltonian mechanics.

Let

$$\Phi : \text{GL}(2n, \mathbb{R}) \longrightarrow \mathbb{R}^{4n^2}$$

be the continuous mapping defined by $\Phi(M) = M^T J M - J$. Since $S \in \text{Sp}(n)$ if and only if $S^T J S = J$ we have $\text{Sp}(n) = \Phi^{-1}(0)$ and $\text{Sp}(n)$ is thus a closed subgroup of $\text{GL}(2n, \mathbb{R})$, hence a “classical Lie group”. The set of all real matrices $X$ such that the exponential $\exp(tX)$ is in $\text{Sp}(n)$ is the Lie algebra of $\text{Sp}(n)$. We will call it the “symplectic algebra” and denote it by $\mathfrak{sp}(n)$:

$$X \in \mathfrak{sp}(n) \iff S_t = \exp(tX) \in \text{Sp}(n) \text{ for all } t \in \mathbb{R}. \quad (1.33)$$

The one-parameter family $(S_t)$ thus defined is a group: $S_t S_{t'} = S_{t+t'}$ and $S_t^{-1} = S_{-t}$.

The following result gives an explicit description of the elements of the symplectic algebra:

**Proposition 26** Let $X$ be a real $2n \times 2n$ matrix.

(i) We have

$$X \in \mathfrak{sp}(n) \iff X J + J X^T = 0 \iff X^T J + J X = 0. \quad (1.34)$$

(ii) Equivalently, $\mathfrak{sp}(n)$ consists of all block-matrices $X$ such that

$$X = \begin{bmatrix} U & V \\ W & -U^T \end{bmatrix} \text{ with } V = V^T \text{ and } W = W^T. \quad (1.35)$$
Proof. Let \((S_t)\) be a differentiable one-parameter subgroup of \(\text{Sp}(n)\) and a \(2n \times 2n\) real matrix \(X\) such that \(S_t = \exp(tX)\). Since \(S_t\) is symplectic we have \(S_tJ(S_t)^T = J\) that is
\[
\exp(tX)J \exp(tX^T) = J.
\]
Differentiating both sides of this equality with respect to \(t\) and then setting \(t = 0\) we get \(XJ + JX^T = 0\), and applying the same argument to the transpose \(S_t^T\) we get \(X^TJ + JX = 0\) as well. Suppose conversely that \(X\) is such that \(XJ + JX^T = 0\) and let us show that \(X \in \mathfrak{sp}(n)\). For this it suffices to prove that \(S_t = \exp(tX)\) is in \(\text{Sp}(n)\) for every \(t\). The condition \(X^TJ + JX = 0\) is equivalent to \(X^T = JXJ\) hence \(S_t^T = \exp(tJXJ)\). Since \(J^2 = -I\) we have \((JXJ)^k = (-1)^{k+1}JX^k J\) and hence
\[
\exp(tJXJ) = -\sum_{k=0}^{\infty} \frac{(-t)^k}{k!} (JXJ)^k = -Je^{-tX}J.
\]
It follows that
\[
S_t^TJS_t = (-Je^{-tX}J)Je^{tX} = J
\]
so that \(S_t \in \text{Sp}(n)\) as claimed. □

Remark 27 The symmetric matrices of order \(n\) forming a \(n(n+1)/2\)-dimensional vector space (1.35) implies, by dimension count, that \(\mathfrak{sp}(n)\) has dimension \(n(2n+1)\). Since \(\text{Sp}(n)\) is connected we consequently have
\[
\dim \text{Sp}(n) = \dim \mathfrak{sp}(n) = n(2n+1).
\] (1.36)

One should be careful to note that the exponential mapping
\[
\exp : \mathfrak{sp}(n) \longrightarrow \text{Sp}(n)
\]
is neither surjective nor injective. This is easily seen in the case \(n = 1\). We claim that
\[
S = \exp X \text{ with } X \in \mathfrak{sp}(1) \implies \text{Tr } S \geq -2.
\] (1.37)
(We are following Frankel’s argument in [21].) In view of (1.35) we have \(X \in \mathfrak{sp}(1)\) if and only \(\text{Tr } X = 0\), so that Hamilton-Cayley’s equation for \(X\) is just \(X^2 + \lambda I = 0\) where \(\lambda = \det X\). Expanding \(\exp X\) in power series it is easy to see that
\[
\exp X = \cos \sqrt{\lambda} I + \frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda} X \text{ if } \lambda > 0
\]
\[
\exp X = \cosh \sqrt{-\lambda} I + \frac{1}{\sqrt{-\lambda}} \sinh \sqrt{-\lambda} X \text{ if } \lambda < 0.
\]
Since \(\text{Tr } X = 0\) we see that in the case \(\lambda > 0\) we have
\[
\text{Tr}(\exp X) = 2 \cos \sqrt{\lambda} \geq -2
\]
and in the case $\lambda < 0$

$$\text{Tr}(\exp X) = 2 \cosh \sqrt{\lambda} \geq 1.$$  

However:

**Proposition 28** A symplectic matrix $S$ is symmetric positive definite if and only if $S = \exp X$ with $X \in \mathfrak{sp}(n)$ and $X = X^T$. The mapping $\exp$ is a diffeomorphism

$$\mathfrak{sp}(n) \cap \text{Sym}(2n, \mathbb{R}) \longrightarrow \text{Sp}(n) \cap \text{Sym}_+(2n, \mathbb{R})$$

($\text{Sym}_+(2n, \mathbb{R})$ is the set of positive definite symmetric matrices).

**Proof.** If $X \in \mathfrak{sp}(n)$ and $X = X^T$ then $S$ is both symplectic and symmetric positive definite. Assume conversely that $S$ symplectic and symmetric positive definite. The exponential mapping is a diffeomorphism $\exp : \text{Sym}(2n, \mathbb{R}) \longrightarrow \text{Sym}_+(2n, \mathbb{R})$ (the positive definite symmetric matrices) hence there exits a unique $X \in \text{Sym}(2n, \mathbb{R})$ such that $S = \exp X$. Let us show that $X \in \mathfrak{sp}(n)$. Since $S = S^T$ we have $SJS = J$ and hence $S = -JSJ^{-1}$. Because $-J = J^{-1}$ it follows that

$$\exp X = J^{-1}(\exp(-X))J = \exp(-J^{-1}XJ)$$

and $J^{-1}XJ$ being symmetric, we conclude that $X = J^{-1}XJ$ that is $JX = -XJ$, showing that $X \in \mathfrak{sp}(n)$. □

We will refine the result above in Subsection 1.5, Proposition 31, by using the Cartan decomposition theorem. This will in particular allow us to obtain a precise formula for calculating $X$ in terms of the logarithm of $S = \exp X$.

### 1.5 Factorization Results in $\text{Sp}(n)$

Factorization (or “decomposition”) theorems for matrices are very useful since they often allow to reduce lengthy or complicated calculations to simpler typical cases. In this section we study three particular factorization procedures symplectic matrices.

**Polar and Cartan decomposition in $\text{Sp}(n)$**

Any matrix $M \in \text{GL}(m, \mathbb{R})$ can be written uniquely as $M = RP$ (or $PR$) where $R$ is orthogonal and $P$ positive definite: this is the classical polar decomposition theorem from elementary linear algebra. Let us specialize this result to the symplectic case. We begin with a rather weak result:

**Proposition 29** For every $S \in \text{Sp}(n)$ there exists a unique $U \in \text{U}(n)$ and a unique $R \in \text{Sp}(n)$, $R$ symmetric positive definite, such that $S = RU$ (resp. $S = UR$).
Proof. Set $R = S^T S$ and define $U$ by $S = (S^T S)^{-1/2} U$. Since $(S^T S)^{-1/2} \in \text{Sp}(n)$ in view of Corollary 25, we have $U \in \text{Sp}(n)$. On the other hand

$$UU^T = (S^T S)^{-1/2} S S^T (S^T S)^{-1/2} = I$$

so that we actually have

$$U \in \text{Sp}(n) \cap O(2n) = U(n).$$

That we can alternatively write $S = UR$ (with different choices of $U$ and $R$ than above) follows by applying the result above to $S^T$.

We are going to precise Proposition 29. For this we need a suitable notion of logarithm for invertible matrices. Recall from elementary linear algebra that if $K = \mathbb{R}$ or $\mathbb{C}$ and $M$ is an invertible $m \times m$ matrix with entries in $K$ then there exists a $m \times m$ matrix $L$ such that $M = e^L$.

Let us define

$$\text{Log} \ M = \int_{-\infty}^{0} \left[ (\lambda I - M)^{-1} - (\lambda - 1)^{-1} I \right] d\lambda; \quad (1.38)$$

it is straightforward to check that when $m = 1$ and $M$ is a scalar $\lambda > 0$ formula (1.38) reduces to the usual logarithm $\text{Log} \ \lambda$.

It turns out that formula (1.38) defines a bona fide logarithm for matrices having no eigenvalues on the negative half-axis:

**Proposition 30**  Assume that $M$ has no eigenvalues $\lambda \leq 0$. Then

(i) $\text{Log} \ M$ defined by (1.38) exists;

(ii) We have

$$e^{\text{Log} \ M} = M, \quad (\text{Log} \ M)^T = \text{Log} \ M^T$$

and also

$$\text{Log} \ M^{-1} = -\text{Log} \ M, \quad \text{Log} (AMA^{-1}) = A(\text{Log} \ M)A^{-1} \quad (1.39)$$

for every invertible matrix $A$.

**Proof.** It is no restriction to assume that $M = \lambda I + N$ with $\lambda > 0$. Set

$$f(M) = \int_{-\infty}^{0} \left[ (\lambda - M)^{-1} - (\lambda - 1)^{-1} I \right] d\lambda.$$ 

We have

$$(\lambda I - M)^{-1} = ((\lambda - \mu)I - N)^{-1} = \sum_{j=0}^{k_0} (\lambda - \mu)^{-k+1} N^j$$
and hence
\[ f(M) = \int_{-\infty}^{0} \left( \frac{1}{\lambda - \mu} - \frac{1}{\lambda - 1} \right) \, I d\lambda + \sum_{k=1}^{k_0} \left( \int_{-\infty}^{0} (\lambda - \mu)^{-k-1} \, d\lambda \right) N^k \]
that is, calculating explicitly the integrals,
\[ f(M) = (\log \mu) I + \sum_{k=1}^{k_0} \frac{(-1)^{k+1}}{k} (\mu^{-1} N)^k \]
\[ = (\log \mu) I + \sum_{k=1}^{k_0} \frac{(-1)^{k+1}}{k} (\mu^{-1} M - I)^k. \]

Direct substitution of the sum in the right-hand side in the power series for the exponential yields the matrix \( \mu^{-1} M \). Hence \( \exp f(M) = M \) which we set out to prove. Formulae (1.39) readily follow from definition (1.38) of the logarithm, and so does the equality \( (\log M)^T = \log M^T \).

The following consequence of Proposition 30, which refines Proposition 28, will be instrumental in the proof of the symplectic version of Cartan’s decomposition theorem from the theory of Lie groups (Varadarajan [73]):

**Proposition 31** If \( S \in \text{Sp}(n) \) is positive definite, then \( X = \log S \) belongs to the symplectic Lie algebra \( \mathfrak{sp}(n) \). That is, for every \( S \in \text{Sp}(n) \cap \text{Sym}_+(2n, \mathbb{R}) \) (the set of symmetric positive-definite symplectic matrices) we have
\[ S = e^{\log S}, \quad \log S \in \mathfrak{sp}(n). \]

**Proof.** Since \( S \) is symplectic we have \( S^{-1} = JS^T J^{-1} \). Taking the logarithm of both sides of this equality, and using Proposition 30 together with the equality \( J^{-1} = -J \) we get
\[ X = -J (\log S^T)^{-1} J (\log S^T) J. \]
We claim that \( XJ + JX^T = 0 \). The result will follow. We have
\[ XJ = -J (\log S^T) (J^{-1} (\log S^T) J) J = -J (\log S^{-1}) J \]
hence, using the fact that \( \log S^T = (\log S)^T \),
\[ XJ = (\log S) J = -JX^T \]
proving our claim.

Let us refine the results above by using Cartan’ decomposition theorem from the theory of Lie groups:

**Proposition 32** Every \( S \in \text{Sp}(n) \) can be written \( S = U e^X \) where \( U \in U(n) \) and \( X = \frac{1}{2} \log (S^T S) \) is in \( \mathfrak{sp}(n) \cap \text{Sym}(2n, \mathbb{R}) \).
Proof. The symplectic matrix $S^T S$ has no negative eigenvalues hence its logarithm $\log(S^T S)$ exists and is in $\mathfrak{sp}(n)$ in view of Proposition 31. It is moreover obviously symmetric. It follows that $X \in \mathfrak{sp}(n)$ and hence $e^X$ and $R$ are both in $\text{Sp}(n)$. Since we also have $U \in O(2n)$ in view of Cartan’s theorem, the proposition follows since we have $\text{Sp}(n) \cap O(2n) = U(n)$. □

A first consequence of the results above is that the symplectic group $\text{Sp}(n)$ is contractible to its subgroup $U(n)$ (which, by the way, gives a new proof of the fact that $\text{Sp}(n)$ is connected):

Corollary 33

(i) The standard symplectic group $\text{Sp}(n)$ can be retracted to the unitary group $U(n)$.

(ii) The set $\text{Sp}(n) \cap \text{Sym}_+(2n, \mathbb{R})$ is contractible to a point.

Proof. Proof of (i). Let $t \mapsto S(t), 0 \leq t \leq 1$, be a loop in $\text{Sp}(n)$. In view of Proposition 32 we can write $S(t) = U(t)e^{X(t)}$ where $U(t) \in U(n)$ and $X(t) = \frac{1}{2} \log(S^T(t)S(t))$. Since $t \mapsto S(t)$ is continuous, so is $t \mapsto X(t)$ and hence also $t \mapsto U(t)$. Consider now the continuous mapping $h : [0,1] \times [0,1] \rightarrow \text{Sp}(n)$ defined by

$$h(t,t') = U(t)e^{(1-t')X(t)} , \quad 0 \leq t \leq t' \leq 1.$$ 

This mapping is a homotopy between the loops $t \mapsto h(t,0) = S(t)$ and $t \mapsto h(t,1) = R(t)$. Obviously $h(t,t') \in \text{Sp}(n)$ hence (i). Part (ii) follows, taking $R(t) = 1$ in the argument above. □

It follows from Corollary 33 that the fundamental group $\pi_1[\text{Sp}(n)]$ is isomorphic to $\pi_1[\text{U}(n, \mathbb{C})]$, that is to the integer group $(\mathbb{Z}, +)$. Let us make a precise construction of the isomorphism $\pi_1[\text{Sp}(n)] \cong \pi_1[\text{U}(n, \mathbb{C})]$.

Proposition 34

The mapping $\Delta : \text{Sp}(n) \rightarrow S^1$ defined by $\Delta(S) = \det u$ where $u$ is the image in $\text{U}(n, \mathbb{C})$ of $U = S(S^T S)^{-1/2} \in \text{U}(n)$ induces an isomorphism

$$\Delta_* : \pi_1[\text{Sp}(n)] \cong \pi_1[\text{U}(n, \mathbb{C})]$$

and hence an isomorphism $\pi_1[\text{Sp}(n)] \cong \pi_1[S^1] \cong (\mathbb{Z}, +)$.

Proof. In view of Corollary 33 above and its proof any loop $t \mapsto S(t) = R(t)e^{X(t)}$ in $\text{Sp}(n)$ is homotopic to the loop $t \mapsto R(t)$ in $\text{U}(n)$. Now $S^T(t)S(t) = e^{2X(t)}$ (because $X(t)$ is in $\mathfrak{sp}(n) \cap \text{Sym}(2n, \mathbb{R})$) and hence

$$R(t) = S(t)(S^T(t)S(t))^{-1/2}.$$ 

The result follows, composing $\Delta_*$ with the isomorphism $\pi_1[\text{U}(n, \mathbb{C})] \cong \pi_1[S^1]$ induced by the determinant map (see Lemma 62 in Subsection 2.2).

□

Let us now introduce the useful notion of free symplectic matrix, and the associated concept of generating function.
Free symplectic matrices

The notion of free symplectic matrix plays a very important role in many practical issues. For instance, it is the key to the definition of the metaplectic group [50, 26]. A noticeable fact is, in addition, that every symplectic matrix can be written as the product of exactly two free symplectic matrices.

**Definition 35** Let \( \ell \) be an arbitrary Lagrangian plane in \((\mathbb{R}^{2n}_z, \sigma)\) and \( S \in \text{Sp}(n) \). We say that the matrix \( S \) is “free relatively to \( \ell \)” if \( S \ell \cap \ell = 0 \). When \( \ell = \ell_P = 0 \times \mathbb{R}_n^p \) we simply say that \( S \) is a “free symplectic matrix”.

That it suffices to consider free symplectic matrices up to conjugation follows from the fact that \( S \in \text{Sp}(n) \) is free relatively to \( \ell \) if and only if \( S^{-1}SS_0 \) is a free symplectic matrix for every \( S_0 \in \text{Sp}(n) \) such that \( S_0 \ell = \ell_P \).

Writing \( S \) as a block matrix one has:

\[
S = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ is free } \iff \det B \neq 0. \tag{1.40}
\]

Suppose in fact that \( z \in S\ell_P \cap \ell_P \). This is equivalent to \( x = 0 \) and \( Bp = 0 \), that is to \( z = 0 \). It follows from condition (1.40) that

\[
S \text{ is free } \iff \det \left( \frac{\partial x}{\partial p'}(z_0) \right) \neq 0. \tag{1.41}
\]

This suggests the following extension of definition 35:

**Definition 36** Let \( f \) be a symplectomorphism of \((\mathbb{R}^{2n}_z, \sigma)\) defined in a neighborhood of some point \( z_0 \). We will say that \( f \) is free at the point \( z_0 \) if

\[
\det (\partial x/\partial p'(z_0)) \neq 0.
\]

Equivalently: the symplectic matrix \( S = Df(z_0) \) is free.

The equivalence of both conditions follows from the observation that the Jacobian matrix

\[
Df(z_0) = \begin{bmatrix} \frac{\partial x}{\partial x'}(z_0) & \frac{\partial x}{\partial p'}(z_0) \\ \frac{\partial p}{\partial x'}(z_0) & \frac{\partial p}{\partial p'}(z_0) \end{bmatrix}
\]

is indeed free if and only if its upper right corner \( \frac{\partial x}{\partial p'}(z_0) \) is invertible.

A very useful property is that every symplectic matrix is the product of two free symplectic matrices. This is a particular case of the following very useful result which will yield a precise factorization result for symplectic matrices, and is in addition the key to many of the properties of the metaplectic group we will study later on:

**Proposition 37** For every \((S, \ell_0) \in \text{Sp}(n) \times \text{Lag}(n)\) there exist two matrices \( S_1, S_2 \) such that \( S = S_1S_2 \) and \( S_1 \ell_0 \cap \ell_0 = S_2 \ell_0 \cap \ell_0 = 0 \). In particular, choosing \( \ell_0 = \ell_P \), every symplectic matrix is the product of two free symplectic matrices.
Proof. The second assertion follows from the first choosing $\ell_0 = 0 \times \mathbb{R}^n$. Recall that $\text{Sp}(n)$ acts transitively on the set of all pairs $(\ell, \ell')$ such that $\ell \cap \ell' = 0$. Choose $\ell'$ transverse to both $\ell_0$ and $S\ell$. There exists $S_1 \in \text{Sp}(n)$ such that $S_1(\ell_0, \ell') = (\ell', S\ell_0)$, that is $S_1\ell_0 = \ell'$ and $S\ell_0 = S_1\ell'$. Since $\text{Sp}(n)$ acts transitively on $\text{Lag}(n)$ we can find $S_2$ such that $\ell' = S_2\ell_0$ and hence $S\ell_0 = S_1S_2\ell_0$. It follows that there exists $S'' \in \text{Sp}(n)$ such that $S''\ell_0 = \ell_0$ and $S = S_1S_2S''$. Set $S_2 = S_2S''$. Then $S = S_1S_2$ and we have

$$S_1\ell_0 \cap \ell_0 = \ell' \cap \ell_0 = 0, \quad S_2\ell_0 \cap \ell_0 = S_2'\ell_0 \cap \ell_0 = \ell' \cap \ell_0 = 0.$$  

The proposition follows. □

Free generating functions

Another interesting property of free symplectic matrices are that they can be “generated” by a function $W$ defined on $\mathbb{R}^n_x \times \mathbb{R}^n_p$, in the sense that:

$$(x, p) = S(x', p') \iff p = \partial_x W(x, x') \quad \text{and} \quad p' = -\partial_{x'} W(x, x'). \quad (1.42)$$

Suppose that

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad (1.43)$$

is a free symplectic matrix. We claim that a generating function for $S$ is the quadratic form

$$W(x, x') = \frac{1}{2} \langle DB^{-1}x, x \rangle - \langle B^{-1}x, x' \rangle + \frac{1}{2} \langle B^{-1}Ax', x' \rangle. \quad (1.44)$$

In fact,

$$\partial_x W(x, x') = DB^{-1}x - (B^{-1})^T x'$$

$$\partial_{x'} W(x, x') = -B^{-1}x' + B^{-1}Ax'$$

and hence, solving in $x$ and $p$,

$$x = Ax' + Bp', \quad p = Cx' + Dp'.$$

Notice that the matrices $DB^{-1}$ and $B^{-1}A$ are symmetric in view of (1.20)). In fact if conversely $W$ is a quadratic form of the type

$$W(x, x') = \frac{1}{2} \langle Px, x \rangle - \langle Lx, x' \rangle + \frac{1}{2} \langle Qx', x' \rangle \quad (1.45)$$

with $P = P^T$, $Q = Q^T$, and $\det L \neq 0$, then the matrix

$$S_W = \begin{bmatrix} L^{-1}Q & L^{-1}P \\ PL^{-1}Q - L^T & L^{-1}P \end{bmatrix} \quad (1.46)$$

is a free symplectic matrix whose generating function is (1.45). To see this, it suffices to remark that we have

$$(x, p) = S_W(x', p') \iff p = Px - L^T x' \text{ and } p' = Lx - Qx'$$
and to solve the equations $p = Px - LT'x'$ and $p' = Lx - Qx'$ in $x, p$.

If $S_W$ is a free symplectic matrix, then its inverse $(S_W)^{-1}$ is also a free symplectic matrix, in fact:

$$(S_W)^{-1} = S_W^*, \quad W^*(x, x') = -W(x', x). \tag{1.47}$$

This follows from the observation that if

$$S = S_W = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

is free then its inverse

$$S_W^{-1} = \begin{bmatrix} D^T & -B^T \\ -C^T & A^T \end{bmatrix}$$

(see (1.22)) is also free. It is generated by the function

$$W^*(x, x') = -\frac{1}{2} \langle A^T(B^T)^{-1}x, x \rangle + \langle (B^T)^{-1}x, x' \rangle - \frac{1}{2} \langle (B^T)^{-1}D^T x', x' \rangle$$
$$= -\frac{1}{2} \langle B^{-1}Ax, x \rangle + \langle B^{-1}x', x \rangle - \frac{1}{2} \langle DB^{-1}x', x' \rangle$$
$$= -W(x', x).$$

There is thus a bijective correspondence between free symplectic matrices in $\text{Sp}(n)$ and quadratic polynomials of the type $W$ above. Since every such polynomial is determined by a triple $(P, L, Q)$, $P$ and $Q$ symmetric and $\det L \neq 0$ it follows that the subset of $\text{Sp}(n)$ consisting of all free symplectic matrices is a submanifold of $\text{Sp}(n)$ with dimension $(n + 1)(2n - 1)$. In particular, $\text{Sp}_0(n)$ has codimension 1 in $\text{Sp}(n)$.

Recall that if $P$ and $L$ are, respectively, a symmetric and an invertible $n \times n$ matrix then

$$V_P = \begin{bmatrix} I & 0 \\ -P & I \end{bmatrix}, \quad U_P = \begin{bmatrix} -P & I \\ -I & 0 \end{bmatrix}, \quad M_L = \begin{bmatrix} L^{-1} & 0 \\ 0 & LT \end{bmatrix}. \tag{1.48}$$

**Proposition 38** If $S$ is a free symplectic matrix (1.40), then

$$S = V_{-DB^{-1}}M_{B^{-1}}U_{-B^{-1}A} \tag{1.49}$$

and

$$S = V_{-DB^{-1}}M_{B^{-1}}J V_{-B^{-1}A}. \tag{1.50}$$

**Proof.** We begin by noting that

$$S = \begin{bmatrix} I & 0 \\ DB^{-1} & I \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & DB^{-1}A - C \end{bmatrix} \begin{bmatrix} B^{-1}A & I \\ -I & 0 \end{bmatrix} \tag{1.51}$$

for any matrix (1.40), symplectic or not. If now $S$ is symplectic, then the middle factor in the right-hand-side of (1.51) also is symplectic, since the
first and the third factors obviously are. Taking the condition $AD^T - BCT = I$ in (1.21) into account, we have $DB^{-1}A - C = (B^T)^{-1}$ and hence

$$\begin{bmatrix} B & 0 \\ 0 & DB^{-1}A - C \end{bmatrix} = \begin{bmatrix} B & 0 \\ 0 & (B^T)^{-1} \end{bmatrix}$$

so that

$$S = \begin{bmatrix} I & 0 \\ DB^{-1} & I \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & (B^T)^{-1} \end{bmatrix} \begin{bmatrix} B^{-1}A & I \\ -I & 0 \end{bmatrix}. \quad (1.52)$$

The factorization (1.49) follows (both $DB^{-1}$ and $B^{-1}A$ are symmetric, as a consequence of the relations $B^TD = D^TB$ and $B^TA = A^TB$ in (1.20)). Noting that

$$\begin{bmatrix} B^{-1}A & I \\ -I & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ B^{-1}A & I \end{bmatrix}$$

the factorization (1.50) follows as well. \(\blacksquare\)

Conversely, if a matrix $S$ can be written in the form $V_PM_LJV_Q$, then it is a free symplectic matrix. In fact:

$$S = SW = \begin{bmatrix} L^{-1}Q & L^{-1} \\ PL^{-1}Q - LT & L^{-1}P \end{bmatrix} \quad (1.53)$$

as is checked by a straightforward calculation.

From this result together with Proposition 37 follows that every element of $\text{Sp}(n)$ is the product of symplectic matrices of the type $V_P$, $M_L$ and $J$. More precisely:

**Corollary 39** Each of the sets

$$\{V_P, M_L, J : P = P^T, \; \det L \neq 0\}$$

and

$$\{U_P, M_L : P = P^T, \; \det L \neq 0\}$$

generates $\text{Sp}(n)$.

**Proof.** Taking $\ell_0 = 0 \times \mathbb{R}^n$ in Proposition 37 every $S \in \text{Sp}(n)$ is the product of two free symplectic matrices. It now suffices to apply Proposition 38. \(\blacksquare\)

### 1.6 Hamiltonian Mechanics

In this section we review the basics of Hamiltonian mechanics and show that it is actually tightly linked to symplectic geometry. For details and proofs see for instance Abraham and Marsden [1], Arnol’d [3], Hofer and Zehnder [38].
Hamiltonian vector fields and flows

In what follows $H$ denotes an infinitely differentiable function $\mathbb{R}^{2n} \rightarrow \mathbb{R}$ ("Hamiltonian").

By definition the Hamilton equations associated with $H$ is the system of $2n$ ordinary differential equations

$$
\dot{x}_j = \frac{\partial H}{\partial p_j}(x, p) , \quad \dot{p}_j = -\frac{\partial H}{\partial x_j}(x, p)
$$

(1.54)

$(1 \leq j \leq n)$. In terms of the standard symplectic matrix $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ this system can be rewritten for short as

$$
\dot{z}(t) = J \partial_z H(z(t))
$$

(1.55)

where $\partial_z$ is the gradient operator in the variables $x_1, ..., x_n; p_1, ..., p_n$.

We will make in what follows the simplifying assumption that the solutions of Hamilton’s equations exist and are unique for every initial condition $z = z(0)$.

**Definition 40** The operator $J \partial_z$ is called “symplectic gradient”. The vector field $X_H = J \partial_z H$ is called the “Hamiltonian vector field” associated with $H$.

In terms of the coordinates $x, p$ the Hamiltonian vector field is given by

$$
X_H = (\partial_p H, -\partial_z H).
$$

(1.56)

Recalling from the elementary theory of dynamical systems that the “flow” of a vector field $X$ on $\mathbb{R}^{2n}_x$ is the family $(f_t)$ of mappings $f_t : \mathbb{R}^{2n}_x \rightarrow \mathbb{R}^{2n}_x$ such that

$$
\frac{d}{dt} f_t(z) = X(f_t(z))
$$

or, equivalently,

$$
\dot{z}(t) = X(z(t)) , \quad z(t) = f_t(z)
$$

we see that the flow $(f_t^H)$ determined by the Hamiltonian $H$ is obtained by solving Hamilton’s equations: let $t \mapsto z(t)$ be the solution of the Cauchy problem

$$
\dot{z}(t) = J \partial_z H(z(t)) , \quad z(0) = z;
$$

then $z(t) = f_t^H(z)$. Notice that Hamiltonian flows satisfy, as all flows do, the group properties:

$$
 f_t^H \circ f_t'^H = f_{t+t'}^H, \quad (f_t^H)^{-1} = f_{-t}^H, \quad f_0^H = I.
$$

(1.57)

The fundamental link between Hamiltonian mechanics and symplectic geometry lies in the fact that each $f_t^H$ is symplectomorphism. Let us prove the following complete result, the first part of which is important in the understanding of the theory of periodic orbits:
**Proposition 41**  For fixed $z$ set $S_t^H(z) = Df_t^H(z)$.

(i) The function $t \mapsto S_{t,t'}(z)$ satisfies the differential equation

$$\frac{d}{dt} S_t^H(z) = JD^2H(z(t))S_t^H(z) \quad , \quad S_0^H(z) = I$$  \hspace{1cm} (1.58)

where $D^2H(z(t))$ is the Hessian matrix of $H$ calculated at $z(t) = f^H(z)$.

(ii) We have $S_t^H(z) \in \text{Sp}(n)$ for every $z$ and $t$ hence $f_t^H$ is a symplectomorphism.

**Proof.**  Proof of (i). Taking Hamilton’s equation into account the time-derivative of the Jacobian matrix $S_t(z)$ is

$$\frac{d}{dt} S_t(z) = \frac{d}{dt} (Df_t^H(z)) = D\left( \frac{d}{dt} f_t^H(z) \right)$$

that is

$$\frac{d}{dt} S_t(z) = D(X_H(f_t^H(z))).$$

Using the fact that $X_H = J\partial_z H$ together with the chain rule, we have

$$D(X_H(f_t^H(z))) = D(J\partial_z H)(f_t^H(z), t)$$

$$= JD(\partial_z H)(f_t^H(z), t)$$

$$= J(D^2H)(f_t^H(z), t)Df_t^H(z)$$

hence $S_t(z)$ satisfies the variational equation (1.58), proving the statement (i).

Proof of (ii). Set $S_t = S_t(z)$ and $A_t = (S_t)^T JS_t$. Using the product rule together with (1.58) we have

$$\frac{dA_t}{dt} = \frac{d(S_t)^T}{dt} JS_t + (S_t)^T J \frac{dS_t}{dt}$$

$$= (S_t)^T D^2H(z, t)S_t - (S_t)^T D^2H(z, t)S_t$$

$$= 0.$$

It follows that the matrix $A_t = (S_t)^T JS_t$ is constant in $t$, hence $A_t(z) = A_0(z) = J$ so that $(S_t)^T JS_t = J$ proving that $S_t \in \text{Sp}(n)$.  \Box

**Remark 42**  The differential equation (1.58) is called the “variational equation” in the literature on dynamical systems. It describes the time-evolution of the linearized flow.

**Remark 43**  All the results and definitions above can be extended to the case where $H$ is time-dependent, that is defined on $\mathbb{R}_z^{2n} \times \mathbb{R}_t$. One must however slightly redefine the notions of Hamiltonian vector field and flow. (For instance the group property (1.57) does not hold any longer. This is remedied at by introducing the notion of time-dependent flow $(f_{t,t'}^H)$. See for instance [1, 38]).
The monodromy matrix

Let $H$ be a Hamiltonian function as above, and $(f^H_t)$ the flow determined by the associated vector field $X_H = J\partial_z$.

**Definition 44** Let $z_0 \in \mathbb{R}_z^{2n}$. The mapping

$$
\gamma : \mathbb{R} \rightarrow \mathbb{R}_z^{2n}, \quad \gamma(t) = f^H_t(z_0)
$$

is called “(Hamiltonian) orbit through $z_0$”. If there exists $T > 0$ such that $f^{H+T}_t(z_0) = f^H_t(z_0)$ for all $t \in \mathbb{R}$ one says that the orbit $\gamma$ through $z_0$ is “periodic with period $T$”. [The smallest possible period is called “primitive period”].

The following properties are obvious:

- Let $\gamma, \gamma'$ be two orbits of $H$. Then the ranges $\text{Im} \gamma$ and $\text{Im} \gamma'$ are either disjoint or identical.

- The value of $H$ along any orbit is constant (“theorem of conservation of energy”).

The first property follows from the uniqueness of the solutions of Hamilton’s equations, while the second property follows from the chain rule, setting $\gamma(t) = (x(t), p(t))$:

$$
\frac{d}{dt} H(\gamma(t)) = \langle \partial_x H(\gamma(t)), \dot{x}(t) \rangle + \langle \partial_p H(\gamma(t)), \dot{p}(t) \rangle
$$

$$
= - \langle \dot{p}(t), \dot{x}(t) \rangle + \langle \dot{x}(t), \dot{p}(t) \rangle
$$

$$
= 0
$$

where we have taken into account Hamilton’s equations.

Assume now that $\gamma$ is a periodic orbit through $z_0$. We will use the notation $S_t(z_0) = Df^H_t(z_0)$.

**Definition 45** Let $T$ be the primitive period of the periodic orbit $\gamma$ through $z_0$. The symplectic matrix $S_T(z_0)$ is called “monodromy matrix”. The eigenvalues of $S_T(z_0)$ are called the “Floquet multipliers” of $\gamma$.

The following property is well-known in Floquet theory, and independent of the Hamiltonian nature of the periodic orbit:

**Lemma 46** Let $S_T(z_0)$ be the monodromy matrix of the periodic orbit $\gamma$. We have

$$
S_{t+T}(z_0) = S_t(z_0)S_T(z_0)
$$

(1.59)

for all $t \in \mathbb{R}$. In particular $S_{NT}(z_0) = S_T(z_0)^N$ for every integer $N$. 

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**Proof.** We have, in view of the first formula (1.57), taking into account the equality \( f_{HT}(z_0) = z_0 \):

\[
f_{t+T}(z_0) = f_t(f_T(z_0))
\]
hence, by the chain rule,

\[
Df_{t+T}(z_0) = Df_t(f_T(z_0))Df_T(z_0)
\]
that is (1.59) since \( f_{HT}(z_0) = z_0 \).

Evidently the orbit through any point \( z(t) \) of the periodic orbit \( \gamma \) is also periodic. The Floquet multipliers do however not depend on the choice of origin of the orbit, and are thus a feature of the orbit itself.

**Lemma 47** (i) Monodromy matrices corresponding to the choice of different origins on the periodic orbit are conjugate of each other in \( \text{Sp}(n) \) hence the Floquet multipliers do not depend on the choice of origin on the periodic orbit. (ii) Each periodic orbit has an even number \( > 0 \) of Floquet multiplier equal to one.

**Proof.** (i) We begin by noting that if \( z_0 \) and \( z_1 \) are points on the same orbit \( \gamma \) then there exists \( t_0 \) such that \( z_0 = f_{t_0}(z_1) \). We have, by the first group property (1.57),

\[
f_t(f_{t_0}(z_1)) = f_{t_0}(f_t(z_1))
\]
hence, applying the chain rule of both sides of this equality,

\[
Df_t(f_{t_0}(z_1))Df_{t_0}(z_1) = Df_{t_0}(f_t(z_1))Df_t(z_1).
\]
Choosing \( t = T \) we have \( f_T(z_0) = z_0 \) and hence

\[
S_T(f_{t_0}(z_1))S_{t_0}(z_1) = S_{t_0}(z_1)S_T(z_1)
\]
that is, since \( f_{t_0}(z_1) = z_0 \),

\[
S_T(z_0)S_{t_0}(z_1) = S_{t_0}(z_1)S_T(z_1).
\]
It follows that the monodromy matrices \( S_T(z_0) \) and \( S_T(z_1) \) are conjugate and thus have the same eigenvalues. (ii) We have, using the chain rule together with the relation \( f_t \circ f_t = f_{t+t'} \),

\[
\left. \frac{d}{dt'} f_t(f_t(z_0)) \right|_{t'=0} = Df_t(z_0)X_H(z_0) = X_H(f_t(z_0))
\]

hence \( S_{t_0}(z_0)X_H(z_0) = X_H(z_0) \) setting \( t = T \); \( X_H(z_0) \) is thus an eigenvector of \( S_{t_0}(z_0) \) with eigenvalue one. The Lemma follows the eigenvalues of a symplectic matrix occurring in quadruples \((\lambda, 1/\lambda, \bar{\lambda}, 1/\bar{\lambda})\).

One has the following theorem, going back to Poincaré (see Abraham and Marsden [1]):
**Theorem 48** Let $E_0 = H(z_0)$ be the value of $H$ along a periodic orbit $\gamma_0$. Assume that $\gamma_0$ has exactly two Floquet multipliers equal to one. Then there exists a unique smooth 1-parameter family $(\gamma_E)$ of periodic orbits of $E$ with period $T$ parametrized by the energy $E$, and each $\gamma_E$ is isolated on the hypersurface $\Sigma_E = \{z : H(z) = E\}$ among those periodic orbits having periods close to the period $T_0$ of $\gamma_0$. Moreover $\lim_{E \to E_0} T = T_0$.

One shows, using “normal form” techniques that when the conditions of the theorem above are fulfilled, the monodromy matrix of $\gamma_0$ can be written as

$$
S_T(z_0) = S_0^T \begin{bmatrix} U & 0 \\ 0 & \tilde{S}(z_0) \end{bmatrix} S_0
$$

with $S_0 \in \text{Sp}(n)$, $\tilde{S}(z_0) \in \text{Sp}(n - 1)$ and $U$ is of the type $\begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}$ for some real number $\beta$. The $2(n - 1) \times 2(n - 1)$ symplectic matrix $\tilde{S}(z_0)$ is called the stability matrix of the isolated periodic orbit $\gamma_0$. It plays a fundamental role not only in the study of periodic orbits, but also in semiclassical mechanics (“Gutzwiller’s formula” [35]). We will return to the topic when we discuss the Conley–Zehnder index in Chapter 3.
2 Multi-Oriented Symplectic Geometry

Multi-oriented symplectic geometry, also called \(q\)-symplectic geometry, is a topic which has not been studied as it deserves in the mathematical literature; see however Leray [50], Dazord [17], de Gosson [25, 26]. The idea is the following: one begins by observing that since symplectic matrices have determinant one, the action of \(\text{Sp}(n)\) on a Lagrangian plane preserves the orientation of that Lagrangian plane. Thus, ordinary symplectic geometry is not only the study of the action

\[
\text{Sp}(n) \times \text{Lag}(n) \rightarrow \text{Lag}(n)
\]

but it is actually the study of the action

\[
\text{Sp}(n) \times \text{Lag}_2(n) \rightarrow \text{Lag}_2(n)
\]

where \(\text{Lag}_2(n)\) is the double covering of \(\text{Lag}(n)\). In a more general way, \(q\)-symplectic geometry will be the study of the action

\[
\text{Sp}_q(n) \times \text{Lag}_{2q}(n) \rightarrow \text{Lag}_{2q}(n)
\]

where \(\text{Sp}_q(n)\) is the \(q\)-th order covering of \(\text{Sp}(n)\) and \(\text{Lag}_{2q}(n)\) is the \(2q\)-th order covering of \(\text{Lag}(n)\).

The study of \(q\)-symplectic geometry makes use of an important generalization of the Maslov index, the “Leray index”. That index plays a crucial role in at least two other areas of mathematics and mathematical physics:

- It is instrumental in giving the correct phase shifts through caustics in semiclassical quantization (Leray [50], Maslov [56, 57], Mischenko \textit{et al.} [59]) because it allows to define the argument of the square root of a de Rham form on a Lagrangian manifold.

- It allows a simple and elegant calculation of Maslov indices of both Lagrangian and symplectic paths. These indices play an essential role in the study of “spectral flow” properties related to the theory of the Morse index. In [10] Booss–Bavnbek and Wojciechowski study related properties for Dirac operators.

2.1 The Signature of a Triple of Lagrangian Planes

In this section we introduce a very useful measure of the relative position of a triple of Lagrangian planes, due to Wall [72], and later redefined by Kashiwara (as exposed in Lion–Vergne [54]). Related notions are introduced in Dazord [17] and Demazure [18]. The Wall–Kashiwara index, as we will call, it is a refinement of the notion of inertia of Leray [50] in the sense that is defined for arbitrary triples, while Leray’s definition only works when some transversality condition is imposed (the same restriction applies to the index used in Guillemin–Sternberg [33]).
First properties

Let us introduce the following terminology and notation: let $Q$ be a quadratic form on some real Euclidean space. The associated symmetric matrix $M = D^2Q$ has $\mu^+$ positive eigenvalues and $\mu^-$ negative eigenvalues. We will call the difference $\mu^+ - \mu^-$ the signature of the quadratic form $Q$ and denote it by $\text{sign} Q$:

$$\text{sign} Q = \mu^+ - \mu^-.$$

**Definition 49** Let $(\ell, \ell', \ell'')$ be an arbitrary triple of Lagrangian planes in a symplectic space $(E, \omega)$. The “Wall–Kashiwara signature” (or, for short: signature) of the triple $(\ell, \ell', \ell'')$ is the signature of the quadratic form

$$Q(z, z', z'') = \omega(z, z') + \omega(z', z'') + \omega(z'', z)$$

on $\ell \oplus \ell' \oplus \ell''$.

Let us illustrate this definition in the case $n = 1$, with $\omega = -\det$. The quadratic form $Q$ is here

$$Q(z, z', z'') = -\det(z, z') - \det(z', z'') - \det(z'', z).$$

Choosing $\ell = \ell_X = \mathbb{R}_x \times 0$, $\ell'' = \ell_P = 0 \times \mathbb{R}_p$, and $\ell' = \ell_a : p = ax$, we have

$$Q = -axx' - p''x' + p''x.$$

After diagonalization this quadratic form becomes

$$Q = Z^2 - (X^2 + (\text{sign} a)Y^2)$$

and hence, by a straightforward calculation:

$$\tau(\ell_X, \ell_a, \ell_P) = \begin{cases} -1 & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ +1 & \text{if } a < 0 \end{cases}$$

(2.1)

(this formula will be generalized to $(\mathbb{R}^{2n}_z, \sigma)$ in Corollary 52). The signature $\tau(\ell, \ell', \ell'')$ of three lines is thus 0 if any two of them coincide, $-1$ if the line $\ell'$ lies “between” $\ell$ and $\ell''$ (the plane being oriented in the usual way), and $+1$ if it lies outside. An essential observation is that we would get the same values for an arbitrary triple $\ell, \ell', \ell''$ of lines having the same relative positions as $\ell_X, \ell_a, \ell_P$ because one can always reduce the general case to that of the triple $(\ell_X, \ell_a, \ell_P)$, by using a matrix with determinant one. Thus the signature is here what one sometimes call the “cyclic order” of three lines. It is easy to check that the following formula holds:

$$\tau(\ell, \ell', \ell'') = 2 \left[ \frac{\theta - \theta'}{2\pi} \right]_{\text{anti}} - 2 \left[ \frac{\theta - \theta''}{2\pi} \right]_{\text{anti}} + 2 \left[ \frac{\theta' - \theta''}{2\pi} \right]_{\text{anti}}$$

(2.2)
The line \( \ell : x \cos \alpha + p \sin \alpha = 0 \) is here identified with \( \theta = 2\alpha \) and \([\cdot]_{\text{anti}} \) is the symmetrized integer part function, that is
\[
[s]_{\text{anti}} = \frac{1}{2}(\lfloor s \rfloor - \lfloor -s \rfloor), \quad [s] = k \text{ if } k \leq s < k + 1
\]
(for \( k \) an integer).

Let us now return to the general case. The following properties of the signature are immediate:

**WK1** \( \tau \) is \( \text{Sp}(E, \omega) \)-invariant: for every \( S \in \text{Sp}(E, \omega) \) and \( \ell, \ell' \) and \( \ell'' \in \text{Lag}(E, \omega) \) we have
\[
\tau(S\ell, S\ell', S\ell'') = \tau(\ell, \ell', \ell'') \quad (2.3)
\]
(because \( \sigma(Sz, Sz') = \sigma(z, z') \), and so on).

**WK2** \( \tau \) is totally antisymmetric: for any permutation \( p \) of the set \( \{1, 2, 3\} \) we have
\[
\tau(\ell_{p(1)}, \ell_{p(2)}, \ell_{p(3)}) = (-1)^{\text{sgn}(p)}\tau(\ell_1, \ell_2, \ell_3) \quad (2.4)
\]
where \( \text{sgn}(p) = 0 \) if \( p \) is even, 1 if \( p \) is odd (this immediately follows from the antisymmetry of \( \sigma \)).

**WK3** Let \( \tau' \) and \( \tau'' \) be the signature in \( \text{Lag}(E', \omega') \) and \( \text{Lag}(E'', \omega'') \) respectively, and \( \tau \) the signature in \( \text{Lag}(E, \omega) \) with \( (E, \omega) = (E', \omega') \oplus (E'', \omega'') \). Then
\[
\tau(\ell'_1 \oplus \ell''_1, \ell'_2 \oplus \ell''_2, \ell'_3 \oplus \ell''_3) = \tau'(\ell'_1, \ell'_2, \ell'_3) + \tau''(\ell''_1, \ell''_2, \ell''_3)
\]
for \( (\ell'_1, \ell'_2, \ell'_3) \in (\text{Lag}(E', \omega'))^3 \) and \( (\ell''_1, \ell''_2, \ell''_3) \in (\text{Lag}(E'', \omega''))^3 \).

At this point we want to mention that the properties listed above characterize the Wall–Kashiwara signature up to a factor: Cappell, Lee, and Miller [14] have shown that if \( (\chi_n)_{n \geq 1} \) is a family of functions \( \chi_n : (\text{Lag}(n))^3 \rightarrow \mathbb{Z} \) satisfying (WK1)–(WK2), then each \( \chi_n \) is proportional to the Wall–Kashiwara signature \( \tau_n = \tau \) on \( \text{Lag}(n) \).

Here are two results which sometimes simplify calculations of the signature:

**Proposition 50** Assume that \( \ell \cap \ell'' = 0 \), then \( \tau(\ell, \ell', \ell'') \) is the signature of the quadratic form
\[
Q'(z') = \omega(P(\ell, \ell'')z', z') = \omega(z', P(\ell'', \ell)z')
\]
on \( \ell' \), where \( P(\ell, \ell'') \) is the projection onto \( \ell \) along \( \ell'' \) and \( P(\ell'', \ell) = I - P(\ell, \ell'') \) is the projection on \( \ell'' \) along \( \ell \).
Proof. We have

\[ Q(z, z', z'') = \omega(z, z') + \omega(z', z'') + \omega(z'', z) = \omega(z, P(\ell', \ell)z') + \omega(P(\ell, \ell'')z', z'') + \omega(z'', z) = \omega(P(\ell, \ell'')z', P(\ell', \ell)z') - \omega(z - P(\ell, \ell'')z', z'' - P(\ell'', \ell)z'). \]

Let \( u = z - P(\ell, \ell'')z' \), \( u' = z' \), \( u'' = z'' - P(\ell'', \ell)z' \). The signature of \( Q \) is then the signature of the quadratic form

\[ (u, u', u'') \mapsto \omega(P(\ell, \ell')u', P(\ell'', \ell)u') - \omega(u, u'') \]

hence the result since the signature of the quadratic form \( (u, u'') \mapsto \omega(u, u'') \) is obviously equal to zero.

Proposition 51 Let \((\ell, \ell', \ell'')\) be a triple of Lagrangian planes such that an \( \ell = \ell \cap \ell' + \ell \cap \ell'' \). Then \( \tau(\ell, \ell', \ell'') = 0 \).

Proof. Let \( E' \subset \ell \cap \ell' \) and \( E'' \subset \ell \cap \ell'' \) be subspaces such that \( \ell = E' \oplus E'' \). Let

\[ (z, z', z'') \in \ell \times \ell' \times \ell'' \]

and write \( z = u' + u'' \), \((u, u') \in E' \times E'' \). We have

\[ \sigma(z, z') = \sigma(u' + u'', z') = \sigma(u'', z') \]
\[ \sigma(z'', z) = \sigma(z'', u' + u'') = \sigma(u'', z) \]

and hence

\[ Q(z, z', z'') = \sigma(u'', z') + \sigma(z', z'') + \sigma(z'', u'). \]

Since \( \sigma(u', u'') = 0 \) this is

\[ Q(z, z', z'') = \sigma(z' - u', z'' - u'') \]

so that \( \tau(\ell, \ell', \ell'') \) is the signature of the quadratic form \((y', y'') \mapsto \sigma(y', y'')\) on \( \ell' \times \ell'' \). This signature is equal to zero hence the result.

The following consequence of Proposition 50 generalizes formula (2.1):

Corollary 52 Let \((E, \omega)\) be the standard symplectic space \((\mathbb{R}^{2n}_\mathbb{Z}, \sigma)\). Let \( \ell_X = \mathbb{R}^n \times 0 \), \( \ell_P = 0 \times \mathbb{R}^n \), and \( \ell_A = \{(x, Ax) : x \in \mathbb{R}^n\} \), \( A \) being a symmetric linear mapping \( \mathbb{R}^n \longrightarrow \mathbb{R}^n \). Then

\[ \tau(\ell_P, \ell_A, \ell_X) = \text{sign}(A) \quad \tau(\ell_X, \ell_A, \ell_P) = -\text{sign}(A). \quad (2.5) \]

Proof. Formulae (2.5) are equivalent in view of the antisymmetry of \( \tau \). In view of the proposition above \( \tau(\ell_P, \ell_A, \ell_X) \) is the signature of the quadratic form \( Q' \) on \( \ell_A \) given by

\[ Q'(z) = \sigma(P(\ell_P, \ell_A, \ell_X)z, P(\ell_X, \ell_P)z) \]

hence \( Q'(z) = \langle x, Ax \rangle \) and the corollary follows.

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The cocycle property of $\tau$

Less obvious—but of paramount importance for the general theory of the Leray index is the following “cocycle property” of the Wall–Kashiwara signature.

**Theorem 53** For $\ell_1, \ell_2, \ell_3, \ell_4$ in $\text{Lag}(E, \omega)$ we have

$$\tau(\ell_1, \ell_2, \ell_3) - \tau(\ell_2, \ell_3, \ell_4) + \tau(\ell_1, \ell_3, \ell_4) - \tau(\ell_1, \ell_2, \ell_4) = 0.$$  

(2.6)

**Proof.** We begin by rewriting the quadratic form $Q$ defining $\tau$ in a more tractable form. Let $\ell, \ell', \ell''$ be three arbitrary Lagrangian planes and choose a symplectic basis $B = \{e_1, ..., e_n\} \cup \{f_1, ..., f_n\}$ of $(E, \omega)$ such that $\ell \cap \ell_0 = \ell' \cap \ell_0 = \ell'' \cap \ell_0 = 0$ where $\ell_0 = \text{Span}\{f_1, ..., f_n\}$. Let $\dim E = 2n$ and write a vector $z$ in the basis $B$ as

$$z = \sum_{i=1}^{n} x_i e_i + \sum_{j=1}^{n} p_{ji} f_j,$$

there exist symmetric matrices $M, M', M''$ such that

$$\ell : p = Mx, \quad \ell' : p = M'x, \quad \ell'' : p = M''x$$

(Proposition 14, (ii)). The integer $\tau(\ell, \ell', \ell'')$ being a symplectic invariant, it is the signature of the quadratic form

$$R(z, z', z'') = \sigma(x, Mx; x', Mx') + \sigma(x', M'x'; x'', M''x'') + \sigma(x'', M''x''; x, Mx)$$

which we can rewrite after a straightforward calculation as

$$R(z, z', z'') = \frac{1}{2} X^T R X, \quad X = (x, x', x'')$$

where $R$ is the symmetric matrix

$$R = \begin{bmatrix}
0 & M - M' & M'' - M \\
M - M' & 0 & M' - M'' \\
M'' - M & M' - M'' & 0
\end{bmatrix}.$$

The quadratic form $R$ has same signature $\tau(\ell, \ell', \ell'')$ as $R \circ V$ for any invertible matrix $V$. Choosing

$$V = \begin{bmatrix}
0 & I & I \\
I & 0 & I \\
I & I & 0
\end{bmatrix}$$
the matrix of the quadratic form $\mathcal{R} \circ V$ is
\[
\frac{1}{2} V^T R V = \begin{bmatrix}
M' - M'' & 0 & 0 \\
0 & M'' - M & 0 \\
0 & 0 & M - M'
\end{bmatrix}
\]
and hence
\[
\tau(\ell, \ell', \ell'') = \text{sign}(M - M') + \text{sign}(M' - M'') + \text{sign}(M'' - M). \tag{2.7}
\]
The theorem now easily follows: writing, with obvious notations
\[
\tau(\ell_1, \ell_2, \ell_3) = \text{sign}(M_1 - M_2) + \text{sign}(M_2 - M_3) + \text{sign}(M_3 - M_1).
\]
\[
\tau(\ell_2, \ell_3, \ell_4) = \text{sign}(M_2 - M_3) + \text{sign}(M_3 - M_4) + \text{sign}(M_4 - M_2).
\]
\[
\tau(\ell_1, \ell_3, \ell_4) = \text{sign}(M_1 - M_3) + \text{sign}(M_3 - M_4) + \text{sign}(M_4 - M_1)
\]
we get, since $\text{sign}(M_i - M_j) = -\text{sign}(M_j - M_i)$:
\[
\tau(\ell_1, \ell_2, \ell_3) - \tau(\ell_2, \ell_3, \ell_4) + \tau(\ell_1, \ell_3, \ell_4) = \\
\text{sign}(M_1 - M_2) + \text{sign}(M_2 - M_4) + \text{sign}(M_4 - M_1)
\]
that is
\[
\tau(\ell_1, \ell_2, \ell_3) - \tau(\ell_2, \ell_3, \ell_4) + \tau(\ell_1, \ell_3, \ell_4) = \tau(\ell_1, \ell_2, \ell_4).
\]

Formula (2.6) has a straightforward combinatorial interpretation. Recall the following terminology from algebraic topology (see for instance Spanier [68]): let $X$ be a set, $k$ an integer $\geq 0$, $(G,+)$ an Abelian group. By definition, a $(G$-valued) $k$-cochain on $X$ (or just cochain when the context is clear) is a mapping
\[
c : X^{k+1} \longrightarrow G.
\]
To every $k$-cochain one associates its coboundary: it is the $(k+1)$-cochain $\partial c$ defined by
\[
\partial c(x_0, ..., x_{k+1}) = \sum_{j=0}^{k+1} (-1)^j c(x_0, ..., \hat{x}_j, ..., x_{k+1}), \tag{2.8}
\]
where the cap $\hat{}$ suppresses the term it covers. The operator
\[
\partial_k : \{k\text{-cochains}\} \longrightarrow \{(k+1)\text{-cochains}\}
\]
defined by (2.8) is called the coboundary operator. We will use the collective notation $\partial$ whenever its range is obvious. The coboundary operator satisfies the important (but easy to prove) equality $\partial^2 c = 0$ for every cochain $c$. A cochain $c$ is called coboundary if there exists a cochain $m$ such that $c = \partial m$. 42
A cochain $c$ is called a cocycle if $\partial c = 0$. Obviously every coboundary is a cocycle.

With this terminology the Wall–Kashiwara index is a 2-cocycle on $\text{Lag}(E, \omega)$ and we have

$$\partial \tau(\ell_1, \ell_2, \ell_3) = \tau(\ell_1, \ell_2, \ell_3) - \tau(\ell_2, \ell_3, \ell_4) + \tau(\ell_1, \ell_3, \ell_4) - \tau(\ell_1, \ell_2, \ell_4)$$

so that Theorem 53 can be restated in concise form as:

$$\partial \tau = 0.$$ 

**Topological properties of $\tau$**

Consider three lines $\ell, \ell', \ell''$ through the origin in the symplectic plane. As discussed in the beginning of the section the signature $\tau(\ell, \ell', \ell'')$ determines the relative positions of these lines. If we now move these three linear continuously, in such a way that their intersections do not change, the signature will remain unaltered. The same property remains true in higher dimensions. To prove this, we need the following elementary lemma which describes the kernel of the quadratic form defining $\tau$. In order to avoid a deluge of multiple "primes" in the proof we slightly change notations and write $(\ell_1, \ell_2, \ell_3)$ instead of $(\ell, \ell', \ell'')$ so that the defining quadratic form becomes

$$Q(z_1, z_2, z_3) = \sigma(z_1, z_2) + \sigma(z_2, z_3) + \sigma(z_3, z_1)$$

with $(z_1, z_2, z_3) \in \ell_1 \times \ell_2 \times \ell_3$.

Recall that the kernel of a quadratic form is the kernel of the matrix of the associated bilinear form.

**Lemma 54** Let $\text{Ker} Q$ be the kernel of the quadratic form $Q$. There exists an isomorphism

$$\text{Ker} Q \cong (\ell_1 \cap \ell_2) \times (\ell_2 \cap \ell_3) \times (\ell_3 \cap \ell_1).$$  \hfill (2.9)

**Proof.** Let $A$ be the matrix of $Q$. The condition $u \in \text{Ker} Q$ is equivalent to

$$v^T Au = 0 \quad \text{for all } v \in \ell_1 \times \ell_2 \times \ell_3.$$  \hfill (2.10)

In view of the obvious identity

$$(u + v)A(u + v)^T = vAv^T$$

valid for every $u$ in $\text{Ker} Q$, formula (2.10) is equivalent to the condition:

$$(u + v)A(u + v)^T - vAv^T = 0 \quad \text{for all } v \in \ell_1 \times \ell_2 \times \ell_3$$  \hfill (2.11)

that is, to

$$Q(z_1 + z'_1, z_2 + z'_2, z_3 + z'_3) - Q(z'_1 + z'_2, z'_3)$$

$$= \omega(z_1, z'_2) + \omega(z_2, z'_3) + \omega(z'_1 + z_2) + \omega(z'_2, z_3) + \omega(z'_3, z_1)$$

$$= \omega(z_1 - z_3, z'_2) + \omega(z_2 - z_1, z'_3) + \omega(z_3 - z_2, z'_1)$$

$$= 0.$$ \hfill (2.12)
Taking successively $z'_1 = z'_3 = 0$, $z'_1 = z'_2 = 0$, and $z'_2 = z'_3 = 0$ the equality (2.12) then implies

$$
\omega(z_1 - z_3, z'_2) = 0 \text{ for all } z'_2 \in \ell_2,
$$

$$
\omega(z_2 - z_1, z'_3) = 0 \text{ for all } z'_3 \in \ell_3,
$$

$$
\omega(z_3 - z_2, z'_3) = 0 \text{ for all } z'_1 \in \ell_1,
$$

hence, since $\ell_1, \ell_2, \ell_3$ are Lagrangian planes:

$$(z_3 - z_2, z_1 - z_3, z_2 - z_1) \in \ell_1 \times \ell_2 \times \ell_3.$$  

It follows that

$$
z_1 + z_2 - z_3 = z_1 + (z_2 - z_3) = (z_1 - z_3) + z_2 \in \ell_1 \cap \ell_2
$$

$$
z_2 + z_3 - z_1 = z_2 + (z_3 - z_1) = (z_2 - z_1) + z_3 \in \ell_2 \cap \ell_3
$$

$$
z_3 + z_1 - z_2 = z_3 + (z_1 - z_2) = (z_3 - z_2) + z_1 \in \ell_3 \cap \ell_1.
$$

The restriction to $\text{Ker} \ Q$ of the automorphism of $z_3$ defined by

$$(z_1, z_2, z_3) \mapsto (z'_1, z'_2, z'_3)$$

with

$$
z'_1 = z_1 + z_2 - z_3, \quad z'_2 = z_2 + z_3 - z_1, \quad z'_3 = z_3 + z_1 - z_2
$$

is thus an isomorphism of $\text{Ker} \ Q$ onto $(\ell_1 \cap \ell_2) \times (\ell_2 \cap \ell_3) \times (\ell_3 \cap \ell_1)$. 

We are now in position to prove the main topological property of the signature. Let us introduce the following notation: if $k, k', k''$ are three integers such that $0 \leq k, k', k'' \leq n$. we define a subset $\text{Lag}^3_{k,k',k''}(n)$ of $\text{Lag}^3(n)$ by

$$
\text{Lag}^3_{k,k',k''}(E, \omega) = \{(\ell, \ell', \ell'') : \dim(\ell \cap \ell') = k, \dim(\ell' \cap \ell'') = k', \dim(\ell'' \cap \ell) = k''\}.
$$

**Proposition 55** The Wall–Kashiwara signature has the following properties:

(i) It is locally constant on each set $\text{Lag}^3_{k,k',k''}(E, \omega)$.

(ii) If the triple $(\ell, \ell', \ell'')$ move continuously in $\text{Lag}^3(E, \omega)$ in such a way that $\dim(\ell \cap \ell')$, $\dim(\ell' \cap \ell'') = k'$, and $\dim(\ell'' \cap \ell)$ do not change then $\tau(\ell, \ell', \ell'')$ remains constant.

(iii) We have

$$
\tau(\ell, \ell', \ell'') = n + \dim(\ell \cap \ell') + \dim(\ell' \cap \ell'') + \dim(\ell'' \cap \ell) \quad \text{mod} \ 2 \quad (2.13)
$$

**Proof.** Properties (i) and (ii) are equivalent since $\text{Lag}(E, \omega)$ (and hence also $\text{Lag}^3(E, \omega)$) is connected. Property (iii) implies (i), and hence (ii). It is therefore sufficient to prove the congruence (2.13). Let $A$ be the matrix

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in the proof of Lemma 54. In view of the isomorphism statement (2.9) we have
\[
\text{rank}(A) = 3n - (\dim \ell \cap \ell' + \dim \ell' \cap \ell'' + \dim \ell'' \cap \ell).
\]
Let \((\tau^+, \tau^-)\) be the signature of \(A\), so that (by definition) \(\tau(\ell, \ell', \ell'') = \tau^+ - \tau^-\). Since \(\text{rank}(A) = \tau^+ + \tau^-\) we thus have
\[
\tau(\ell, \ell', \ell'') \equiv \text{rank}(A) \mod 2
\]

hence (2.13).

**Remark 56** Define a 1-cochain \(\dim\) on \(\text{Lag}(n)\) by \(\dim(\ell, \ell') = \dim(\ell \cap \ell')\). In view of the obvious relation
\[
\dim \ell \cap \ell' + \dim \ell' \cap \ell'' + \dim \ell'' \cap \ell \equiv \dim \ell \cap \ell' - \dim \ell' \cap \ell'' + \dim \ell'' \cap \ell \mod 2
\]
we can rewrite formula (2.13) as
\[
\tau(\ell, \ell', \ell'') \equiv \partial \dim(\ell, \ell', \ell'') \mod 2.
\]
(2.14)

For short: \(\tau = \partial \dim\), mod 2.

### 2.2 The Souriau Mapping and the Maslov Index

The locution *Maslov index* has become a household name in mathematics. It is actually a collective denomination for a whole constellation of discrete-values functions defined on loops (or, more generally, on paths) in \(\text{Lag}(n)\) or \(\text{Sp}(n)\), and which can be viewed as describing the number of times a given loop (or path) intersects some particular locus in \(\text{Lag}(n)\) or \(\text{Sp}(n)\) known under the omnibus name of “caustic”.

In this section we will only deal with the simplest notion of Maslov index, that of loops in \(\text{Lag}(n)\), and whose definition is due to Maslov and Arnol’d. We will generalize the notion to paths in both \(\text{Lag}(n)\) and \(\text{Sp}(n)\) when we will deal with semiclassical mechanics.

There are several (but equivalent) ways of introducing the Maslov index on \(\text{Lag}(n)\). The simplest (especially for explicit calculation) makes use of the fact that we can identify the Lagrangian Grassmannian \(\text{Lag}(n)\) with a set of matrices, using the so-called “Souriau mapping”. This will provide us not only with a simple way of defining correctly the Maslov index of Lagrangian loops, but will also allow us to construct the “Maslov bundle” in Subsection 2.3.

**The Souriau mapping**

Recall that the mapping
\[
A + iB \mapsto \begin{bmatrix} A & -B \\ B & A \end{bmatrix}
\]
identifies $U(n, \mathbb{C})$ with a subgroup $U(n)$ of $\text{Sp}(n)$. That subgroup consists of all

$$U = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$$

where $A$ and $B$ satisfy the conditions

$$A^T A + B^T B = I, \quad A^T B = B^T A \quad (2.15a)$$

$$A A^T + B B^T = I, \quad A B^T = B A^T. \quad (2.15b)$$

Also recall that the unitary group $U(n, \mathbb{C})$ acts transitively on the Lagrangian Grassmannian $\text{Lag}(n)$ by the law $u \ell = U \ell$ where $U \in U(n)$ is associated to $u \in U(n, \mathbb{C})$.

We denote by $W(n, \mathbb{C})$ the set of all symmetric unitary matrices:

$$W(n, \mathbb{C}) = \{ w \in U(n, \mathbb{C}) : w = w^T \}$$

and by $W$ the image of $w \in W(n, \mathbb{C})$ in $U(n) \subset \text{Sp}(n)$. The set of all such matrices $W$ is denoted by $W(n)$. Applying the conditions (2.15) we thus have

$$W = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \in W(n) \iff \begin{cases} A^2 + B^2 = I \\ AB = BA \\ A = A^T, B = B^T. \end{cases}$$

Observe that neither $W(n, \mathbb{C})$ nor $W(n)$ are groups: the product of two symmetric matrices is in general not symmetric.

Interestingly enough, the set $W(n, \mathbb{C})$ is closed under the operation of taking square roots:

**Lemma 57** For every $w \in W(n, \mathbb{C})$ [respectively $W \in W(n)$] there exists $u \in W(n, \mathbb{C})$ [resp. $U \in W(n)$] such that $w = u^2$ [respectively $W = U^2$].

**Proof.** Let $w = A + iB$. The condition $ww^* = I$ implies that $AB = BA$. It follows that the symmetric matrices $A$ and $B$ can be diagonalized simultaneously: there exists $R \in \text{O}(n, \mathbb{R})$ such that $G = RAR^T$ and $H = RBR^T$ are diagonal. Let $g_j$ and $h_j$ ($1 \leq j \leq n$) be the eigenvalues of $G$ and $H$, respectively. Since $A^2 + B^2 = I$ we have $g_j^2 + h_j^2 = 1$ for every $j$. Choose now real numbers $x_j$, $y_j$ such that $x_j^2 - y_j^2 = g_j$ and $2x_jy_j = h_j$ for $1 \leq j \leq n$ and let $X$ and $Y$ be the diagonal matrices whose entries are these numbers $x_j, y_j$. Then $(X + iY)^2 = G + iH$, and $u = R^T (X + iY) R$ is such that $u^2 = w$. ■

We are now going prove that $\text{Lag}(n)$ can be identified with $W(n, \mathbb{C})$ (and hence with $W(n)$). Let us begin with a preparatory remark:

**Remark 58** Suppose that $u \ell_P = \ell_P$. Writing $u = A + iB$ this implies $B = 0$, and hence $u \in \text{O}(n, \mathbb{R})$. This is immediately seen by noting that
the condition $u\ell_P = \ell_P$ can be written in matrix form as: for every $p$ there exists $p'$ such that

$$\begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} 0 \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ p' \end{bmatrix}.$$ 

**Theorem 59** (i) For $\ell \in \text{Lag}(n)$ and $u \in U(n, \mathbb{C})$ such that $\ell = u\ell_P$ the product $w = uu^T$ only depends on $\ell$ and not on the choice of $u$. The correspondence $\ell \mapsto uu^T$ is thus the mapping

$$w(\cdot) : \text{Lag}(n) \longrightarrow \text{W}(n, \mathbb{C}) \; , \; w(\ell) = w = uu^T$$

for every $u \in U(n, \mathbb{C})$.

**Proof.** Proof of (i). Let us show that if two unitary matrices $u$ and $u'$ are both such that $u\ell_P = u'\ell_P$ then $uu^T = u'(u'^T)$. This will prove the first statement. The condition $u\ell_P = u'\ell_P$ is equivalent to $u^{-1}u'\ell_P = \ell_P$. In view of the preparatory remark above, this implies that we have $u^{-1}u' = h$ for some $h \in \text{O}(n, \mathbb{R})$. Writing $u' = uh$ we have

$$u'(u')^T = (uh)(uh)^T = u(hh^T)u^T = uu^T$$

since $hh^T = I$.

Proof of (ii). Let us show that the mapping $w(\cdot)$ is surjective. In view of Lemma 57 for every $w \in \text{W}(n, \mathbb{C})$ there exists $u \in \text{W}(n, \mathbb{C})$ such that $w = u^2 = uu^T$ (since $u$ is symmetric). The Lagrangian plane $\ell = u\ell_P$ is then given by $w(\ell) = w$, hence the surjectivity. To show that $w(\cdot)$ is injective it suffices to show that if $uu^T = u'u'^T$, then $u\ell_P = u'\ell_P$, or equivalently, that $(u')^{-1}u \in \text{O}(n, \mathbb{R})$. Now, the condition $uu^T = u'u'^T$ implies that $(u')^{-1}u = u'((u')^{-1}u)^T$ and hence

$$(u')^{-1}u ((u')^{-1}u)^T = (u')^{-1}u (u'^T(u'^T)^{-1})^T = I$$

that is $(u')^{-1}u \in \text{O}(n, \mathbb{R})$ as claimed. There remains to prove formula (2.17). Assume that $\ell = u'\ell_P$. Then $u\ell = uu'\ell_P$ and hence

$$w(u\ell) = (uu')(uu')^T = u(u'(u'^T))u^T = u'(u'^T)$$

as claimed. $lacksquare$

The Souriau mapping is a very useful tool when one wants to study transversality properties for Lagrangian planes. For instance

$$\ell \cap \ell' = 0 \iff \det(w(\ell) - w(\ell')) \neq 0.$$  \hfill (2.18)

This is immediately seen by noticing that the condition $\det(w(\ell) - w(\ell')) \neq 0$ is equivalent to saying that $w(\ell)(w(\ell'))^{-1}$ does not have $+1$ as an eigenvalue.
The equivalence (2.18) is in fact a particular case of the more general result. We denote by $W(\ell)$ the image of $w(\ell)$ in $U(n)$:

$$w(\ell) = X + iY \iff W(\ell) = \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} \quad (2.19)$$

**Proposition 60** For any two Lagrangian planes $\ell$ and $\ell'$ in we have

$$\text{rank}(W(\ell) - W(\ell')) = 2(n - \dim(\ell \cap \ell')). \quad (2.20)$$

**Proof.** Since $\text{Sp}(n)$ acts transitively on $\text{Lag}(n)$ it suffices to consider the case $\ell' = \ell_P$, in which case formula (2.20) reduces to

$$\text{rank}(w(\ell) - I) = 2(n - \dim(\ell \cap \ell_P)).$$

Let $w(\ell) = uu^T$ where $u = A + iB$. Then, using the relations (2.15),

$$w(\ell) - I = -2(B^T B - iA^T B) = -2B^T (B - iA)$$

hence, with notation (2.19).

$$W(\ell) - I = -2 \begin{bmatrix} B^T & 0 \\ 0 & B^T \end{bmatrix} \begin{bmatrix} B & A \\ -A & B \end{bmatrix}. $$

It follows that

$$\text{rank}(W(\ell) - I) = 2 \text{rank } B,$$

which is equivalent to (2.20). □

The Souriau mapping $w(\cdot)$ can also be expressed in terms of projection operators on Lagrangian planes. Let $\ell$ be a Lagrangian plane and denote by $P_{\ell}$ the orthogonal projection in $\mathbb{R}_{2n}^2$ on $\ell$:

$$P_{\ell}^2 = P_{\ell}, \; \text{Ker}(P_{\ell}) = J\ell, \; (P_{\ell})^T = P_{\ell}.$$

We have:

**Proposition 61** The image $W(\ell)$ of $w(\ell)$ in $U(n)$ is given by

$$W(\ell) = (I - 2P_{\ell})C \quad (2.21)$$

where $P_{\ell}$ is the orthogonal projection in $\mathbb{R}_{2n}^2$ on $\ell$ and $C = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$ is the “conjugation matrix”.

**Proof.** Since $U(n)$ acts transitively on $\text{Lag}(n)$, there exists $U \in U(n)$ such that $\ell = U\ell_P$. Writing

$$U = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$$

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it follows from the relations (2.15) that the vector \((Ax,Bx)\) is orthogonal to \(\ell\). One immediately checks that the projection operator on \(\ell\) has matrix

\[
P_\ell = \begin{bmatrix} BB^T & -AB^T \\ -BA^T & AA^T \end{bmatrix}
\]

and hence

\[
(I - 2P_\ell)C = \begin{bmatrix} AA^T - BB^T & -2AB^T \\ 2BA^T & BB^T - AA^T \end{bmatrix}
\]

that is:

\[
(I - 2P_\ell)C = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} A^T & -B^T \\ B^T & A^T \end{bmatrix} = W(\ell).
\]

**Definition and properties of the Maslov index**

We are going to use the Souriau mapping to show that \(\pi_1[\text{Lag}(n)]\) is isomorphic to the integer group \((\mathbb{Z}, +)\). This will also allow us to define the Maslov index of a loop in \(\text{Lag}(n)\).

Let us begin with a preliminary result, interesting by itself. It is a “folk theorem” that the Poincaré group \(\pi_1[U(n, \mathbb{C})]\) is isomorphic to the integer group \((\mathbb{Z}, +)\). Let us give a detailed proof of this property. This will at the same time give an explicit isomorphism we will use to define the Maslov index. We recall that the special unitary group \(\text{SU}(n, \mathbb{C})\) is connected and simply connected (see e.g. Leray [50], Ch. I, §2,3).

**Lemma 62** The mapping \(\pi_1[U(n, \mathbb{C})] \rightarrow \mathbb{Z}\) defined by

\[
\gamma \mapsto \frac{1}{2\pi i} \oint_\gamma \frac{d(\det u)}{\det u}
\]

is an isomorphism, and hence \(\pi_1[U(n, \mathbb{C})] \cong (\mathbb{Z}, +)\).

**Proof.** The kernel of the epimorphism \(u \mapsto \det u\) is \(\text{SU}(n, \mathbb{C})\) so that we have a fibration \(U(n, \mathbb{C})/\text{SU}(n, \mathbb{C}) = S^1\). The homotopy sequence of that fibration contains the following exact sequence

\[
\pi_1[\text{SU}(n, \mathbb{C})] \xrightarrow{i} \pi_1[U(n, \mathbb{C})] \xrightarrow{f} \pi_1[S^1] \xrightarrow{\pi_0[\text{SU}(n, \mathbb{C})]}
\]

where \(f\) is induced by \(U(n, \mathbb{C})/\text{SU}(n, \mathbb{C}) = S^1\). Since \(\text{SU}(n, \mathbb{C})\) is both connected and simply connected \(\pi_0[\text{SU}(n, \mathbb{C})]\) and \(\pi_1[\text{SU}(n, \mathbb{C})]\) are trivial, and the sequence above reduces to

\[
0 \rightarrow \pi_1[U(n, \mathbb{C})] \xrightarrow{f} \pi_1[S^1] \rightarrow 0
\]

hence \(f\) is an isomorphism. The result now follows from the fact that the mapping \(\pi_1[S^1] \rightarrow \mathbb{Z}\) defined by

\[
\alpha \mapsto \frac{1}{2\pi i} \int_\alpha \frac{dz}{z}
\]

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is an isomorphism $\pi_1[S^1] \cong \mathbb{Z}$.

Next result is important since it shows, among other things, that the fundamental group of the Lagrangian Grassmannian $\text{Lag}(n)$ is isomorphic to the integer group $(\mathbb{Z}, +)$. That isomorphism $\text{Lag}(n) \cong (\mathbb{Z}, +)$ is, by definition, the Maslov index:

**Theorem 63** (i) The mapping

$$\pi_1[W(n, \mathbb{C})] \ni \gamma_W \mapsto \frac{1}{2\pi i} \int_{\gamma_W} \frac{d(\det w)}{\det w} \in \mathbb{Z} \quad (2.23)$$

is an isomorphism $\pi_1[W(n, \mathbb{C})] \cong (\mathbb{Z}, +)$.

(ii) The composition of this isomorphism with the isomorphism $\pi_1[\text{Lag}(n)] \cong \pi_1[W(n, \mathbb{C})]$ induced by the Souriau mapping is an isomorphism

$$m_{\text{Lag}} : \pi_1[\text{Lag}(n)] \cong (\mathbb{Z}, +). \quad (2.24)$$

(iii) In fact $\pi_1[\text{Lag}(n)]$ has a generator $\beta$ such that $m_{\text{Lag}}(\beta^r) = r$ for every $r \in \mathbb{Z}$.

**Proof.** The statement (ii) is an obvious consequence of the statement (i). Let us prove (i). Since $W(n, \mathbb{C}) \subset U(n, \mathbb{C})$ it follows from Lemma 62 that

$$\frac{1}{2\pi i} \int_{\gamma_W} \frac{d(\det w)}{\det w} \in \mathbb{Z}$$

for every $\gamma_W \in \pi_1[W(n, \mathbb{C})]$ and that the homomorphism (2.23) is injective. Let us show that this homomorphism is also surjective. It suffices for that to exhibit the generator $\beta$ in (iii). Writing $(x, p) = (x_1, p_1; \ldots; x_n, p_n)$ the direct sum $\text{Lag}(1) \oplus \cdots \oplus \text{Lag}(1)$ ($n$ terms) is a subset of $\text{Lag}(n)$. Consider the loop $\beta_{(1)} : t \mapsto e^{2\pi i t}, 0 \leq t \leq 1$, in $W(1, \mathbb{C}) \equiv \text{Lag}(1)$. Set now $\beta = \beta_{(1)} \oplus I_{2n-2}$ where $I_{2n-2}$ is the identity in $W(n-1, \mathbb{C})$. We have $\beta^r = \beta_{(1)} \oplus I_{2n-2}$ and

$$m_{\text{Lag}}(\beta^r) = \frac{1}{2\pi i} \int_0^1 \frac{d(e^{2\pi i rt})}{e^{2\pi i rt}} = r$$

which was to be proven.

The isomorphism $\pi_1[\text{Lag}(n)] \cong (\mathbb{Z}, +)$ constructed in Theorem 63 is precisely the Maslov index of the title of this section:

**Definition 64** (i) The mapping $m_{\text{Lag}}^{(n)}$ which to every loop $\gamma$ in $\text{Lag}(n)$ associates the integer

$$m_{\text{Lag}}^{(n)}(\gamma) = \frac{1}{2\pi i} \oint_{\gamma} \frac{d(\det w)}{\det w} \quad (2.25)$$
is called “Maslov index” on $\text{Lag}(n)$. When the dimension $n$ is understood we denote it by $m_{\text{Lag}}^{(n)}$ as in Theorem 63.

(ii) The loop $\beta = \beta(1) \oplus I_{2n-2}$ in $\text{Lag}(n)$ is called the generator of $\pi_1[\text{Lag}(n)]$ whose natural image in $\mathbb{Z}$ is $+1$.

We will construct an extension of $m_{\text{Lag}}^{(n)}$ under the name “Leray index”. It will lead us to the definition of quite general Lagrangian intersection indices.

If $\gamma$ and $\gamma'$ are loops with same origin we denote by $\gamma \ast \gamma'$ their concatenation, that is the loop $\gamma$ followed by the loop $\gamma'$:

$$\gamma \ast \gamma'(t) = \begin{cases} 
\gamma(2t) & \text{for } 0 \leq t \leq 1/2 \\
\gamma(2t-1) & \text{for } 1/2 \leq t \leq 1.
\end{cases}$$

The following result characterizes the Maslov index:

**Proposition 65** The family $(m_{\text{Lag}}^{(n)})_{n \in \mathbb{N}}$ is the only family of mappings $m_{\text{Lag}}^{(n)} : \text{Lag}(n) \longrightarrow \mathbb{Z}$ having the following properties:

(i) Homotopy: two loops $\gamma$ and $\gamma'$ in $\text{Lag}(n)$ are homotopic if and only if $m_{\text{Lag}}^{(n)}(\gamma) = m_{\text{Lag}}^{(n)}(\gamma')$;

(ii) Additivity under concatenation: for all loops $\gamma$ and $\gamma'$ in $\text{Lag}(n)$ with same origin

$$m_{\text{Lag}}^{(n)}(\gamma \ast \gamma') = m_{\text{Lag}}^{(n)}(\gamma) + m_{\text{Lag}}^{(n)}(\gamma');$$

(iii) Normalization: the generator $\beta$ of $\pi_1[\text{Lag}(n)]$ has Maslov index $m_{\text{Lag}}^{(n)}(\beta) = +1$.

(iv) Dimensional additivity: identifying $\text{Lag}(n_1) \oplus \text{Lag}(n_2)$ with a subset of $\text{Lag}(n)$, $n = n_1 + n_2$, we have

$$m_{\text{Lag}}^{(n)}(\gamma_1 \oplus \gamma_2) = m_{\text{Lag}}^{(n_1)}(\gamma_1) + m_{\text{Lag}}^{(n_2)}(\gamma_2)$$

if $\gamma_j$ is a loop in $\text{Lag}(n_j)$, $j = 1, 2$.

**Proof.** The additivity properties (ii) and (iv) are obvious and so is the normalization property (iii) using formula (2.25). That $m_{\text{Lag}}^{(n)}(\gamma)$ only depends on the homotopy class of the loop $\gamma$ is clear from the definition of the Maslov index as being a mapping $\pi_1[\text{Lag}(n)] \longrightarrow (\mathbb{Z}, +)$ and that $m_{\text{Lag}}^{(n)}(\gamma) = m_{\text{Lag}}^{(n)}(\gamma')$ implies that $\gamma$ and $\gamma'$ are homotopic follows from the injectivity of that mapping. Let us finally prove the uniqueness of $(m_{\text{Lag}}^{(n)})_{n \in \mathbb{N}}$. Suppose there is another family of mappings $m_{\text{Lag}}^{(n)} : \text{Lag}(n) \longrightarrow \mathbb{Z}$ having the same property. Then the difference $(\delta_{\text{Lag}}^{(n)})_{n \in \mathbb{N}}$ has the properties (i), (ii), (iv) and (iii) is replaced by $\delta_{\text{Lag}}^{(1)}(\beta(1)) = 0$. Every loop $\gamma$ in $\text{Lag}(n)$ being homotopic to $\beta^r$ for some $r \in \mathbb{Z}$ it follows from the concatenation property (ii) that $\delta_{\text{Lag}}^{(n)}(\gamma) = \delta_{\text{Lag}}^{(n)}(\beta^r) = 0$. □
Remark 66 Notice that we did not use in the proof of uniqueness in Proposition 65 the dimensional additivity property: properties (i), (ii), and (iii) thus characterize the Maslov index.

The Maslov index on \( \text{Sp}(n) \)

Let \( \gamma : [0, 1] \rightarrow \text{Sp}(n) \) be a loop of symplectic matrices: \( \gamma(t) \in \text{Sp}(n) \) and \( \gamma(0) = \gamma(1) \). The orthogonal part of the polar decomposition \( \gamma(t) = U(t) e^{X(t)} \) is given by the formula

\[
U(t) = \gamma(t)(\gamma(t)^T \gamma(t))^{-1/2}. \tag{2.26}
\]

Definition 67 The Maslov index of the symplectic loop \( \gamma \) is the integer

\[
m_{\text{Sp}}(\gamma) = \frac{1}{2\pi}(\theta(1) - \theta(0))
\]

where \( \theta \) is the continuous function \( [0, 1] \rightarrow \mathbb{R} \) defined by \( \det u(t) = e^{i\theta(t)} \) where \( u(t) \) is the image in \( \text{U}(n, \mathbb{C}) \) of the matrix \( U(t) \in \text{U}(n) \) defined by (2.26).

Let us exhibit a particular generator of \( \pi_1[\text{Sp}(n)] \). In addition to the fact that it allows easy calculations of the Maslov index it will be useful in the study of general symplectic intersection indices in Chapter 3.

Let us rearrange the coordinates in \( \mathbb{R}^{2n} \) and identify \((x, p)\) with the vector \((x_1, p_1, \ldots, x_n, p_n)\). Denoting by \( \text{Sp}(1) \) the symplectic group acting on pairs \((x_j, p_j)\) the direct sum

\[
\text{Sp}(1) \oplus \text{Sp}(1) \oplus \cdots \oplus \text{Sp}(1) \quad (n \text{ terms})
\]

is identified with a subgroup of \( \text{Sp}(n) \) in the obvious way. We will denote by \( J_1 \) the standard \( 2 \times 2 \) symplectic matrix:

\[
J_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\]

With these notations we have (cf. the proof of Theorem 63):

Proposition 68 (i) The fundamental group \( \pi_1[\text{Sp}(n)] \) is generated by the loop

\[
\alpha : t \mapsto e^{2\pi t J_1} \oplus I_{n-2}, \quad 0 \leq t \leq 1 \tag{2.27}
\]

where \( I_{2n-2} \) is the identity on \( \mathbb{R}^{2n-2} \).

(ii) The Maslov index of any symplectic loop \( \gamma \) is \( m_{\text{Sp}}(\gamma) = r \) where the integer \( r \) is defined by the condition: \( \gamma \) is homotopic to \( \alpha^r \).

Proof. Proof of (i). Clearly \( J_1 \in \text{sp}(1) \) hence \( \alpha(t) \in \text{Sp}(n) \). Since

\[
e^{2\pi t J_1} = \begin{bmatrix} \cos 2\pi t & -\sin 2\pi t \\ \sin 2\pi t & \cos 2\pi t \end{bmatrix}
\]
we have $\alpha(0) = \alpha(1)$. Now $\alpha(t)^T \alpha(t)$ is the identity hence
\[
\alpha(t)(\alpha(t)^T \alpha(t))^{-1/2} = \alpha(t) = \begin{bmatrix} A(t) & -B(t) \\ B(t) & A(t) \end{bmatrix}
\]
where $A(t)$ and $B(t)$ are the diagonal matrices
\[
A(t) = \text{diag} [\cos(2\pi t), 1, ..., 1] \\
B(t) = \text{diag} [\sin(2\pi t), 0, ..., 0].
\]
Let $\Delta$ be the mapping $\text{Sp}(n) \to S^1$ defined in Corollary 34 above. We have
\[
\Delta(\alpha(t)) = \det(A(t) + iB(t)) = e^{2\pi it} \tag{2.28}
\]
hence $t \mapsto \Delta(\alpha(t))$ is the generator of $\pi_1[S^1]$. The result follows.

**Proof of (ii).** It suffices to show that $m_{\text{Sp}}(\alpha) = 1$. But this immediately follows from formula (2.28). 

**Definition 69**

(i) The loop $\alpha$ defined by (2.27) will be called the “generator of $\pi_1[\text{Sp}(n)]$ whose image in $\mathbb{Z}$ is $+1$”.

(ii) Let $\gamma$ be an arbitrary loop in $\text{Sp}(n)$. The integer $r$ such that $\gamma$ is homotopic to $\alpha^r$ is called “Maslov index of $\gamma$”.

The following result is the analogue of Proposition 65. Its proof being quite similar we do not give it:

**Proposition 70** The family $(m_{\text{Sp}}^{(n)})_{n \in \mathbb{N}}$ Maslov index is the only family of mappings $m_{\text{Sp}}^{(n)} : \text{Sp}(n) \to \mathbb{Z}$ having the following properties:

(i) Homotopy: two loops $\gamma$ and $\gamma'$ in $\text{Sp}(n)$ are homotopic if and only if $m_{\text{Sp}}^{(n)}(\gamma) = m_{\text{Sp}}^{(n)}(\gamma')$;

(ii) Additivity under concatenation: for all loops $\gamma$ and $\gamma'$ in $\text{Sp}(n)$ with same origin
\[
m_{\text{Sp}}^{(n)}(\gamma \ast \gamma') = m_{\text{Sp}}^{(n)}(\gamma) + m_{\text{Sp}}^{(n)}(\gamma');
\]

(iii) Normalization: the generator $\alpha$ of $\pi_1[\text{Sp}(n)]$ has Maslov index $m_{\text{Sp}}^{(n)}(\beta) = +1$.

(iv) Dimensional additivity: identifying $\text{Sp}(n_1) \oplus \text{Sp}(n_2)$ with a subset of $\text{Lag}(n)$, $n = n_1 + n_2$ we have
\[
m_{\text{Sp}}^{(n)}(\gamma_1 \oplus \gamma_2) = m_{\text{Sp}}^{(n_1)}(\gamma_1) + m_{\text{Sp}}^{(n_2)}(\gamma_2)
\]
if $\gamma_j$ is a loop in $\text{Sp}(n_j)$, $j = 1, 2$. 

53
2.3 The Leray Index

Following the ideas of Maslov [56] and Arnol’d [4] Leray constructed in [50, 51, 52] an index $m$ having the property

$$m(\ell_\infty, \ell'_\infty) - m(\ell_\infty, \ell''_\infty) + m(\ell'_\infty, \ell''_\infty) = \text{Inert}(\ell, \ell', \ell'')$$

for all triples $(\ell_\infty, \ell'_\infty, \ell''_\infty)$ with pairwise transversal projections:

$$\ell \cap \ell' = \ell' \cap \ell'' = \ell'' \cap \ell = 0.$$

(2.29)

The integer $\text{Inert}(\ell, \ell', \ell'')$ is the index of inertia of the triple $(\ell, \ell', \ell'')$ (Leray [50]). It is defined as follows: the conditions

$$(z, z', z'') \in \ell \times \ell' \times \ell'' \ , \ z + z' + z'' = 0$$

define three isomorphisms $z \mapsto -\rightarrow z', z' \mapsto -\rightarrow z'', z'' \mapsto -\rightarrow z$ whose product is the identity. It follows that

$$\sigma(z, z') = \sigma(z', z'') = \sigma(z'', z)$$

is the value of a quadratic form at $z \in \ell$ (or $z' \in \ell'$, or $z'' \in \ell''$). These quadratic forms have the same index of inertia, denoted by $\text{Inert}(\ell, \ell', \ell'')$.

Since Leray’s index of inertia $\text{Inert}(\ell, \ell', \ell'')$ is defined in terms of quadratic forms which only exist when the transversality conditions (2.29) are satisfied, it is not immediately obvious how to extend $m(\ell_\infty, \ell'_\infty)$ to arbitrary pairs $(\ell_\infty, \ell'_\infty)$. The extension we will present first appeared in [23] and was then detailed in [25]. (Dazord has constructed in [17] a similar index using methods from algebraic topology). The main idea is to use the signature $\tau(\ell, \ell', \ell'')$ instead of $\text{Inert}(\ell, \ell', \ell'')$. This idea probably goes back to Lion and Vergne [54], albeit in a somewhat incomplete form: see the discussion in [23, 24]. For a very detailed study of related indices see the deep paper [14] by Cappell et al.

The theory of the Leray index is a beautiful generalization of the theory of the Maslov index of Lagrangian loops. It is a very useful mathematical object, which can be used to express various other indices: Lagrangian and symplectic path intersection indices, and, as we will see, the Conley–Zehnder index.

The problem

Recall that the Wall–Kashiwara signature associates to every triple $(\ell, \ell', \ell'')$ of Lagrangian planes in $(\mathbb{R}^{2n}, \sigma)$ an integer $\tau(\ell, \ell', \ell'')$ which is the signature of the quadratic form

$$(z, z', z'') \mapsto \sigma(z, z') + \sigma(z', z'') + \sigma(z'', z)$$

on $\ell \oplus \ell' \oplus \ell''$. Besides being antisymmetric and $\text{Sp}(n)$-invariant, $\tau$ is a cocycle, that is:
\[
\tau(\ell_1, \ell_2, \ell_3) - \tau(\ell_2, \ell_3, \ell_4) + \tau(\ell_1, \ell_3, \ell_4) - \tau(\ell_1, \ell_2, \ell_4) = 0.
\] (2.30)

As shortly mentioned in the statement of Theorem 53 this property can be expressed in terms of the coboundary operator \(\partial\) in the abbreviated form

\[
\partial\tau(\ell_1, \ell_2, \ell_3, \ell_4) = 0.
\]

Let us look for primitives of the cocycle \(\tau\). By primitive we mean a “1-cochain”

\[
\mu : \text{Lag}(n) \times \text{Lag}(n) \longrightarrow \mathbb{Z}
\]
such that

\[
\partial\mu(\ell_1, \ell_2, \ell_3) = \tau(\ell_1, \ell_2, \ell_3)
\]

where \(\partial\) is the usual “coboundary operator”. That primitives exist is easy to see: for instance, for every fixed Lagrangian plane \(\ell\) the cochain \(\mu_\ell\) defined by

\[
\mu_\ell(\ell_1, \ell_2) = \tau(\ell, \ell_1, \ell_2)
\] (2.31)
satisfies, in view of (2.30),

\[
\mu_\ell(\ell_1, \ell_2) - \mu_\ell(\ell_1, \ell_3) + \mu_\ell(\ell_2, \ell_3) = \tau(\ell, \ell_1, \ell_2) - \tau(\ell, \ell_1, \ell_3) + \tau(\ell, \ell_2, \ell_3)
\]

and hence \(\partial\mu_\ell = \tau\). We however want the primitive we are looking for to satisfy, in addition, topological properties consistent with those of the signature \(\tau\). Recall that we showed that \(\tau(\ell_1, \ell_2, \ell_3)\) remains constant when the triple \((\ell_1, \ell_2, \ell_3)\) moves continuously in such a way that the dimensions of the intersections \(\dim(\ell_1, \ell_2), \dim(\ell_1, \ell_3), \dim(\ell_2, \ell_3)\) do not change. It is therefore reasonable to demand that the primitive \(\mu\) also is locally constant on all pairs \((\ell_1, \ell_2)\) such that \(\dim(\ell_1, \ell_2)\) is fixed. It is easy to see why the cochain (2.31) does not satisfy this property: assume, for instance, that the pair \((\ell_1, \ell_2)\) moves continuously while remaining transversal: \(\ell_1 \cap \ell_2 = 0\). Then, \(\tau(\ell, \ell_1, \ell_2)\) would –if the desired condition is satisfied– remain constant. This is however not the case, since the signature of a triple of Lagrangian planes changes when we change the relative positions of the involved planes (see Subsection 2.1). It turns out that we will actually never be able to find a cochain \(\mu\) on \(\text{Lag}(n)\) which is both a primitive of \(\tau\) and satisfies the topological condition above: to construct such an object we have to pass to the universal covering \(\text{Lag}_\infty(n)\) (“Maslov bundle”) of \(\text{Lag}(n)\).

Let \(\pi : \text{Lag}_\infty(n) \longrightarrow \text{Lag}(n)\) be the universal covering of the Lagrangian Grassmannian \(\text{Lag}(n)\). We will write \(\ell = \pi(\ell_\infty)\).

**Definition 71** The Leray index is the unique mapping

\[
\mu : (\text{Lag}_\infty(n))^2 \longrightarrow \mathbb{Z}
\]
having the two following properties:

(i) $\mu$ is locally constant on the set $\{(\ell_\infty, \ell'_\infty) : \ell \cap \ell' = 0\}$;
(ii) $\partial \mu : (\text{Lag}_\infty(n))^3 \to \mathbb{Z}$ descends to $\text{(Lag}(n))^3$ and is equal to $\tau$:

$$
\mu(\ell_\infty, \ell'_\infty) - \mu(\ell_\infty, \ell''_\infty) + \mu(\ell'_\infty, \ell''_\infty) = \tau(\ell, \ell', \ell'').
$$

(2.32)

Notice that property (2.32) implies, together with the antisymmetry of the signature, that if the Leray index exists then it must satisfy

$$
\mu(\ell_\infty, \ell'_\infty) = -\mu(\ell'_\infty, \ell_\infty)
$$

(2.33)

for all pairs $(\ell_\infty, \ell'_\infty)$.

Admittedly, the definition above is not very constructive. And, by the way, why is $\mu$ (provided that it exists) unique? This question is at least easily answered. Suppose there are two mappings $\mu$ and $\mu'$ satisfying the same conditions as above: for all triples $(\ell_\infty, \ell'_\infty, \ell''_\infty)$

$$
\mu(\ell_\infty, \ell'_\infty) - \mu(\ell_\infty, \ell''_\infty) + \mu(\ell'_\infty, \ell''_\infty) = \tau(\ell, \ell', \ell'')
$$

$$
\mu'(\ell_\infty, \ell'_\infty) - \mu'(\ell_\infty, \ell''_\infty) + \mu'(\ell'_\infty, \ell''_\infty) = \tau(\ell, \ell', \ell'').
$$

It follows that $\delta = \mu - \mu'$ is such that

$$
\delta(\ell_\infty, \ell'_\infty) = \delta(\ell_\infty, \ell''_\infty) - \delta(\ell'_\infty, \ell''_\infty).
$$

(2.34)

Since $\mu$ and $\mu'$ are locally constant on $\{(\ell_\infty, \ell'_\infty) : \ell \cap \ell' = 0\}$ the same is true of $\delta$. Choosing $\ell''$ such that $\ell'' \cap \ell = \ell'' \cap \ell' = 0$ we see that in fact $\delta$ is locally constant on all of $(\text{Lag}_\infty(n))^2$. Now $\text{Lag}_\infty(n)$, and hence $(\text{Lag}_\infty(n))^2$, is connected so that $\delta$ is actually constant. Its constant value is

$$
\delta(\ell_\infty, \ell_\infty) = \delta(\ell_\infty, \ell''_\infty) - \delta(\ell'_\infty, \ell''_\infty) = 0
$$

hence $\mu = \mu'$ and the Leray index is thus uniquely characterized by properties (i) and (ii) in the definition above.

We will see that the action of fundamental group of $\text{Lag}(n)$ on $\text{Lag}_\infty(n)$ is reflected on the Leray index by the formula

$$
\mu(\beta^r \ell_\infty, \beta^r \ell'_\infty) = \mu(\ell_\infty, \ell'_\infty) + 2(r - r')
$$

(2.35)

where $\beta$ is the generator of $\pi_1[\text{Lag}(n)] \cong (\mathbb{Z}, +)$ whose image in $\mathbb{Z}$ is $+1$.

This formula shows that the Leray index is effectively defined on $(\text{Lag}_\infty(n))^2$ (i.e., it is “multi-valued” on $(\text{Lag}(n))^2$). It also shows why we could not expect to find a function having similar properties on $\text{Lag}(n)$ itself: if such a function $\mu'$ existed, we could “lift” it to a function on $(\text{Lag}_\infty(n))^2$ in an obvious way by the formula $\mu'(\ell_\infty, \ell'_\infty) = \mu'(\ell, \ell')$. But the uniqueness of the Leray index would then imply that $\mu' = \mu$ which is impossible since $\mu'$ cannot satisfy (2.35).

An important consequence of this uniqueness is the invariance of the Leray index is under the action of the universal covering group $\text{Sp}_\infty(n)$ of $\text{Sp}(n)$:

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Proposition 72  For all \((S_\infty, \ell_\infty, \ell'_\infty) \in \text{Sp}_\infty(n) \times (\text{Lag}_\infty(n))^2\) we have
\[
\mu(S_\infty \ell_\infty, S_\infty \ell'_\infty) = \mu(\ell_\infty, \ell'_\infty). \tag{2.36}
\]

Proof. Set, for fixed \(S_\infty \in \text{Sp}_\infty(n), \mu'(\ell_\infty, \ell'_\infty) = \mu(S_\infty \ell_\infty, S_\infty \ell'_\infty).\) We have
\[
\mu'(\ell_\infty, \ell'_\infty) - \mu'(\ell_\infty, \ell''_\infty) + \mu'(\ell'_\infty, \ell''_\infty) = \tau(S\ell, S\ell', S\ell'')
\]
where \(S \in \text{Sp}(n)\) is the projection of \(S_\infty.\) In view of the symplectic invariance of the Wall–Kashiwara signature we have \(\tau(S\ell, S\ell', S\ell'')\) and hence
\[
\mu'(\ell_\infty, \ell'_\infty) - \mu'(\ell_\infty, \ell''_\infty) + \mu'(\ell'_\infty, \ell''_\infty) = \tau(\ell, \ell', \ell'').
\]
Since on the other hand \(S\ell \cap S\ell' = 0\) if and only if \(\ell \cap \ell' = 0,\) the index \(\mu'\) is locally constant on \(\{(\ell_\infty, \ell'_\infty) : \ell \cap \ell' = 0\}\) and must thus be equal to \(\mu,\) that is (2.36).

There now remains the hard part of the work, namely the explicit construction of the Leray index. Let us show how this can be done in the case \(n = 1.\) The general case will definitely require more work. The Lagrangian Grassmannian \(\text{Lag}(1)\) consists of all straight lines through the origin in the symplectic plane \((\mathbb{R}^2, -\det).\) Let \(\ell\) and \(\ell'\) be the lines with equations
\[
x \cos \alpha + p \sin \alpha = 0, \quad x \cos \alpha' + p \sin \alpha' = 0
\]
and identify \(\ell_\infty\) and \(\ell'_\infty\) with \(\theta = 2\alpha\) and \(\theta' = 2\alpha'.\) Denoting by \([r]\) the integer part of \(r \in \mathbb{R}\) we then have
\[
\mu(\theta, \theta') = \begin{cases} 
2\left[\frac{\theta - \theta'}{2\pi}\right] + 1 & \text{if } \theta - \theta' \notin \pi \mathbb{Z} \\
2k & \text{if } \theta - \theta' = k\pi
\end{cases} \tag{2.37}
\]
Introducing the antisymmetric integer part function
\[
[r]_{\text{anti}} = \frac{1}{2}([r] - [-r]) = \begin{cases} 
[r] + \frac{1}{2} & \text{if } r \notin \mathbb{Z} \\
r & \text{if } r \in \mathbb{Z}
\end{cases}
\]
definition (2.37) can be rewritten in compact form as
\[
\mu(\theta, \theta') = 2\left[\frac{\theta - \theta'}{2\pi}\right]_{\text{anti}}. \tag{2.38}
\]
The coboundary \(\partial \mu\) is the function
\[
\partial \mu(\theta, \theta', \theta'') = 2\left[\frac{\theta - \theta'}{2\pi}\right]_{\text{anti}} - 2\left[\frac{\theta - \theta''}{2\pi}\right]_{\text{anti}} + 2\left[\frac{\theta' - \theta''}{2\pi}\right]_{\text{anti}}
\]
and this is precisely the signature \(\tau(\ell, \ell', \ell'')\) in view of formula (2.2) in Section 2.1.
To generalize this construction to arbitrary \(n\) we need a precise “numerical” description of the universal covering of \(\text{Lag}(n).\)
The Maslov bundle

The Maslov bundle is, by definition, the universal covering $\text{Lag}_\infty(n)$ of the Lagrangian Grassmannian $\text{Lag}(n)$.

The homomorphism

$$\pi_1[U(n, \mathbb{C})] \ni \gamma \mapsto k_\gamma \in \mathbb{Z}$$

defined by

$$k_\gamma = \frac{1}{2\pi i} \int_{\gamma} \frac{d(\det u)}{\det u}$$

is an isomorphism $\pi_1[U(n, \mathbb{C})] \cong (\mathbb{Z}, +)$. Set now

$$U_\infty(n, \mathbb{C}) = \{(u, \theta) : u \in U(n, \mathbb{C}), \det u = e^{i\theta}\}$$

and equip this set with the topology induced by the product $U(n, \mathbb{C}) \times \mathbb{R}$. Define a projection $\pi_\infty : U_\infty(n, \mathbb{C}) \to U(n, \mathbb{C})$ by $\pi_\infty(U, \theta) = W$, and let the group $\pi_1[U(n, \mathbb{C})]$ act on $U_\infty(n, \mathbb{C})$ by the law

$$\gamma(u, \theta) = (u, \theta + 2k_\gamma \pi).$$

That action is clearly transitive, hence $U_\infty(n, \mathbb{C})$ is the universal covering group of $U(n, \mathbb{C})$, the group structure being given by

$$(U, \theta)(U', \theta') = (UU', \theta + \theta').$$

Let us now identify the Maslov bundle with a subset of $U_\infty(n, \mathbb{C})$:

**Proposition 73** The universal covering of $W(n, \mathbb{C})$ is the set

$$W_\infty(n, \mathbb{C}) = \{(w, \theta) : w \in W(n, \mathbb{C}), \det w = e^{i\theta}\}$$

equipped with the topology induced by $U_\infty(n)$, together with the projection $\pi_\infty : W_\infty(n, \mathbb{C}) \to W(n, \mathbb{C})$ defined by $\pi_\infty(w, \theta) = w$.

**Proof.** It is sufficient to check that $W_\infty(n, \mathbb{C})$ is connected: $W_\infty(n)$ will then cover all coverings of $W(n, \mathbb{C})$ and will thus be the universal covering. Let $U_\infty(n, \mathbb{C})$ act on $W_\infty(n, \mathbb{C})$ via the law

$$(u, \varphi)(w, \theta) = (uwu^T, \theta + 2\varphi).$$

(2.39)

The stabilizer of $(I, 0)$ in $U_\infty(n)$ under this action is the subgroup of $U_\infty(n)$ consisting of all pairs $(U, \varphi)$ such that $UU^T = I$ and $\varphi = 0$ (and hence $\det U = 1$). It can thus be identified with the rotation group $\text{SO}(n)$ and hence

$$W_\infty(n, \mathbb{C}) = U_\infty(n, \mathbb{C})/\text{SO}(n, \mathbb{R}).$$

Since $U_\infty(n, \mathbb{C})$ is connected, so is $W_\infty(n, \mathbb{C})$. ■

The Maslov bundle $\text{Lag}_\infty(n)$ is the universal covering of the Lagrangian Grassmannian. Quite abstractly, it is constructed as follows: choose a “base
point” $\ell_0$ in $\text{Lag}(n)$: it is any fixed Lagrangian plane. Let $\ell$ be an arbitrary element of $\text{Lag}(n)$. Since $\text{Lag}(n)$ is path-connected, there exists at least one continuous path $\lambda : [0,1] \rightarrow \text{Lag}(n)$ going from $\ell_0$ to $\ell$: $\lambda(0) = \ell_0$ and $\lambda(1) = \ell$. We say that two such paths $\lambda$ and $\lambda'$ are ‘homotopic with fixed endpoints’ if one of them can be continuously deformed into the other while keeping its origin $\ell_0$ and its endpoint $\ell$ fixed. Homotopy with fixed endpoints is an equivalence relation. Denote the equivalence class of the path $\lambda$ by $\ell_\infty$.

The universal covering of $\text{Lag}_\infty(n)$ is the set of all the equivalence classes $\ell_\infty$ as $\ell$ ranges over $\text{Lag}(n)$. The mapping $\pi_\infty : \text{Lag}_\infty(n) \rightarrow \text{Lag}(n)$ which to $\ell_\infty$ associates the endpoint $\ell$ of a path $\lambda$ in $\ell_\infty$ is called the ‘covering projection’. One shows that it is possible to endow the set $\text{Lag}_\infty(n)$ with a topology for which it is both connected and simply connected, and such that every $\ell \in \text{Lag}(n)$ has an open neighborhood $\mathcal{U}_\ell$ such that $\pi_\infty^{-1}(\mathcal{U}_\ell)$ is the disjoint union of open neighborhoods $\mathcal{U}_\ell^{(1)}, ..., \mathcal{U}_\ell^{(k)}$ of the points of $\pi_\infty^{-1}(\ell)$, and the restriction of $\pi_\infty$ to each $\mathcal{U}_\ell^{(k)}$ is a homeomorphism $\mathcal{U}_\ell^{(k)} \rightarrow \mathcal{U}_\ell$.

Explicit construction of the Leray index

Let us now give the promised explicit formula for the Leray index in arbitrary dimension. Assume first that $\ell \cap \ell' = 0$. Then we define an integer $\mu(\ell_\infty, \ell'_\infty)$ by the formula

$$\mu(\ell_\infty, \ell'_\infty) = \frac{1}{\pi} \left[ \theta - \theta' + i \text{Tr} \log(-w(w')^{-1}) \right]$$  \hspace{1cm} (2.40)

where we are identifying $\ell_\infty$ with $(w, \theta), w$ being the image of $\ell$ in $W(n, \mathbb{C})$ by the Souriau mapping and $\det w = e^{i\theta}$. (Notice that the transversality condition $\ell \cap \ell' = 0$ is equivalent to $-w(w')^{-1}$ having no negative eigenvalue).

If $\ell \cap \ell' \neq 0$ one chooses $\ell''$ such that $\ell \cap \ell'' = \ell' \cap \ell'' = 0$ and one then calculates $\mu(\ell_\infty, \ell''_\infty)$ using the property

$$\mu(\ell_\infty, \ell''_\infty) = -\mu(\ell'_\infty, \ell''_\infty) - \mu(\ell''_\infty, \ell_\infty) + \tau(\ell, \ell', \ell'')$$  \hspace{1cm} (2.41)

and the expressions for $\mu(\ell_\infty, \ell''_\infty)$ and $\mu(\ell'_\infty, \ell''_\infty)$ given by (2.40).

We will see that the function $\mu$ is actually the Leray index in arbitrary dimension. In particular we will always have

$$\mu(\ell_\infty, \ell'_\infty) - \mu(\ell_\infty, \ell''_\infty) + \mu(\ell'_\infty, \ell''_\infty) = \tau(\ell, \ell', \ell'').$$  \hspace{1cm} (2.42)

Let us begin by showing that $\mu(\ell_\infty, \ell'_\infty)$ always is an integer:

**Proposition 74** We have

$$\mu(\ell_\infty, \ell'_\infty) \equiv n \mod 2 \text{ when } \ell \cap \ell' = 0.$$  \hspace{1cm} (2.43)

More generally

$$\mu(\ell_\infty, \ell'_\infty) \equiv n + \dim(\ell \cap \ell') \mod 2.$$  \hspace{1cm} (2.44)
Proof. Setting \( \mu = \mu(\ell_\infty, \ell'_\infty) \) we have
\[
e^{i\pi \mu} = \exp[i(\theta - \theta')][\exp[\text{Tr} \log(-w(w')^{-1})]]^{-1}
= \exp[i(\theta - \theta')][\det(-w(w')^{-1})]^{-1}
= \exp[i(\theta - \theta')][(-1)^n \exp[-i(\theta - \theta')]
= (-1)^n
\]

hence (2.43). Formula (2.44) follows, using (2.42) together with the value modulo 2 of the index \( \tau \) given by formula (2.13) in Section 2.1).

Let us now prove the main result of this subsection:

**Theorem 75** The Leray index \( \mu \) exists and is calculates as follows:

(i) If \( \ell \cap \ell' = 0 \) then
\[
\mu(\ell_\infty, \ell'_\infty) = \frac{1}{\pi} [\theta - \theta' + i \text{Tr} \log(-w(w')^{-1})] ;
\]

(ii) In the general case choose \( \ell'' \) such that \( \ell \cap \ell'' = \ell' \cap \ell'' = 0 \) and calculate \( \mu(\ell_\infty, \ell'_\infty) \) using the formula:
\[
\mu(\ell_\infty, \ell'_\infty) = \mu(\ell_\infty, \ell''_\infty) - \mu(\ell'_\infty, \ell''_\infty) + \tau(\ell, \ell', \ell'')
\]
[the right-hand side is independent of the choice of \( \ell'' \)].

**Proof.** It is clear that \( \mu \) defined by (2.40) is locally constant on the set of all \( (\ell, \ell') \) such that \( \ell \cap \ell' = 0 \). Let us prove that
\[
\mu(\ell_\infty, \ell'_\infty) - \mu(\ell_\infty, \ell''_\infty) + \mu(\ell'_\infty, \ell''_\infty) = \tau(\ell, \ell', \ell'')
\]
when
\[
\ell \cap \ell' = \ell \cap \ell'' = \ell' \cap \ell'' = 0;
\]
the formula will then hold in the general case as well in view of (2.41). We begin by noting that the function \( \mu \) is locally constant on its domain. It follows that the composed mapping \( S_\infty \mapsto \mu(S_\infty \ell_\infty, S_\infty \ell'_\infty) \) is (for fixed \( (\ell_\infty, \ell'_\infty) \) such that \( \ell \cap \ell' = 0 \)) is locally constant on \( \text{Sp}_\infty(n) \). Since \( \text{Sp}(n) \) is connected this mapping is in fact constant so we have \( \mu(S_\infty \ell_\infty, S_\infty \ell'_\infty) = \mu(\ell_\infty, \ell'_\infty) \) and it is thus sufficient to show that
\[
\mu(S_\infty \ell_\infty, S_\infty \ell'_\infty) - \mu(S_\infty \ell_\infty, S_\infty \ell''_\infty) + \mu(S_\infty \ell'_\infty, S_\infty \ell''_\infty) = \tau(\ell, \ell', \ell'')
\]
for some convenient \( S_\infty \in \text{Sp}(n) \). Since \( \text{Sp}(n) \) acts transitively on pairs of Lagrangian planes then exists \( S \in \text{Sp}(n) \) such that \( S(\ell, \ell'') = (\ell_P, \ell_X) \) where \( \ell_X = \mathbb{R}^n \times 0 \) and \( \ell_P = 0 \times \mathbb{R}^n \). The transversality condition \( \ell' \cap \ell = \ell' \cap \ell'' = 0 \) then implies that
\[
\ell' = \{ (x, p) : p = Ax \} = \ell_A
\]
for some symmetric matrix $A$ with $\det A \neq 0$. We have thus reduced the proof to the case $(\ell, \ell', \ell'') = (\ell_P, \ell_A, \ell_X)$ and we have to show that

$$\mu(\ell_P, \ell_A, \ell_X) - \mu(\ell_P, \ell_X, \ell_A) + \mu(\ell_A, \ell_X, \ell_A) = \tau(\ell_P, \ell_A, \ell_X).$$

(2.45)

Now, in view of formula (2.5) (Corollary 52, Section 2.1) we have

$$\tau(\ell_P, \ell_A, \ell_X) = \text{sign}(A) = p - q$$

where $p$ (resp. $q$) is the number of $> 0$ (resp. $< 0$) eigenvalues of the symmetric matrix $A$. Let us next calculate $\mu(\ell, \ell''') = \mu(\ell_P, \ell_A, \ell_X)$. Identifying $\text{Lag}_\infty(n)$ with $W_\infty(n)$ there exist integers $k$ and $k'$ such that

$$\ell_P = (I, 2k\pi)$$

and hence

$$\mu(\ell_P, \ell_A, \ell_X) = \frac{1}{\pi}(2k\pi - (2k' + n)\pi + i \text{Tr Log } I) = 2(k - k') - n.$$

Let us now calculate $\mu(\ell_P, \ell_A, \ell_X)$. Recall that $V_{-A}$ and $M_L$ ($\det L \neq 0$) denote the symplectic matrices defined by (1.48) in Subsection 1.5:

$$V_{-A} = \begin{bmatrix} I & 0 \\ A & I \end{bmatrix}, \quad M_L = \begin{bmatrix} L^{-1} & 0 \\ 0 & L^T \end{bmatrix}.$$

We begin by noting that we have $\ell_A = V_{-A} \ell_X$ hence $M_L \ell_A = V_{A'} \ell_X$, using the intertwining formula

$$M_L V_{-A} = V_{-A'} M_L, \quad A' = L^T A L.$$

We may thus assume, replacing $\ell_A$ by $M_L^{-1} \ell_A$ and $A$ by $L^T A L$ where $L$ diagonalizes $A$, that

$$A = \text{diag}[+1, ..., +1, -1, ..., -1]$$

with $p$ plus signs and $q = n - p$ minus signs. Let now $\mathcal{B} = \{e_1, ..., e_n; f_1, ..., f_n\}$ be the canonical symplectic basis of $(\mathbb{R}^{2n}, \sigma)$. The $n$ vectors

$$g_i = \frac{1}{\sqrt{2}}(e_i + f_i), \quad 1 \leq i \leq p$$

$$g_j = \frac{1}{\sqrt{2}}(e_j + f_j), \quad p + 1 \leq j \leq n$$

(with obvious conventions if $p = n$ or $q = n$) form an orthonormal basis of $\ell_A = V_{-A} \ell_X$. We thus have $\ell_A = U \ell_P$ where

$$UR = \frac{1}{\sqrt{2}} \begin{bmatrix} A & I \\ -I & A \end{bmatrix} \in U(n).$$
The identification of $U$ with $u = \frac{1}{\sqrt{2}} (A - iI)$ in $U(n, \mathbb{C})$ identifies $\ell_A$ with $uu^T = -iA$. We have $\det(-iA) = i^{q-p}$ hence

$$
\ell_{A, \infty} \equiv (iA, \frac{1}{2}(q - p)\pi + 2r\pi)
$$

for some $r \in \mathbb{Z}$. To calculate $\mu(\ell_{P, \infty}, \ell_{A, \infty})$ we need to know

$$
\text{Log}(-iA) = \text{Log}(-i \text{diag}[+1, ..., +1, -1, ..., -1])
$$

and hence, by definition (2.40) of $\mu$,

$$
\mu(\ell_{P, \infty}, \ell_{A, \infty}) = \frac{1}{\pi} \left[ 2k\pi - \left( \frac{1}{2}(q - p)\pi + 2r\pi \right) + i \text{Tr} \text{Log}(-iA) \right]
$$

Similarly

$$
\mu(\ell_{A, \infty}, \ell_{X, \infty}) = -\mu(\ell_{X, \infty}, \ell_{A, \infty})
$$

hence

$$
\mu(\ell_{P, \infty}, \ell_{A, \infty}) - \mu(\ell_{P, \infty}, \ell_{X, \infty}) + \mu(\ell_{A, \infty}, \ell_{X, \infty}) = p - q = \tau(\ell_P, \ell_A, \ell_X)
$$

which was to be proven. \(\blacksquare\)

The following consequence of the theorem above describes the action of $\pi_1[\text{Lag}(n)] \cong (\mathbb{Z}, +)$ on the Leray index, and shows why the Leray index is an extension of the usual Maslov index defined and studied in Section 2.2:

**Corollary 76** Let $\beta$ be the generator of $\pi_1[\text{Lag}(n)]$ whose natural image in $\mathbb{Z}$ is +1. We have

$$
\mu(\beta^r \ell_\infty, \beta^{r'} \ell'_\infty) = \mu(\ell_\infty, \ell'_\infty) + 2(r - r') \quad (2.46)
$$

for all $(\ell_\infty, \ell'_\infty) \in (\text{Lag}(n))^2$ and $(r, r') \in \mathbb{Z}^2$ and hence

$$
\mu(\gamma \ell_\infty, \ell'_\infty) - \mu(\ell_\infty, \ell'_\infty) = 2m(\gamma) \quad (2.47)
$$

for every loop $\gamma$ in $\text{Lag}(n)$ ($m(\gamma)$ the Maslov index of $\gamma$).
Proof. Formula (2.47) follows from formula (2.46) since every loop $\gamma$ is homotopic to $\beta^r$ for some $r \in \mathbb{Z}$. Let us first prove (2.46) when $\ell \cap \ell' = 0$. Assume that $\ell_\infty \equiv (w, \theta)$ and $\ell'_\infty \equiv (w', \theta')$ with $w, w' \in W(n, \mathbb{C})$, $\det w = e^{i\theta}$, and $\det w' = e^{i\theta'}$. Then 

$$\beta^r \ell_\infty \equiv (w, \theta + 2r\pi), \quad \beta^r' \ell'_\infty \equiv (w', \theta' + 2r'\pi)$$

and hence, by definition (2.40)

$$\mu(\beta^r \ell_\infty, \beta^r' \ell'_\infty) = \frac{1}{\pi} [\theta + 2r - \theta' - 2r' + i \text{Tr} \log(-w(w')^{-1})]$$

$$= \mu(\ell_\infty, \ell'_\infty) + 2(r - r').$$

The general case immediately follows using formula (2.41) and the fact that $\beta^r \ell_\infty$ and $\beta^r' \ell'_\infty$ have projections $\ell$ and $\ell'$ on $\text{Lag}(n)$. $
$

2.4 $q$-Symplectic Geometry

Now we can study the action of $\text{Sp}_q(n)$ on $\text{Lag}_{2q}(n)$. Due to the properties of the fundamental groups of $\text{Sp}(n)$ and $\text{Lag}(n)$ the general case will easily follow from the case $q = +\infty$. We begin by identifying $\text{Lag}_\infty(n)$ with $\text{Lag}(n) \times \mathbb{Z}$ and $\text{Sp}_\infty(n)$ with a subgroup of $\text{Sp}(n) \times \mathbb{Z}$ equipped with a particular group structure.

The identification $\text{Lag}_\infty(n) = \text{Lag}(n) \times \mathbb{Z}$

The title of this Subsection is at first sight provocative: how can we identify the Maslov bundle $\text{Lag}_\infty(n)$, which is a connected manifold, with a Cartesian product where one of the factors is a discrete space? The answer is that we will identify $\text{Lag}_\infty(n)$ and $\text{Lag}(n) \times \mathbb{Z}$ as sets, not as topological spaces, and equip $\text{Lag}(n) \times \mathbb{Z}$ with the transported topology—which is of course not the product topology.

Let us justify this in detail. We denote by $\partial \text{dim}$ the coboundary of the 1-cocycle $\text{dim}(\ell, \ell') = \text{dim} \ell \cap \ell'$ on $\text{Lag}(n)$. It is explicitly given by

$$\partial \text{dim}(\ell, \ell', \ell'') = \text{dim} \ell \cap \ell' - \text{dim} \ell \cap \ell'' + \text{dim} \ell' \cap \ell''.$$  

Definition 77 (i) The function $m : \text{Lag}_\infty(n) \to \mathbb{Z}$ defined by

$$m(\ell_\infty, \ell'_\infty) = \frac{1}{2} (\mu(\ell_\infty, \ell'_\infty) + n + \text{dim} \ell \cap \ell')$$

is called the “reduced Leray index” on $\text{Lag}_\infty(n)$.

(ii) The function $(\text{Lag}(n))^3 \to \mathbb{Z}$ defined by

$$\text{Inert}(\ell, \ell', \ell'') = \frac{1}{2} (\tau(\ell, \ell', \ell'') + n + \partial \text{dim}(\ell, \ell', \ell''))$$

where $\tau$ is the signature is called the “index of inertia” of $(\ell, \ell', \ell'')$. 

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That $m(\ell_\infty, \ell'_\infty)$ is an integer follows from the congruence (2.44) in Proposition 74 (Subsection 2.3). That $\text{Inert}(\ell, \ell', \ell'')$ also is an integer follows from the congruence (2.13) in Proposition 55 (Subsection 2.1). These congruences, together with the antisymmetry of the signature $\tau$ moreover imply that

$$m(\ell_\infty, \ell'_\infty) + m(\ell'_\infty, \ell_\infty) = n + \dim \ell \cap \ell'$$

(2.49)

for all $(\ell'_\infty, \ell_\infty) \in (\text{Lag}_\infty(n))^2$.

**Proposition 78** The reduced Leray index has the following properties:

(i) For all $(\ell_\infty, \ell'_\infty, \ell''_\infty) \in (\text{Lag}_\infty(n))^3$

$$m(\ell_\infty, \ell'_\infty) - m(\ell_\infty, \ell''_\infty) + m(\ell'_\infty, \ell''_\infty) = \text{Inert}(\ell, \ell', \ell'');$$

(2.50)

(ii) Let $\beta$ be the generator of $\pi_1[\text{Lag}(n)]$ whose natural image in $\mathbb{Z}$ is $+1$; then

$$m(\beta r \ell_\infty, \beta r' \ell'_\infty) = m(\ell_\infty, \ell'_\infty) + r - r'$$

(2.51)

for all $(r, r') \in \mathbb{Z}^2$.

**Proof.** Formula (2.50) is equivalent to the property (2.32) of the Leray index. Formula (2.51) follows from (2.46) in Corollary 76. ■

**Remark 79** Formula (2.51) shows that the range of the mapping

$$(\ell_\infty, \ell'_\infty) \mapsto m(\ell_\infty, \ell'_\infty)$$

is all of $\mathbb{Z}$.

Let us state and prove the main result of this subsection:

**Theorem 80** Let $\ell_{\alpha, \infty}$ be an arbitrary element of $\text{Lag}_\infty(n)$ and define a mapping

$$\Phi_\alpha : \text{Lag}_\infty(n) \rightarrow \text{Lag}(n) \times \mathbb{Z}$$

by the formula

$$\Phi_\alpha(\ell_\infty) = (\ell, m(\ell_\infty, \ell_{\alpha, \infty})) \quad , \quad \ell = \pi_{\text{Lag}}(\ell_\infty).$$

(i) The mapping $\Phi_\alpha$ is a bijection whose restriction to the subset $\{\ell_\infty : \ell \cap \ell_\alpha = 0\}$ of $\text{Lag}_\infty(n)$ is a homeomorphism onto $\{\ell : \ell \cap \ell_\alpha = 0\} \times \mathbb{Z}$.

(ii) The set of all bijections $(\Phi_\alpha)_{\ell_{\alpha, \infty}}$ form a system of local charts of $\text{Lag}_\infty(n)$ whose transitions $\Phi_{\alpha \beta} = \Phi_\alpha \Phi_{\beta}^{-1}$ are the functions

$$\Phi_{\alpha \beta}(\ell, \lambda) = (\ell, \lambda + \text{Inert}(\ell, \ell_\alpha, \ell_\beta) - m(\ell_{\alpha, \infty}, \ell_{\beta, \infty})).$$

(2.52)

**Proof.** Proof of (i). Assume that $\Phi_\alpha(\ell_\infty) = \Phi_\alpha(\ell'_\infty)$. Then $\ell = \ell'$ and $m(\ell_\infty, \ell_{\alpha, \infty}) = m(\ell'_\infty, \ell_{\alpha, \infty})$. Let $r \in \mathbb{Z}$ be such that $\ell''_\infty = \beta r \ell_\infty$ ($\beta$ the generator of $\pi_1[\text{Lag}(n)]$). In view of formula (2.51) we have

$$m(\ell'_\infty, \ell_{\alpha, \infty}) = m(\beta r \ell_\infty, \ell_{\alpha, \infty}) = m(\ell_\infty, \ell_{\alpha, \infty}) + r$$

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hence \( r = 0 \) and \( \ell'_\infty = \ell_\infty \), so that \( \Phi_\alpha \) is injective. Let us show it is surjective. For \((\ell, k) \in \text{Lag}(n) \times \mathbb{Z}\) choose \( \ell_\infty \in \text{Lag}_\infty(n) \) such that \( \ell = \pi_\text{Lag}(\ell_\infty) \). If \( m(\ell_\infty, \ell_\alpha, \infty) = k \) we are done. If \( m(\ell_\infty, \ell_\alpha, \infty) \neq k \) replace \( \ell_\infty \) by \( \beta^r \ell_\infty \) such that \( m(\ell_\infty, \ell_\alpha, \infty) + r = k \) (cf. Remark 79). The Leray index \( \mu \) is locally constant on the set

\[
\{(\ell_\infty, \ell'_\infty) : \ell \cap \ell' = 0\} \subset (\text{Lag}_\infty(n))^2
\]
hence so is \( m \). It follows that the restriction of \( \Phi_\alpha \) to \( \{\ell_\infty : \ell \cap \ell_\alpha = 0\} \) indeed is a homeomorphism onto its image \( \{\ell : \ell \cap \ell_\alpha = 0\} \times \mathbb{Z} \).

**Proof of (ii).** The mapping \( \Phi_{\alpha\beta} = \Phi_\alpha \Phi_\beta^{-1} \) takes \((\ell, \lambda) = (\ell, m(\ell_\infty, \ell_\beta, \infty))\)
to \((\ell', \lambda') = (\ell, m(\ell_\infty, \ell_\alpha, \infty))\) hence

\[
\Phi_{\alpha\beta}(\ell_\infty) = (\ell, \lambda + m(\ell_\infty, \ell_\alpha, \infty) - m(\ell_\infty, \ell_\beta, \infty))
\]
which is the same thing as (2.52) in view of formula (2.50). □

We are going to perform a similar identification for the universal covering of the symplectic group. This will allow us to exhibit precise formulas for \( q \)-symplectic geometry.

**The universal covering** \( \text{Sp}_\infty(n) \)

Recall that the Leray index is \( \text{Sp}_\infty(n) \)-invariant:

\[
\mu(S_\infty \ell_\infty, S_\infty \ell'_\infty) = \mu(\ell_\infty, \ell'_\infty)
\]
for all \((S_\infty, \ell_\infty, \ell'_\infty) \in \text{Sp}_\infty(n) \times \text{Lag}_\infty^2(n)\) (Proposition 72, Subsection 2.3).

**Definition 81** Let \( \ell \in \text{Lag}(n) \). We call “Maslov index” on \( \text{Sp}_\infty(n) \) relative to \( \ell \) the mapping \( \mu_{\ell} : \text{Sp}_\infty(n) \to \mathbb{Z} \) defined by

\[
\mu_{\ell}(S_\infty) = \mu(S_\infty \ell_\infty, \ell_\infty)
\]
(2.53)
where \( \ell_\infty \) is an arbitrary element of \( \text{Lag}_\infty(n) \) with projection \( \pi_\text{Lag}(\ell_\infty) = \ell \).

This definition makes sense in view of the following observation: suppose that we change \( \ell_\infty \) into another element \( \ell'_\infty \) with same projection \( \ell \). Then there exists an integer \( m \) such that \( \ell_\infty = \beta^m \ell'_\infty \) and

\[
\mu(S_\infty \ell_\infty, \ell_\infty) = \mu(S_\infty (\beta^r \ell'_\infty), \beta^r \ell'_\infty)
\]

\[
= \mu(S_\infty (\beta^r \ell'_\infty), \ell'_\infty) - 2r
\]

\[
= \mu((\beta^r \ell'_\infty, S^{-1}_\infty \ell'_\infty)) - 2r
\]

\[
= \mu(\ell'_\infty, S^{-1}_\infty \ell'_\infty) + 2r - 2r
\]

\[
= \mu(S_\infty \ell'_\infty, \ell'_\infty)
\]
where we have used successively (2.46), (2.66), again (2.46), and finally the \( \text{Sp}_\infty(n) \)-invariance (2.53) of the Leray index.

Here are a few properties which immediately follow from those of the Leray index:
In view of property (2.44) (Proposition 74) of the Leray index we have

\[ \mu_\ell(S_\infty) \equiv n - \dim(S_\ell \cap \ell) \mod 2 \tag{2.54} \]

for all \( S_\infty \in \text{Sp}_\infty(n) \).

- The antisymmetry (2.33) of the Leray index implies that we have

\[ \mu_\ell(S^{-1}_\infty) = -\mu_\ell(S_\infty) \quad \mu_\ell(I_\infty) = 0 \tag{2.55} \]

\( (I_\infty \text{ the unit of Sp}_\infty(n)) \).

- Let \( \alpha \) be the generator of \( \pi_1[\text{Sp}(n)] \cong (\mathbb{Z},+) \) whose image in \( \mathbb{Z} \) is +1. Then

\[ \mu_\ell(\alpha^r S_\infty) = \mu_\ell(S_\infty) + 4r \tag{2.56} \]

for every \( S_\infty \in \text{Sp}(n) \) and \( r \in \mathbb{Z} \); this immediately follows from formula (2.46) for the action of \( \pi_1[\text{Lag}(n)] \) on the Leray index.

The following properties of \( \mu_\ell \) are immediate consequences of the characteristic properties of the Leray index:

**Proposition 82** (i) For all \( S_\infty, S'_\infty \) in \( \text{Sp}_\infty(n) \)

\[ \mu_\ell(S_\infty S'_\infty) = \mu_\ell(S_\infty) + \mu_\ell(S'_\infty) + \tau_\ell(S, S') \tag{2.57} \]

where \( \tau_\ell : (\text{Sp}(n))^2 \longrightarrow \mathbb{Z} \) is defined by

\[ \tau_\ell(S, S') = \tau(\ell, S_\ell, S S'_\ell). \tag{2.58} \]

(ii) The function \( (S_\infty, \ell, \ell') \longmapsto \mu_\ell(S) - \tau(S_\ell, S'_\ell, \ell'') \) is locally constant on the set

\[ \{ (S_\infty, \ell, \ell') : S S_\ell \cap \ell'' = \ell \cap \ell'' = 0 \} \subset \text{Sp}_\infty(n) \times (\text{Lag}(n))^2. \]

In particular, \( \mu_\ell \) is locally constant on \( \{ S_\infty : \dim(S_\ell \cap \ell) = 0 \} \).

(iii) We have

\[ \mu_\ell(S_\infty) - \mu_\ell(S'_\infty) = \tau(S_\ell, S_\ell') - \tau(S_\ell, S'_\ell, \ell') \tag{2.59} \]

for every \( S_\infty \in \text{Sp}(n) \) and \( (\ell, \ell') \in (\text{Lag}(n))^2 \).

**Proof.** Proof of (i). By definition of \( \mu_\ell \)

\[ \mu_\ell(S_\infty S'_\infty) - \mu_\ell(S_\infty) - \mu_\ell(S'_\infty) = \mu(S_\ell S'_\ell, S_\ell \cap \ell') - \mu(S_\ell \ell_\infty, S_\ell) - \mu(S_\ell \ell_\infty, S'_\ell) - \mu(S_\ell \ell_\infty, S'_\ell). \]

that is, using the \( \text{Sp}_\infty(n) \)-invariance and the antisymmetry of \( \mu \):

\[ \mu_\ell(S_\infty S'_\infty) - \mu_\ell(S_\infty) - \mu_\ell(S'_\infty) = \mu(S_\ell S'_\ell, S_\ell \ell_\infty) + \mu(S_\ell S'_\ell, S_\ell \ell_\infty) - \mu(S_\ell S'_\ell, S_\ell \ell_\infty) - \mu(S_\ell S'_\ell, S_\ell \ell_\infty). \]
In view of the cocycle property (2.42) of the Leray index the right-hand side of this equality is equal to

$$\tau(SS'\ell, \ell, S\ell) = \tau(\ell, S\ell, SS'\ell) = \tau(\ell, S,)$$

hence (2.57). Property (ii) immediately follows from the two following observations: the Leray index is locally constant on

$$\{(\ell, \ell') : \ell \cap \ell' = 0\} \subset (\text{Lag}(n))^2$$

and the signature $$\tau(\ell, \ell', \ell'')$$ is locally constant on

$$\{(\ell, \ell', \ell'') : \ell \cap \ell' = \ell' \cap \ell'' = \ell'' \cap \ell = 0\} \subset (\text{Lag}(n))^3.$$ 

Proof of (iii). Using again the property (2.42) and the $$\text{Sp}_\infty(n)$$-invariance of $$\mu$$ we have

$$\mu(S_{\infty}\ell_{\infty}, \ell_{\infty}) - \mu(S_{\infty}\ell_{\infty}, \ell'_{\infty}) + \mu(S_{\infty}\ell_{\infty}, S_{\infty}\ell''_{\infty}) = \tau(S\ell, \ell, \ell')$$

$$\mu(S_{\infty}\ell_{\infty}, S_{\infty}\ell'_{\infty}) - \mu(S_{\infty}\ell_{\infty}, \ell''_{\infty}) + \mu(S_{\infty}\ell''_{\infty}, \ell''_{\infty}) = \tau(S\ell, S\ell', \ell')$$

which yields (2.59) subtracting the first identity from the second. 

The practical calculation of $$\mu_{\ell}(S_{\infty})$$ does not always require the determination of a Leray index. Formula (2.57) can often be used with profit. Here is an example:

**Example 83** Assume that $$n = 1$$. Let $$(-I)_{\infty}$$ be the homotopy class of the symplectic path $$t \mapsto e^{\pi t J}, 0 \leq t \leq 1,$$ joining $$I$$ to $$-I$$ in $$\text{Sp}(1)$$. We have

$$(-I)^2_{\infty} = \alpha$$ (the generator of $$\pi_1[\text{Sp}(n)]$$) hence

$$\mu_{\ell}((-I)^2_{\infty}) = \mu_{\ell}(\alpha) = 4.$$ 

But (2.57) implies that

$$\mu_{\ell}((-I)^2_{\infty}) = 2\mu_{\ell}((-I)_{\infty}) + \tau(\ell, \ell, \ell) = 2\mu_{\ell}((-I)_{\infty})$$

hence $$\mu_{\ell}((-I)_{\infty}) = 2.$$ 

It turns out that the properties (i) and (ii) of the Maslov index $$\mu_{\ell}$$ listed in Proposition 82 characterize that index. More precisely:

**Proposition 84** Assume that $$\mu'_{\ell} : \text{Sp}_{\infty}(n) \to \mathbb{Z}$$ is locally constant on

$$\{S_{\infty} : \dim(S\ell \cap \ell) = 0\}$$ and satisfies

$$\mu_{\ell}(S_{\infty}S'_{\infty}) = \mu_{\ell}(S_{\infty}) + \mu_{\ell}(S'_{\infty}) + \tau_{\ell}(S, S')$$

(2.60)

for all $$S_{\infty}, S'_{\infty}$$ in $$\text{Sp}_{\infty}(n)$$ ($$S = \pi^{\text{Sp}}(S_{\infty}), S' = \pi^{\text{Sp}}(S'_{\infty})$$). Then $$\mu'_{\ell} = \mu_{\ell}.$$
Proof. The function $\delta_\ell = \mu_\ell - \mu'_\ell$ satisfies

$$\delta_\ell(S_\infty S'_\infty) = \delta_\ell(S_\infty) + \delta_\ell(S'_\infty)$$

and is locally constant on $\{S_\infty : \text{dim}(S_\ell \cap \ell) = 0\}$. In view of Proposition 37 every $S \in \text{Sp}(n)$ can be written $S = S_1 S_2$ with $S_1 \ell_0 \cap \ell_0 = S_2 \ell_0 \cap \ell_0 = 0$. Since

$$\delta_\ell(S_\infty) = \delta_\ell(S_{1,\infty}) + \delta_\ell(S_{2,\infty})$$

it follows that $\delta_\ell$ is actually constant on $\text{Sp}_\infty(n)$. Taking $S_\infty = S'_\infty$ we thus have $\delta_\ell(S_\infty) = 0$ for all $S_\infty$ hence $\mu'_\ell = \mu_\ell$.

We have the following result:

**Lemma 85** Let $S_W$ and $S_{W'}$ be two free symplectic matrices and write

$$S_W = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad S_{W'} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}, \quad S_W S_{W'} = \begin{bmatrix} A'' & B'' \\ C'' & D'' \end{bmatrix}.$$ 

Let $\ell_P = 0 \times \mathbb{R}^n$. We have

$$\tau(\ell_P, S_W \ell_P, S_W S_{W'} \ell_P) = \text{sign}(B^{-1} B''(B')^{-1}). \quad (2.61)$$

**Proof.** Let us first prove (2.61) in the particular case where $S_{W'} \ell_P = \ell_X$, that is when $S_{W'}$ is of the type

$$S_{W'} = \begin{bmatrix} A' & B' \\ C' & 0 \end{bmatrix}$$

in which case $B'' = AB'$. Using the $\text{Sp}(n)$-invariance and the antisymmetry of the signature, we have, since $S_{W'} \ell_P = \ell_X$:

$$\tau(\ell_P, S_W \ell_P, S_W S_{W'} \ell_P) = \tau(\ell_X, S_W^{-1} \ell_P, \ell_P). \quad (2.62)$$

The inverse of $S_W$ being the symplectic matrix

$$S_W^{-1} = \begin{bmatrix} D^T & -B^T \\ -C^T & A^T \end{bmatrix}$$

we thus have

$$S_W^{-1} \begin{bmatrix} 0 \\ p \end{bmatrix} = \begin{bmatrix} -B^T p \\ A^T p \end{bmatrix}$$

so the Lagrangian plane $S_W^{-1} \ell_P$ has for equation $A x + B p = 0$. Since $B$ is invertible (because $S_W$ is free) this equation can be rewritten $p = -B^{-1} A x$. Using the second formula (2.5) (Section 2.1) together with the identity (2.62) we have

$$\tau(\ell_P, S_W \ell_P, S_W S_{W'} \ell_P) = -\text{sign}(-B^{-1} A)$$

$$= \text{sign}(B^{-1}(AB')(B')^{-1})$$

$$= \text{sign}(B^{-1}B''(B')^{-1})$$
which proves (2.61) in the case $S_W \ell_P = \ell_X$. The general case reduces to the former, using the fact that the symplectic group acts transitively on all pairs of transverse Lagrangian planes. In fact, since

$$S_W, \ell_P \cap \ell_P = \ell_X \cap \ell_P = 0$$

we can find $S_0 \in \text{Sp}(n)$ such that $(\ell_P, S_W \ell_P) = S_0(\ell_P, \ell_X)$, that is $S_W \ell_P = S_0 \ell_X$ and $S_0 \ell_P = \ell_P$. It follows, using again the antisymmetry and $\text{Sp}(n)$-invariance of $\sigma$ that:

$$\tau(\ell_P, S_W \ell_P, S_W S_W' \ell_P) = \tau(\ell_X, (S_W S_0)^{-1} \ell_P, \ell_P)$$

which is (2.62) with $S_W$ replaced by $S_W S_0$. Changing $S_W'$ into $S_0^{-1} S_W'$ (and hence leaving $S_W S_W'$ unchanged) we are led back to the first case. Since $S_0 \ell_P = \ell_P$, $S_0$ must be of the type

$$S_0 = \begin{bmatrix} L & 0 \\ P & (L^{-1})^T \end{bmatrix}, \quad \det(L) \neq 0, \ P = P^T;$$

writing again $S_W$ in block-matrix form, the products $S S_0$ and $S_0^{-1} S_W'$ are thus of the type

$$S S_0 = \begin{bmatrix} * & B(L^T)^{-1} \\ * & * \end{bmatrix}, \quad S_0^{-1} S_W' = \begin{bmatrix} * & L^{-1} B' \\ * & * \end{bmatrix}$$

(the stars "*" are block-entries that are easily calculated, but that we do not need to write down), and hence

$$\tau(\ell_P, S_W \ell_P, S_W S_W' \ell_P) = \text{sign}(L^T B^{-1} B'' B'^{-1} L) = \text{sign}(B^{-1} B'' B'^{-1})$$

proving (2.61) in the general case. $\blacksquare$

**Remark 86** Formula (2.61) identifies $\tau(\ell_P, S \ell_P, S S' \ell_P)$ with the "composition index" $Q(S, S')$ defined by Robbin and Salamon in [62]. Notice however that $\tau(\ell_P, S \ell_P, S S' \ell_P)$ is defined for all $S, S' \in \text{Sp}(n)$ while $Q(S, S')$ is only defined for $S, S'$ when they are free.

For $\ell \in \text{Lag}(n)$ define a function $m_\ell : \text{Sp}_\infty(n) \rightarrow \mathbb{Z}$ by

$$m_\ell(S_\infty) = \frac{1}{2}(\mu_\ell(S_\infty) + n + \dim(S \ell \cap \ell)) \quad (2.63)$$

that is

$$m_\ell(S_\infty) = m(S_\infty \ell_\infty, \ell_\infty)$$

where $\ell_\infty$ has projection $\pi_{\text{Lag}}(\ell_\infty) = \ell$. Since $m(\ell_\infty, \ell'_\infty) \in \mathbb{Z}$ for all $(\ell_\infty, \ell'_\infty) \in (\text{Lag}_\infty(n))^2$ it follows that $m_\ell(S_\infty) \in \text{Sp}_\infty(n)$.

**Definition 87** The mapping $m_\ell : \text{Sp}_\infty(n) \rightarrow \mathbb{Z}$ defined by (2.63) is called the reduced Maslov index on $\text{Sp}_\infty(n)$ relatively to $\ell \in \text{Lag}(n)$.
The properties of the reduced Maslov index immediately follow from those of $\mu_\ell$. In particular (2.56) implies that
\[
m_\ell(\alpha^r S_\infty) = m_\ell(S_\infty) + 2r
\] (2.64)
for every integer $r$ ($\alpha$ being the generator of $\pi_1[\text{Sp}(n)]$). An immediate consequence of (2.64) is that the value modulo 2 of $m_\ell(S_\infty)$ only depends on the projection $S = \pi^{\text{Sp}}(S_\infty)$. We will denote by $m_\ell(S)$ the corresponding equivalence class:
\[
m \in m_\ell(S) \iff m \equiv m_\ell(S_\infty) \mod 2.
\]

The following result is the symplectic equivalent of Theorem 80. It identifies $\text{Sp}_\infty(n)$ with a subset of $\text{Sp}(n) \times \mathbb{Z}$:

**Theorem 88** For $\ell_\alpha \in \text{Lag}(n)$ define a mapping
\[
\Psi_\alpha : \text{Sp}_\infty(n) \longrightarrow \text{Sp}(n) \times \mathbb{Z}
\]
by the formula
\[
\Psi_\alpha(S_\infty) = (S, m_\ell_\alpha(S_\infty)).
\]

(i) The mapping $\Psi_\alpha$ is a bijection
\[
\text{Sp}_\infty(n) \longrightarrow \{(S, m) : S \in \text{Sp}(n), m \in m_\ell_\alpha(S)\}
\]
whose restriction to the subset $\{S_\infty : S_{\ell_\alpha} \cap \ell_\alpha = 0\}$ is a homeomorphism onto
\[
\text{Sp}_{\ell_\alpha}(n) = \{(S, m) : S \in \text{Sp}(n), S_{\ell_\alpha} \cap \ell_\alpha = 0, m \in m_\ell(S)\}.
\]

(ii) The set of all bijections $(\Psi_\alpha)_{\ell_\alpha}$ form a system of local charts of $\text{Sp}_\infty(n)$ whose transitions $\Psi_\alpha \Psi_\beta^{-1}$ are the functions
\[
\Psi_{\alpha\beta}(S, m) = (S, m + \text{Inert}(S_{\ell_\alpha}, \ell_\alpha, \ell_\beta) - \text{Inert}(S_{\ell_\alpha}, S_{\ell_\beta}, \ell_\beta)).
\]

**Proof.** Proof of (i). By definition of $m_\ell(S)$ the range of $\Psi_\alpha$ consists of all pairs $(S, m)$ with $m \in m_\ell(S)$. Assume that $(S, m_\ell(S_\infty)) = (S', m_\ell(S'_\infty))$. Then $S = S'$ and $S'_\infty = \alpha^r S_\infty$ for some $r \in \mathbb{Z}$ (cf. the proof of (i) in Theorem 80). In view of (2.64) we must have $r = 0$ and hence $S_\infty = S'_\infty$ so that $\Psi_\alpha$ is injective.

Proof of (ii). It is identical to that of the corresponding properties in Theorem 80.

The theorem above allows us to describe in a precise way the composition law of the universal covering group $\text{Sp}_\infty(n)$:

**Corollary 89** Let $\ell_\alpha \in \text{Lag}(n)$. Identifying $\text{Sp}_\infty(n)$ with the subset
\[
\{(S, m) : S \in \text{Sp}(n), m \in m_\ell_\alpha(S)\}
\]
of $\text{Sp}(n) \times \mathbb{Z}$ the composition law of $\text{Sp}_\infty(n)$ is given by the formula
\[
(S, m) *_{\ell_\alpha} (S', m') = (SS', m + m' + \text{Inert}(\ell_\alpha, S_{\ell_\alpha}, SS'_{\ell_\alpha})).
\] (2.65)
Proof. This is obvious since we have
\[ m_{\ell,\alpha}(S_\infty S'_\infty) = m_{\ell,\alpha}(S_\infty) + m_{\ell,\alpha}(S'_\infty) + \text{Inert}(\ell, S_\ell \alpha, SS' \ell) \]
in view of property (2.57) of \( \mu_\ell \) and definition (2.63) of \( m_\ell \). □

Let us now proceed to prove the main results of this section.

The action of \( \text{Sp}_q(n) \) on \( \text{Lag}_2(q)(n) \)

Let \( \text{St}(\ell_P) \) be the isotropy subgroup of \( \ell_P = 0 \times \mathbb{R}_P^n \) in \( \text{Sp}(n) \): \( S \in \text{St}(\ell_P) \) if and only if \( S \in \text{Sp}(n) \) and \( S \ell_P = \ell_P \). The fibration
\[ \text{Sp}(n)/\text{St}(\ell_P) = \text{Lag}(n) \]
defines an isomorphism
\[ \mathbb{Z} \cong \pi_1[\text{Sp}(n)] \longrightarrow \pi_1[\text{Lag}(n)] \cong \mathbb{Z} \]
which is multiplication by 2 on \( \mathbb{Z} \). It follows that the action of \( \text{Sp}(n) \) on \( \text{Lag}(n) \) can be lifted to a transitive action of the universal covering \( \text{Sp}_\infty(n) \) on the Maslov bundle \( \text{Lag}_\infty(n) \) such that
\[ (\alpha S_\infty) \ell_\infty = \beta^2(S_\infty \ell_\infty) = S_\infty(\beta^2 \ell_\infty) \] (2.66)
for all \( (S_\infty, \ell_\infty) \in \text{Sp}_\infty(n) \times \text{Lag}_\infty(n) \). As previously \( \alpha \) (resp. \( \beta \)) is the generator of \( \pi_1[\text{Sp}(n)] \) (resp. \( \pi_1[\text{Lag}(n)] \)) whose natural image in \( \mathbb{Z} \) is +1.

The following theorem describes \( \infty \)-symplectic geometry:

Theorem 90 Let \( \ell_\alpha \in \text{Lag}(n) \). Identifying \( \text{Sp}_\infty(n) \) with the subset
\[ \{(S, m) : S \in \text{Sp}(n), m \in m_{\ell,\alpha}(S)\} \]
of \( \text{Sp}(n) \times \mathbb{Z} \) defined in Theorem 88 and \( \text{Lag}_\infty(n) \) with \( \text{Lag}(n) \times \mathbb{Z} \) as in Theorem 80 the action of \( \text{Sp}_\infty(n) \) on \( \text{Lag}_\infty(n) \) is given by the formula
\[ (S, m) \cdot \ell_\alpha(\ell, \lambda) = (S \ell, m + \lambda - \text{Inert}(S \ell, S \ell_\alpha, \ell_\alpha)). \] (2.67)

Proof. We have \( \lambda = m(\ell, \ell_\alpha, \ell_\alpha, \infty) \) for some \( \ell_\infty \) covering \( \ell \), and \( m = m(S_\infty \ell_\alpha, \ell_\alpha, \ell_\alpha, \infty) \) for some \( S_\infty \) covering \( S \). Let us define the integer \( \partial \) by the condition
\[ m + \lambda + \delta = m(S_\infty \ell, \ell_\alpha, \infty) \]
that is
\[ \delta = m(S_\infty \ell, \ell_\alpha, \infty) - m(\ell, \ell_\alpha, \infty) - m(S_\infty \ell_\alpha, \ell_\alpha, \infty). \]
We have to show that
\[ \delta = - \text{Inert}(S \ell, S \ell_\alpha, \ell_\alpha). \] (2.68)
In view of the $\text{Sp}_\infty(n)$-invariance of the reduced Leray index we have

$$m(\ell_\infty, \ell_{\alpha, \infty}) = m(S_\infty \ell_\infty, S_\infty \ell_{\alpha, \infty})$$

and hence

$$\delta = m(S_\infty \ell_\infty, \ell_{\alpha, \infty}) - m(S_\infty \ell_\infty, S_\infty \ell_{\alpha, \infty}) - m(S_\infty \ell_{\alpha, \infty}, \ell_{\alpha, \infty});$$

on the other hand

$$m(S_\infty \ell_{\alpha, \infty}, \ell_{\alpha, \infty}) + m(\ell_{\alpha, \infty}, S_\infty \ell_{\alpha, \infty}) = n + \dim(S\ell_{\alpha} \cap \ell_{\alpha})$$

(formula (2.49)) so that

$$\delta = m(S_\infty \ell_\infty, \ell_{\alpha, \infty}) - m(S_\infty \ell_\infty, S_\infty \ell_{\alpha, \infty}) + m(\ell_{\alpha, \infty}, S_\infty \ell_{\alpha, \infty}) - n - \dim(S\ell_{\alpha} \cap \ell_{\alpha}).$$

Using property (2.50) of $m$ this can be rewritten

$$\delta = \text{Inert}(S\ell, \ell_{\alpha}, S\ell_{\alpha}) - n - \dim(S\ell_{\alpha} \cap \ell_{\alpha})$$

The equality (2.68) follows noting that by definition of the index of inertia and the antisymmetry of $\tau$

$$\text{Inert}(S\ell, \ell_{\alpha}, S\ell_{\alpha}) - n - \dim(S\ell_{\alpha} \cap \ell_{\alpha}) = -\text{Inert}(S\ell, S\ell_{\alpha}, \ell_{\alpha}).$$

Recall that there is an isomorphism

$$\mathbb{Z} \cong \pi_1[\text{Sp}(n)] \longrightarrow \pi_1[\text{Lag}(n)] \cong \mathbb{Z}$$

which is multiplication by 2 on $\mathbb{Z}$. In fact (formula (2.66))

$$(\alpha S_\infty)\ell_\infty = \beta^2(S_\infty \ell_\infty) = S_\infty(\beta^2 \ell_\infty)$$

for all $(S_\infty, \ell_\infty) \in \text{Sp}_\infty(n) \times \text{Lag}_\infty(n)$. Also recall that the Leray and Maslov indices satisfy

$$\mu(\beta^r \ell_\infty, \beta^{r'} \ell'_\infty) = \mu(\ell_\infty, \ell'_\infty) + 4(r - r')$$

(formula (2.46) and

$$\mu_\ell(\alpha^r S_\infty) = \mu_\ell(S_\infty) + 2r$$

(formula (2.56) for all integers $r$ and $r'$.

Let now $q$ be an integer, $q \geq 1$. We have

$$\pi_1[\text{Lag}(n)] = \{\beta^k : k \in \mathbb{Z}\}$$

hence:

$$\text{Lag}_q(n) = \text{Lag}(n)/\{\beta^q k : k \in \mathbb{Z}\}. \quad (2.69)$$
Similarly, since
\[ \pi_1[\text{Sp}(n)] = \{ \alpha^k : k \in \mathbb{Z} \} \]
we have
\[ \text{Sp}_q(n) = \text{Sp}(n)/\{ \alpha^{qk} : k \in \mathbb{Z} \}. \] (2.70)
Let us now identify \( \pi_1[\text{Lag}(n)] \) with \( \mathbb{Z} \). Recalling that the natural homomorphism \( \pi_1[\text{Sp}(n)] \rightarrow \pi_1[\text{Lag}(n)] \) is multiplication by 2 in \( \mathbb{Z} \) (cf. formula (2.66)) \( \pi_1[\text{Sp}(n)] \) is then identified with \( 2\mathbb{Z} \). This leads us, taking (2.69) and (2.70) into account, to the identifications
\[ \text{Lag}_q(n) = \text{Lag}_\infty(n)/q\mathbb{Z}, \quad \text{Sp}_q(n) = \text{Sp}_\infty(n)/2q\mathbb{Z}. \] (2.71)
The Leray index on \( \text{Lag}_q(n) \) is now defined as being the function
\[ [\mu]_q : (\text{Lag}_q(n))^2 \rightarrow \mathbb{Z} = \mathbb{Z}/q\mathbb{Z} \]
given by
\[ [\mu]_q(\ell(q), \ell'(q)) = \mu(\ell_\infty, \ell'_\infty) \mod q \]
if \( (\ell_\infty, \ell'_\infty) \in (\text{Lag}_\infty(n))^2 \) covers \( (\ell(q), \ell'(q)) \in (\text{Lag}_q(n))^2 \). Similarly the Maslov index relative to \( \ell \in \text{Lag}(n) \) on \( \text{Sp}_q(n) \) is the function
\[ [\mu_\ell]_{2q} : \text{Sp}_q(n) \rightarrow \mathbb{Z}_{2q} = \mathbb{Z}/2q\mathbb{Z} \]
defined by
\[ [\mu_\ell]_{2q}(S(q)) = \mu_\ell(S_\infty) \mod 2q \]
if \( S_\infty \in \text{Sp}_\infty(n) \) covers \( S(q) \in \text{Sp}_q(n) \).

Exactly as was the case for \( \infty \)-symplectic geometry the study of \( q \)-symplectic geometry requires the use of reduced indices:

**Definition 91** The reduced Leray index on \( \text{Lag}_{2q}(n) \) is defined by
\[ [m]_{2q}(\ell(q), \ell'(q)) = m(\ell_\infty, \ell'_\infty) \mod 2q \]
and the reduced Maslov index \( [m_\ell]_{2q} \) on \( \text{Sp}_q(n) \) by
\[ [m_\ell]_{2q}(S(q)) = m_\ell(S_\infty) \mod 2q. \]

Let us denote by \([r]_{2q}\) the equivalence class modulo \( 2q \) of \( r \in \mathbb{Z} \). Corollary 89 and Theorem 90 immediately imply that:

**Corollary 92** Let \( \ell_\alpha \in \text{Lag}(n) \) and identify \( \text{Sp}_q(n) \) with the subset
\[ \{(S, m) : S \in \text{Sp}(n), m \in m_{\ell_\alpha}(S)\} \]
of \( \text{Sp}(n) \times \mathbb{Z}_{2q} \) equipped with the composition law
\[ (S, [m]_{2q}) *_{\ell_\alpha} (S', [m']_{2q}) = (SS', [m + m' + \text{Inert}(\ell_\alpha, S\ell_\alpha, SS'\ell_\alpha)]_{2q}) \]
and \( \text{Lag}_{2q}(n) \) with \( \text{Lag}(n) \times \mathbb{Z}_{2q} \). The action of \( \text{Sp}_q(n) \) on \( \text{Lag}_{2q}(n) \) is then given by the formula
\[ (S, [m]_{2q}) \cdot_{\ell_\alpha} ([\lambda]_{2q}) = (S\ell, [m + \lambda - \text{Inert}(S\ell, \ell_\alpha)]_{2q}). \]

We have now achieved our goal which was to describe the structure of \( q \)-symplectic geometry.

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3 Lagrangian and symplectic Intersection Indices

We will generalize the notion of Maslov index to arbitrary paths (not just loops) in \( \text{Lag}(n) \) and \( \text{Sp}(n) \). We will study two (related) constructions of these “intersection indices”: the Lagrangian and symplectic “Maslov indices”, which extend the usual notion of Maslov index for loops to arbitrary paths in \( \text{Lag}(n) \) and \( \text{Sp}(n) \), and which are directly related to the notion of “spectral flow”, and the Conley–Zehnder index, which plays a crucial role in Morse theory.

The results in the two first Sections have appeared in de Gosson [27] and in M. de Gosson and S. de Gosson [30]. For applications to Morse theory and to functional analysis, see Booss–Bavnbek and Furutani [9].

3.1 Lagrangian Path Intersection Index

A Lagrangian path is a continuous mapping \( \Lambda : [0, 1] \rightarrow \text{Sp}(n) \). The vocation of a Lagrangian intersection index is to keep a precise accountancy of the way that path intersects a given locus (or “Maslov cycle”) in \( \text{Lag}(n) \).

It can be viewed as a generalization of the usual Maslov index in \( \text{Lag}(n) \), to which it reduces (up to the factor two) when \( \Lambda \) is a loop. We begin by briefly discussing the notion of stratum in \( \text{Lag}(n) \).

The strata of \( \text{Lag}(n) \)

Let \( M^m \) be a \( m \)-dimensional topological manifold. A stratification of \( M^m \) is a partition of \( M^m \) in a family \( \{M^k_\alpha\}_{\alpha \in A} \) of connected submanifolds (“strata”) of dimension \( k \leq m \) such that:

- The family \( \{M^k_\alpha\}_{\alpha \in A} \) is a locally finite partition of \( M^m \);
- If \( M^k_\alpha \cap M^{k'}_{\alpha'} \neq \emptyset \) for \( \alpha \neq \alpha' \) then \( M^k_\alpha \subset M^{k'}_{\alpha'} \) and \( k \leq k' \);
- \( \overline{M^k_\alpha} \setminus M^k_{\alpha} \) is a disjoint union of strata of dimension \( < k \).

It turns out that to every Lagrangian plane \( \ell \) we can associate a natural stratification of \( \text{Lag}(n) \):

For \( \ell \in \text{Lag}(n) \) and \( 0 \leq k \leq n \) set

\[
\text{Lag}_\ell(n, k) = \{ \ell' \in \text{Lag}(n) : \dim \ell \cap \ell' = k \}.
\]

We will call \( \text{Lag}_\ell(n; k) \) the stratum of \( \text{Lag}(n) \) of order \( k \), relative to the Lagrangian plane \( \ell \). Clearly

\[
\text{Lag}_\ell(n; k) \cap \text{Lag}_\ell(n; k') = \emptyset \quad \text{if} \quad k \neq k'
\]

and

\[
\text{Lag}(n) = \cup_{0 \leq k \leq n} \text{Lag}_\ell(n; k).
\]
One proves (see e.g. Trèves [70] or Robbin and Salamon [62]) that the sets \( \text{Lag}_\ell(n, k) \) form a stratification of \( \text{Lag}(n) \). Moreover \( \text{Lag}_\ell(n; 0) \) is an open subset of \( \text{Lag}(n) \) and the sets \( \text{Lag}_\ell(n; k) \) are, for \( 0 \leq k \leq n \), connected submanifolds of \( \text{Lag}(n) \) with codimension \( k(k + 1)/2 \).

**Definition 93** The closed set

\[
\Sigma_\ell = \text{Lag}(n) \setminus \text{Lag}_\ell(n; 0) = \overline{\text{Lag}_\ell(n, 1)}
\]

is called the “Maslov cycle relative to \( \ell \)”: it is the set of Lagrangians that are not transverse to \( \ell \). When \( \ell = \ell_P \) we call \( \Sigma_\ell \) simply the “Maslov cycle”, and denote it by \( \Sigma \).

Let us now enunciate a system of “reasonable” axioms that should be satisfied by a generalization of the Maslov index for loops.

**The Lagrangian intersection index**

The definition we give here is slightly more general than those of, for instance, Robbin and Salamon [62]. We do not in particular impose from the beginning any “dimensional additivity”. This property is however satisfied by the explicit indices we construct in next subsection.

Let \( \mathcal{C}(\text{Lag}(n)) \) be the set of continuous paths \( \Lambda : [0, 1] \rightarrow \text{Lag}(n) \). If \( \Lambda \) and \( \Lambda' \) are two consecutive paths (i.e., if \( \Lambda(1) = \Lambda'(0) \)) we shall denote by \( \Lambda \ast \Lambda' \) the concatenation of \( \Lambda \) and \( \Lambda' \), that is, the path \( \Lambda \) followed by the path \( \Lambda' \):

\[
\Lambda \ast \Lambda'(t) = \begin{cases} 
\Lambda(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\
\Lambda'(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1.
\end{cases}
\]

We will use the notation \( \Lambda^o \) for the inverse of the path \( \Lambda \):

\[
\Lambda^o(t) = \Lambda(1 - t), \quad 0 \leq t \leq 1.
\]

Finally, we shall write \( \Lambda \sim \Lambda' \) when the paths \( \Lambda \) and \( \Lambda' \) are homotopic with fixed endpoints.

**Definition 94** A "Lagrangian intersection index" is a mapping

\[
\mu_{\text{Lag}} : \mathcal{C}(\text{Lag}(n)) \times \text{Lag}(n) \rightarrow \mathbb{Z}
\]

having the following four properties:

(L1) Homotopy invariance: If the paths \( \Lambda \) and \( \Lambda' \) have the same endpoints, then \( \mu_{\text{Lag}}(\Lambda, \ell) = \mu_{\text{Lag}}(\Lambda', \ell) \) if and only if \( \Lambda \sim \Lambda' \);

(L2) Additivity under composition: if \( \Lambda \) and \( \Lambda' \) are two consecutive paths, then

\[
\mu_{\text{Lag}}(\Lambda \ast \Lambda', \ell) = \mu_{\text{Lag}}(\Lambda, \ell) + \mu_{\text{Lag}}(\Lambda', \ell)
\]

for every \( \ell \in \text{Lag}(n) \);
(L₃) Zero in strata: if the path Λ remains in the same stratum Lagₖ(n; k), then μₗag(Λ, ℓ) is zero:
\[ \dim(Λ(t) ∩ ℓ) = k \quad (0 ≤ t ≤ 1) \implies μₗag(Λ, ℓ) = 0; \]

(L₄) Restriction to loops: if γ is a loop in Lag(n) then
\[ μₗag(γ, ℓ) = 2m(γ) \]
(m(γ) the Maslov index of γ) for every ℓ ∈ Lag g(n).

We note in particular that the axioms (L₂) and (L₄) imply that an intersection index is antisymmetric in the sense that
\[ μₗag(Λ°, ℓ) = -μₗag(Λ, ℓ). \quad (3.1) \]

Indeed, by (L₂) we have
\[ μₗag(Λ * Λ°, ℓ) = μₗag(Λ, ℓ) + μₗag(Λ°, ℓ) \]
and since the loop Λ * Λ° = γ is homotopic to a point (L₄) implies that
\[ μₗag(Λ * Λ°, ℓ) = 2m(γ) = 0. \]

The system of axioms (L₁)–(L₄) is in fact equivalent to the system of axioms obtained by replacing (L₁) by the apparently stronger condition (3.2) below. Let us first define the notion of “homotopy in strata”:

**Definition 95** Two Lagrangian paths Λ and Λ’ are said to be “homotopic in the strata relative to ℓ” (denoted Λ ≈ₗ Λ’) if there exist a continuous mapping \( h : [0, 1] \times [0, 1] \to \text{Lag}(n) \) such that
\[ h(t, 0) = Λ(t) \quad , \quad h(t, 1) = Λ'(t) \quad \text{for} \quad 0 ≤ t ≤ 1 \]
and two integers \( k_0, k_1 \) (\( 0 ≤ k_0, k_1 ≤ n \)) such that
\[ h(0, s) ∈ \text{Lag}_ℓ(n; k_0) \quad \text{and} \quad h(1, s) ∈ \text{Lag}_ℓ(n; k_1) \quad \text{for} \quad 0 ≤ s ≤ 1. \]

Intuitively Λ ≈ₗ Λ means that Λ and Λ’ are homotopic in the usual sense and that the endpoints Λ(0) and Λ’(0) (resp. Λ(1) and Λ’(1)) remain in the same stratum during the homotopy taking Λ to Λ’.

The intersection indices μₗag have the following property that strengthens (L₁):

**Proposition 96** If the paths Λ and Λ’ are homotopic in strata relative to ℓ, then μₗag(Λ, ℓ) = μₗag(Λ’, ℓ):
\[ Λ ≈ₗ Λ’ \implies μₗag(Λ, ℓ) = μₗag(Λ’, ℓ). \quad (3.2) \]
Proof. Suppose that $\Lambda \approx \ell \Lambda'$ and define the paths $\varepsilon_0$ and $\varepsilon_1$ joining $\Lambda'(0)$ to $\Lambda(0)$ and $\Lambda(1)$ to $\Lambda'(1)$, respectively, by $\varepsilon_0(s) = h(0, 1-s)$ and $\varepsilon_1(s) = h(1, s)$ ($0 \leq s \leq 1$). Then $\Lambda \ast \varepsilon_1 \ast \Lambda'^{-1} \ast \varepsilon_0$ is homotopic to a point, and hence, in view of (L2) and (L4):

$$\mu_{\text{Lag}}(\Lambda, \ell) + \mu_{\text{Lag}}(\varepsilon_1, \ell) + \mu_{\text{Lag}}(\Lambda'^{-1}, \ell) + \mu_{\text{Lag}}(\varepsilon_0, \ell) = 0.$$ 

But, in view of (L3),

$$\mu_{\text{Lag}}(\varepsilon_1, \ell) = \mu_{\text{Lag}}(\varepsilon_0, \ell) = 0$$

and thus

$$\mu_{\text{Lag}}(\Lambda, \ell) + \mu_{\text{Lag}}(\Lambda'^{-1}, \ell) = 0;$$

the conclusion now follows from the antisymmetry property (3.1). 

Let us next construct explicitly a Lagrangian intersection index using the properties of the Leray index previously studied.

Explicit construction

Our approach is purely topological, and does not appeal to any differentiability conditions for the involved paths, as opposed to the construction given in, for instance, Robbin and Salamon [62].

**Theorem 97** For $(\Lambda_{12}, \ell) \in C(\text{Lag}(n)) \times \text{Lag}(n)$ let us define $\ell_\infty$, $\ell_{1,\infty}$ and $\ell_{2,\infty}$ in the following way:

(i) $\ell_\infty$ is an arbitrary element of $\text{Lag}_\infty(n)$ covering $\ell$;

(ii) $\ell_{1,\infty}$ is the equivalence class of an arbitrary path $\Lambda_{01} \in C(\text{Lag}(n))$ joining $\ell_0$ to $\ell_1$;

(iii) $\ell_{2,\infty}$ is the equivalence class of $\Lambda_{02} = \Lambda_{01} \ast \Lambda_{12}$.

Then the formula

$$\mu_{\text{Lag}}(\Lambda_{12}, \ell) = \mu(\ell_{2,\infty}, \ell_\infty) - \mu(\ell_{1,\infty}, \ell_\infty)$$

(3.3)

defines an intersection index on $\text{Lag}(n)$.

**Proof.** Let us first show that $\mu_{\text{Lag}}(\Lambda_{12}, \ell)$ is independent of the choice of the element $\ell_\infty$ of $\text{Lag}_\infty(n)$ covering $\ell$. Assume in fact that

$$\pi_{\text{Lag}}(\ell'_\infty) = \pi_{\text{Lag}}(\ell_\infty) = \ell;$$

then there exists $r \in \mathbb{Z}$ such that $\ell'_\infty = \beta^r \ell_\infty$ ($\beta$ is as usual the generator of $\pi_1[\text{Lag}(n)]$) and hence

$$\mu(\ell_{2,\infty}, \ell'_\infty) = \mu(\ell_{2,\infty}, \ell_\infty) - 2r, \quad \mu(\ell_{1,\infty}, \ell'_\infty) = \mu(\ell_{1,\infty}, \ell_\infty) - 2r$$

in view of property (2.46) of the Leray index; it follows that

$$\mu(\ell_{2,\infty}, \ell'_\infty) - \mu(\ell_{1,\infty}, \ell'_1, \ell_\infty) = \mu(\ell_{2,\infty}, \ell_\infty) - \mu(\ell_{1,\infty}, \ell_\infty).$$
Let us next show that $\mu_{\text{Lag}}(\Lambda_{12}, \ell)$ is also independent of the choice of $\Lambda_{01}$ and hence of the choice of the element $\ell_{1,\infty}$ such that $\pi_{\text{Lag}}(\ell_{1,\infty}) = \ell_1$. Let us replace $\Lambda_{01}$ by a path $\Lambda'_{01}$ with same such that $\ell'_{1,\infty} = \beta^r \ell_{1,\infty}$. $\ell_{2,\infty}$ will thus be replaced by $\ell'_{2,\infty} = \beta^r \ell_{2,\infty}$. Using again (2.46) we have

$$
\mu(\ell'_{2,\infty}, \ell_{\infty}) - \mu(\ell'_{1,\infty}, \ell_{\infty}) = \mu(\ell_{2,\infty}, \ell_{\infty}) - \mu(\ell_{1,\infty}, \ell_{\infty})
$$

hence our claim. It remains to prove that the function $\mu_{\text{Lag}}$ defined by (3.3) satisfies the axioms (L$_1$)–(L$_4$). \textit{Axiom L$_1$.} Let us replace the path $\Lambda_{12}$ by any path $\Lambda'_{12}$ homotopic (with fixed endpoints) to $\Lambda_{12}$. Then $\Lambda_{02} = \Lambda_{01} * \Lambda_{12}$ is replaced by a homotopic path $\Lambda'_{02} = \Lambda_{01} * \Lambda'_{12}$ and the homotopy class $\ell_{2,\infty}$ does not change. Consequently, $\mu_{\text{Lag}}(\Lambda'_{12}, \ell) = \mu_{\text{Lag}}(\Lambda_{12}, \ell)$. \textit{Axiom L$_2$.} Consider two consecutive paths $\Lambda_{12}$ and $\Lambda_{23}$. By definition

$$
\mu_{\text{Lag}}(\Lambda_{23}, \ell) = \mu(\ell_{3,\infty}, \ell_{\infty}) - \mu(\ell'_{2,\infty}, \ell_{\infty})
$$

where $\ell'_{2,\infty}$ is the homotopy class of an arbitrary path $\Lambda'_{02}$ and $\ell_{3,\infty}$ that of $\Lambda'_{02} * \Lambda_{23}$. Let us choose $\Lambda'_{02} = \Lambda_{02}$. Then $\ell'_{2,\infty} = \ell_{2,\infty}$ and

$$
\mu_{\text{Lag}}(\Lambda_{12}, \ell) + \mu_{\text{Lag}}(\Lambda_{23}, \ell) = 
\mu(\ell_{2,\infty}, \ell_{\infty}) - \mu(\ell_{1,\infty}, \ell_{\infty}) + \mu(\ell_{3,\infty}, \ell_{\infty}) - \mu(\ell_{2,\infty}, \ell_{\infty})
$$

that is

$$
\mu_{\text{Lag}}(\Lambda_{12}, \ell) + \mu_{\text{Lag}}(\Lambda_{23}, \ell) = \mu_{\text{Lag}}(\Lambda_{13}, \ell)
$$

which we set out to prove. \textit{Axiom L$_3$.} Let $\Lambda_{12}$ be a path in the stratum $\text{Lag}_{\ell}(n; k)$ and denote by $\ell_{\infty}(t)$ the equivalence class of $\Lambda_{01} * \Lambda_{12}(t)$ for $0 \leq t \leq 1$. The mapping $t \mapsto \ell_{\infty}(t)$ being continuous the composition mapping $t \mapsto \mu(\ell_{\infty}(t), \ell_{\infty})$ is locally constant on the interval $[0, 1]$. It follows that it is constant on that interval since $\text{Lag}_{\ell}(n; k)$ is connected. Its value is

$$
\mu(\ell_{\infty}(0), \ell_{\infty}) = \mu(\ell_{\infty}(1), \ell_{\infty})
$$

hence $\mu(\Lambda_{12}, \ell) = 0$. \textit{Axiom L$_4$.} Let $\gamma \in \pi_1[\text{Lag}(n), \ell_0)]$. In view of formula (2.47) in Corollary 76 we have the equality

$$
\mu_{\text{Lag}}(\gamma, \ell) = \mu(\gamma \ell_{0,\infty}, \ell_{\infty}) - \mu(\ell_{0,\infty}, \ell_{\infty}) = 2m(\gamma)
$$

which concludes the proof. ■

\textbf{Remark 98} We have shown in [29, 30], using the methods above, that the so-called “Hörmander index” $\xi$ intervening in the study of Fourier integral operators is given by

$$
\xi(\ell_1, \ell_2, \ell_3, \ell_4) = \frac{1}{2}(\tau(\ell_1, \ell_2, \ell_3) - \tau(\ell_1, \ell_2, \ell_4)).
$$

(3.4)

Let us now proceed to the study of symplectic intersection indices.
3.2 Symplectic Intersection Indices

The theory of symplectic intersection indices is analogous to the Lagrangian case. In fact each theory can be deduced from the other. For the sake of clarity we however treat the symplectic case independently.

The strata of $\text{Sp}(n)$

Similar definitions are easy to give for the symplectic group $\text{Sp}(n)$. For $\ell \in \text{Lag}(n)$ and $k$ an integer we call the set

$$\text{Sp}_\ell(n; k) = \{ S \in \text{Sp}(n) : \dim S \ell \cap \ell = k \}$$

the *stratum of $\text{Sp}(n)$ of order $k$, relative to the Lagrangian plane $\ell$*. The sets $\text{Sp}_\ell(n; k)$ form a stratification of $\text{Sp}(n)$. Clearly $\text{Sp}_\ell(n; k)$ is empty for $k < 0$ or $k > n$, and we have

$$\text{Sp}_\ell(n; 0) = \text{St}(\ell)$$

(the stabilizer of $\ell$ in $\text{Sp}(n)$). We have of course

$$\text{Sp}(n) = \bigcup_{0 \leq k \leq n} \text{Sp}_\ell(n; k).$$

The strata $\text{Sp}_\ell(n; k)$ are not in general connected. It is in fact easy to see that $\text{Sp}_\ell(n; 0)$ has two connected components (this is more generally true of all $\text{Sp}_\ell(n; k)$).

Construction of a symplectic intersection index

Let us now define symplectic intersection indices. We denote by $C(\text{Sp}(n))$ the set of continuous paths $[0, 1] \rightarrow \text{Sp}(n)$:

$$C(\text{Sp}(n)) = C^0([0, 1], \text{Sp}(n)).$$

**Definition 99** A symplectic intersection index is a mapping

$$\mu_{\text{Sp}} : C(\text{Sp}(n)) \times \text{Lag}(n) \rightarrow \mathbb{Z}$$

$$(\Sigma, \ell) \mapsto \mu_{\text{Sp}}(\Sigma, \ell)$$

satisfying the following four axioms:

1. **Homotopy invariance:** if the symplectic paths $\Sigma$ and $\Sigma'$ are homotopic with fixed endpoints, then $\mu_{\text{Sp}}(\Sigma, \ell) = \mu_{\text{Sp}}(\Sigma', \ell)$ for all $\ell \in \text{Lag}(n)$.

2. **Additivity under concatenation:** if $\Sigma$ and $\Sigma'$ are two consecutive symplectic paths, then

$$\mu_{\text{Sp}}(\Sigma \ast \Sigma', \ell) = \mu_{\text{Sp}}(\Sigma, \ell) + \mu_{\text{Sp}}(\Sigma', \ell)$$

for all $\ell \in \text{Lag}(n)$.
(S₃) Zero in strata: if $\Sigma$ and $\ell$ are such that $\text{Im}(\Sigma \ell) \subset \text{Lag}_\ell(n)$, then $\mu_{\text{Sp}}(\Sigma, \ell) = 0$.

(S₄) Restriction to loops: if $\psi$ is a loop in $\text{Sp}(n)$, then

$$\mu_{\text{Sp}}(\psi, \ell) = 2m(\psi \ell)$$

for all $\ell \in \text{Lag}(n)$. $m(\psi \ell)$ is the Maslov index of the loop $t \mapsto \psi(t)\ell$ in $\text{Lag}(n)$.

The following result is just a symplectic version of Proposition 96:

**Proposition 100** If the symplectic paths $\Sigma$ and $\Sigma'$ are such that $\Sigma \ell$ and $\Sigma' \ell$ are homotopic in strata relative to $\ell$, then $\mu_{\text{Sp}}(\Sigma, \ell) = \mu_{\text{Sp}}(\Sigma', \ell)$.

The data of an intersection index on $\text{Lag}(n)$ is equivalent to that of an intersection index on $\text{Sp}(n)$. Indeed, let $\mu_{\text{Lag}}$ be an intersection index on $\text{Lag}(n)$ and let $\Sigma \in C(\text{Sp}(n))$ be a symplectic path. Then then function

$$C(\text{Sp}(n)) \times \text{Lag}(n) \ni (\Sigma, \ell) \mapsto \mu(\Sigma \ell, \ell) \in \mathbb{Z} \quad (3.5)$$

($\Sigma \ell$ being the path $t \mapsto \Sigma(t)\ell$) is an intersection index on $\text{Sp}(n)$.

Conversely, to each intersection index $\mu_{\text{Sp}}$ we may associate an intersection index $\mu_{\text{Lag}}$ on $\text{Lag}(n)$ in the following way. For each $\ell \in \text{Lag}(n)$ we have a fibration

$$\text{Sp}(n) \longrightarrow \text{Sp}(n)/\text{St}(\ell) = \text{Lag}(n)$$

($\text{St}(\ell)$ the stabilizer of $\ell$ in $\text{Sp}(n)$), hence, every path $\Lambda \in C(\text{Lag}(n))$ can be lifted to a path $\Sigma_\Lambda \in C(\text{Sp}(n))$ such that $\Lambda = \Sigma_\Lambda \ell$. One verifies that the mapping

$$\mu_{\text{Lag}} : C(\text{Lag}(n)) \times \text{Lag}(n) \longrightarrow \mathbb{Z}$$

defined by

$$(\Lambda, \ell) \mapsto \mu_{\text{Sp}}(\Sigma_\Lambda, \ell)$$

is an intersection index on $\text{Lag}(n)$.

In Theorem 97 we expressed a Lagrangian intersection index as the difference between two values of the Leray index. A similar result holds for symplectic intersection indices:

**Proposition 101** Let $\Sigma_{12} \in C(\text{Sp}(n))$ be a symplectic path joining $S_1$ to $S_2$ in $\text{Sp}(n)$. Let $S_{1,\infty}$ be an arbitrary element of $\text{Sp}_\infty(n)$ covering $S_1$ and $S_{2,\infty}$ the homotopy class of $\Sigma_{01} * \Sigma_{12}$ ($\Sigma_{01}$ a representative of $S_{1,\infty}$). The function $\mu_{\text{Sp}} : C(\text{Sp}(n)) \longrightarrow \mathbb{Z}$ defined by

$$\mu_{\text{Sp}}(\Sigma_{12}, \ell) = \mu_\ell(S_{2,\infty}) - \mu_\ell(S_{1,\infty}) \quad (3.6)$$

is an intersection index on $\text{Sp}(n)$.
Proof. Consider \( \ell_\infty \) to be the homotopy class of an arbitrary path \( \Lambda \) joining \( \ell_0 \) (the base point of \( \text{Lag}(n)_\infty \)) to \( \ell \). We have
\[
S_{1,\infty}\ell_\infty = \text{class} \left[ t \mapsto \Sigma_{01}(t)\Lambda(t), \ 0 \leq t \leq 1 \right]
\]
(where “class” means “equivalence class of”) and
\[
S_{2,\infty}\ell_\infty = \text{class} \left[ t \mapsto \begin{cases} \Sigma_{01}(2t)\Lambda(2t), & 0 \leq t \leq \frac{1}{2} \\ \Sigma_{12}(2t-1)\Lambda(2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases} \right]
\]
hence, using (3.5) and (3.3),
\[
\mu_{\text{Sp}}(\Sigma_{12}, \ell) = \mu_{\text{Lag}}(\Sigma_{12}, \ell) = \mu_{\infty}(S_{2,\infty}\ell_\infty, \ell_\infty) - \mu_{\infty}(S_{1,\infty}\ell_\infty, \ell_\infty)
\]
that is (3.6). \( \blacksquare \)

Let us illustrate the notions studied above on a simple example.

Example: spectral flows
Here is a simple application of the constructions above. Let \((A(t))_{0 \leq t \leq 1}\) be a family of real symmetric matrices of order \( n \) depending continuously on \( t \in [0,1] \). By definition the “spectral flow” of \((A(t))_{0 \leq t \leq 1}\) is the integer
\[
\text{SF}(A(t))_{0 \leq t \leq 1} = \text{sign } A(1) - \text{sign } A(0) \quad (3.7)
\]
where \( \text{sign } A(t) \) is the difference between the number of eigenvalues \( > 0 \) and the number of eigenvalues \( < 0 \) of \( A(t) \).

We have the following result, which has been established in a particular case by Duistermaat [19], and which relates the spectral flow to the notions of Lagrangian and symplectic intersection indices:

Proposition 102 Let \( \Lambda \) be the Lagrangian path associated to the family \((A(t))_{0 \leq t \leq 1}\) by
\[
\Lambda(t) = \{(x, A(t)x) : x \in \mathbb{R}^n\}, \quad 0 \leq t \leq 1, \quad (3.8)
\]
Then the spectral flow of \((A(t))_{0 \leq t \leq 1}\) is given by
\[
\text{SF}(A(t))_{0 \leq t \leq 1} = \mu_{\text{Lag}}(\Lambda, \ell_X) \quad (3.9)
\]
or, equivalently
\[
\text{SF}(A(t))_{0 \leq t \leq 1} = \mu_{\text{Sp}}(\Sigma_A, \ell_X) \quad (3.10)
\]
where \( \Sigma_A \) the symplectic path defined by
\[
\Sigma(t) = V_{-A(t)} = \begin{bmatrix} I & 0 \\ A(t) & I \end{bmatrix}
\]
for \( 0 \leq t \leq 1 \).
Proof. Formula (3.10) follows immediately from formula (3.9) observing that

\[
\begin{bmatrix}
I & 0 \\
A(t) & I
\end{bmatrix}
\begin{bmatrix}
x \\
0
\end{bmatrix} =
\begin{bmatrix}
x \\
A(t)x
\end{bmatrix}.
\]

To prove formula (3.9) we begin by noting that \(\dim(\Lambda(t) \cap \ell_P) = n\) for \(0 \leq t \leq 1\), and hence

\[\bar{\mu}_{\text{Lag}}(\Lambda, \ell_P) = 0\] (3.11)

in view of the axiom (L_3) of nullity in the strata. By definition of \(\mu_{\text{Lag}}\) we have, with obvious notations,

\[\mu_{\text{Lag}}(\Lambda, \ell_X) = \mu(\Lambda(1)_{\infty}, \ell_{X, \infty}) - \mu(\Lambda(0)_{\infty}, \ell_{X, \infty}).\]

In view of the property \(\partial \mu = \tau\) of the Leray index

\[
\mu(\Lambda(t)_{\infty}, \ell_{X, \infty}) - \mu(\Lambda(t)_{\infty}, \ell_{P, \infty}) = -\mu(\ell_{X, \infty}, \ell_{P, \infty}) + \tau(\Lambda(t), \ell_X, \ell_P)
\]

for \(0 \leq t \leq 1\), and hence, taking (3.11) into account,

\[
\mu_{\text{Lag}}(\Lambda, \ell_X) = \tau(\Lambda(1), \ell_X, \ell_P) - \tau(\Lambda(0), \ell_X, \ell_P) = \tau(\ell_P, \Lambda(1), \ell_X) - (\ell_P, \Lambda(1), \ell_X).
\]

In view of formula (2.5) in Corollary 52 (Subsection 2.1) we have

\[\tau(\ell_P, \Lambda(t), \ell_X) = \text{sign}(A(t))\]

and formula (3.9) follows. \(\blacksquare\)

The result above is rather trivial in the sense that the spectral flow (3.7) depends only on the extreme values \(A(1)\) and \(A(0)\). The situation is however far more complicated in the case of infinite dimensional symplectic spaces and its analysis requires elaborated functional analytical techniques. Let us very shortly and concisely explain the problem. Let \(H\) be a Hilbert space and \(T\) a Fredholm operator on \(H\): this means that both \(\ker(T)\) and \(\coker(T) = H/\text{Im}(T)\) are finite dimensional spaces. The difference

\[\text{Index}(T) = \dim \ker(T) - \dim \coker(T)\]

is then, by definition, the \textit{index of the operator} \(T\). Suppose now that we are dealing with a whole family \(T(t)\) of Fredholm operators, depending on a real parameter \(t\). One defines the \textit{spectral flow} of that family as being the net change in negative eigenvalues of the \(T(t)\) as \(t\) varies from \(-\infty\) to \(+\infty\). It turns out that it is possible to establish formulas of the type

\[\text{spectral flow} = \text{Maslov index}\]

when the Hilbert space \(H\) is finite-dimensional (see Cappell et al. [14], Robbin and Salamon [62], and the references in these papers). A crucial generalization of theorems of that type have very recently been obtained by Booss-Bavnbek and Furutani [9]. They define an intersection in infinite-dimensional spaces and it seems likely that most of the constructions described above can be generalized to the infinite dimensional case.

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3.3 The Conley–Zehnder Index

The Conley–Zehnder index plays a very important role in the theory of periodic Hamiltonian orbits and related topics (such as Morse theory and Floer homology). In this last section we extend this index in a non-trivial way using again the properties of the Leray index. We foresee that this extension could be used in the study of spectral flows, but we do not give any such application here.

Definition of the Conley–Zehnder index

Let Σ be a continuous path \([0, 1] \rightarrow \text{Sp}(n)\) such that Σ(0) = I and \(\det(\Sigma(1) - I) \neq 0\). Loosely speaking, the Conley–Zehnder index [16, 37] counts algebraically the number of times this path crosses the locus

\[\text{Sp}_0(n) = \{S : \det(S - I) = 0\}.\]

To give a more precise definition we need some additional notations. Let us define

\[\text{Sp}^+(n) = \{S : \det(S - I) > 0\}\]
\[\text{Sp}^-(n) = \{S : \det(S - I) < 0\}.\]

These sets partition \(\text{Sp}(n)\), and \(\text{Sp}^+(n)\) and \(\text{Sp}^-(n)\) are moreover arcwise connected (this is proven in [16]). The symplectic matrices \(S^+ = -I\) and \(S^- = \begin{bmatrix} L & 0 \\ 0 & L^{-1} \end{bmatrix}, \quad L = \text{diag}[2, -1, ..., -1]\)

belong to \(\text{Sp}^+(n)\) and \(\text{Sp}^-(n)\), respectively.

Let us denote by \(\rho\) the mapping \(\text{Sp}(n) \rightarrow S^1\) defined as follows:

\[S \in \text{Sp}(n) \mapsto U = S(S^T S)^{-1/2} \in U(n) \mapsto \det_C U \in S^1\]

where

\[\det_C U = \det(A + iB) \quad \text{if} \quad U = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}.\]

We obviously have \(\rho(S^+) = (-1)^n\) and \(\rho(S^-) = (-1)^{n-1}\).

We now have all we need to define the Conley–Zehnder index. Let us denote by \(\Sigma^\pm(2n, \mathbb{R})\) the space of all paths \(\Sigma : [0, 1] \rightarrow \text{Sp}(n)\) with \(\Sigma(0) = I\) and \(\Sigma(1) \in \text{Sp}^\pm(n)\). Any such path can be extended into a path \(\tilde{\Sigma} : [0, 2] \rightarrow \text{Sp}(n)\) such that \(\tilde{\Sigma}(t) \in \text{Sp}^\pm(n)\) for \(1 \leq t \leq 2\) and \(\tilde{\Sigma}(2) = S^+\) or \(\tilde{\Sigma}(2) = S^-\). The orthogonal part of the polar decomposition of \(\tilde{\Sigma}(t)\) is given by the formula

\[U(t) = \tilde{\Sigma}(t) (\tilde{\Sigma}(t)^T \tilde{\Sigma}(t))^{-1/2}\]

(cf. the definition of the Maslov index on \(\text{Sp}(n)\), formula (2.26)). When \(t\) varies from 0 to 2 the complex number \(\det_C U(t) = e^{i\theta(t)}\) varies from \(e^{i\theta(0)} = 1\) to \(e^{i\theta(2)} = \pm 1\) so that \(\theta(2) \in \pi \mathbb{Z}\).
Definition 103 The mapping $i_{CZ}: C^\pm(2n, \mathbb{R}) \to \mathbb{Z}$ defined by

$$i_{CZ}(\Sigma) = \frac{\theta(2)}{\pi}$$

is called the Conley-Zehnder index on $C^\pm(2n, \mathbb{R})$.

It turns out that $i_{CZ}(\Sigma)$ is invariant under homotopy as long as the end-point $S = \Sigma(1)$ remains in $\text{Sp}^\pm(n)$. In particular it does not change under homotopies with fixed endpoints so we may view $i_{CZ}$ as defined on the subset

$$\text{Sp}^\ast_\infty(n) = \{S_\infty : \text{det}(S - I) \neq 0\}$$

of the universal covering group $\text{Sp}_\infty(n)$. With this convention one proves (see [37]) that the Conley-Zehnder index is the unique mapping

$$i_{CZ}: \text{Sp}^\ast_\infty(n) \to \mathbb{Z}$$

having the following properties:

(CZ$_1$) Antisymmetry: For every $S_\infty$ we have

$$i_{CZ}(S_\infty^{-1}) = -i_{CZ}(S_\infty)$$

where $S_\infty^{-1}$ is the homotopy class of the path $t \mapsto S_t^{-1}$;

(CZ$_2$) Continuity: Let $\Sigma$ be a symplectic path representing $S_\infty$ and $\Sigma'$ a path joining $S$ to an element $S'$ belonging to the same component $\text{Sp}^\pm(n)$ as $S$. Let $S'_\infty$ be the homotopy class of $\Sigma * \Sigma'$. We have

$$i_{CZ}(S_\infty) = i_{CZ}(S'_\infty);$$

(CZ$_3$) Action of $\pi_1[\text{Sp}(n)]$:

$$i_{CZ}(\alpha^r S_\infty) = i_{CZ}(S_\infty) + 2r$$

for every $r \in \mathbb{Z}$.

The uniqueness of a mapping $\text{Sp}^\ast_\infty(n) \to \mathbb{Z}$ satisfying these properties is actually rather obvious: suppose $i'_{CZ}: \text{Sp}^\ast_\infty(n) \to \mathbb{Z}$ has the same properties and set $\delta = i_{CZ} - i'_{CZ}$. In view of (CZ$_3$) we have $\delta(\alpha^r S_\infty) = \delta(S_\infty)$ for all $r \in \mathbb{Z}$ hence $\delta$ is defined on $\text{Sp}^\ast(n) = \text{Sp}^+(n) \cup \text{Sp}^-(n)$ so that $\delta(S_\infty) = \delta(S)$ where $S = S_1$, the endpoint of the path $t \mapsto S_t$. Property (CZ$_2$) implies that this function $\text{Sp}^\ast(n) \to \mathbb{Z}$ is constant on both $\text{Sp}^+(n)$ and $\text{Sp}^-(n)$. We next observe that since $\det S = 1$ we have $\det(S_{-1} - I) = \det(S - I)$ so that $S$ and $S^{-1}$ always belong to the same set $\text{Sp}^+(n)$ or $\text{Sp}^-(n)$ if $\det(S - I) \neq 0$. Property (CZ$_1$) then implies that $\delta$ must be zero on both $\text{Sp}^+(n)$ or $\text{Sp}^-(n)$.

One proves that the Conley-Zehnder in addition satisfies:
(CZ$_4$) **Normalization:** Let $J_1$ be the standard symplectic matrix in $Sp(1)$. If $S_1$ is the path $t \rightarrow e^{\pi t J_1}$ ($0 \leq t \leq 1$) joining $I$ to $-I$ in $Sp(1)$ then $i_{CZ,1}(S_1,\infty) = 1$ ($i_{CZ,1}$ the Conley–Zehnder index on $Sp(1)$);

(CZ$_5$) **Dimensional additivity:** if $S_1,\infty \in Sp^{\ast}_\infty(n_1), S_2,\infty \in Sp^{\ast}_\infty(n_2), n_1 + n_2 = n$, then

$$i_{CZ}(S_1,\infty \oplus S_2,\infty) = i_{CZ,1}(S_1,\infty) + i_{CZ,2}(S_2,\infty)$$

where $i_{CZ,j}$ is the Conley–Zehnder index on $Sp(n_j), j = 1, 2$.

These properties will actually easily follow from the properties of the extended index we will construct. Let us first introduce a useful notion of Cayley transform for symplectic matrices.

**The symplectic Cayley transform**

Our extension of the index $i_{CZ}$ requires a notion of Cayley transform for symplectic matrices. If $S \in Sp(n), \det(S - I) \neq 0$, we call the matrix

$$M_S = \frac{1}{2}J(S + I)(S - I)^{-1}$$

(3.12)

the “symplectic Cayley transform of $S$”. Equivalently:

$$M_S = \frac{1}{2}J + J(S - I)^{-1}.$$  

(3.13)

It is straightforward to check that $M_S$ always is a symmetric matrix: $M_S = M_S^T$. In fact:

$$M_S^T = -\frac{1}{2}J - (S^T - I)^{-1}J = -\frac{1}{2}J + (JS^T - J)^{-1}$$

that is, since $JS^T = S^{-1}J$ and $(S^{-1} - I)^{-1} = (I - S)^{-1}S$:

$$M_S^T = -\frac{1}{2}J + (S^{-1}J - J)^{-1} = -\frac{1}{2}J - J(I - S)^{-1}S.$$  

Noting the trivial identity $(I - S)^{-1}S = -I + (I - S)^{-1}$ we finally obtain

$$M_S^T = \frac{1}{2}J - J(I - S)^{-1} = M_S.$$  

The symplectic Cayley transform has the following properties:

**Lemma 104** (i) We have

$$(M_S + M_{S'})^{-1} = -(S' - I)(SS' - I)^{-1}(S - I)J$$

(3.14)

and the symplectic Cayley transform of the product $SS'$ is (when defined) given by the formula

$$M_{SS'} = M_S + (S^T - I)^{-1}J(M_S + M_{S'})^{-1}J(S - I)^{-1}.$$  

(3.15)

(ii) The symplectic Cayley transform of $S$ and $S^{-1}$ are related by

$$M_{S^{-1}} = -M_S.$$  

(3.16)
Proof. (i) We begin by noting that (3.13) implies that
\[ M_S + M_{S'} = J(I + (S - I)^{-1} + (S' - I)^{-1}) \]  
(3.17)
hence the identity (3.14). In fact, writing \( SS' - I = S(S' - I) + S - I \), we have
\[ (S' - I)(SS' - I)^{-1}(S - I) = (S' - I)(S(S' - I) + S - I)^{-1}(S - I) \]
\[ = ((S - I)^{-1}S(S' - I)(S' - I)^{-1} + (S' - I)^{-1})^{-1} \]
\[ = ((S - I)^{-1}S + (S' - I)^{-1}) \]
\[ = I + (S - I)^{-1} + (S' - I)^{-1}. \]
The equality (3.14) follows in view of (3.17). Let us prove (3.15). Equivalently
\[ M_S + M = M_{SS'} \]
(3.18)
where \( M \) is the matrix defined by
\[ M = (S^T - I)^{-1}J(M_S + M_{S'})^{-1}J(S - I)^{-1} \]
that is, in view of (3.14),
\[ M = (S^T - I)^{-1}J(S' - I)(SS' - I)^{-1}. \]
Using the obvious relations \( S^T = JS^{-1}J \) and \( (-S^{-1} + I)^{-1} = S(S - I)^{-1} \) we have
\[ M = (S^T - I)^{-1}J(S' - I)(SS' - I)^{-1} \]
\[ = -J(-S^{-1} + I)^{-1}(S' - I)(SS' - I)^{-1} \]
\[ = -JS(S - I)^{-1}(S' - I)(SS' - I)^{-1} \]
that is, writing \( S = S - I + I \),
\[ M = -J(S' - I)(SS' - I)^{-1} - J(S - I)^{-1}(S' - I)(SS' - I)^{-1}. \]
Replacing \( M_S \) by its value (3.13) we have
\[ M_S + M = \]
\[ J(\frac{1}{2}I + (S - I)^{-1} - (S' - I)(SS' - I)^{-1} - (S - I)^{-1}(S' - I)(SS' - I)^{-1}). \]
Noting that
\[ (S - I)^{-1} - (S' - I)(SS' - I)^{-1} = \]
\[ (S - I)^{-1}(SS' - I - S' + I)(SS' - I)^{-1} \]
that is
\[
(S - I)^{-1} - (S - I)^{-1}(S' - I)(SS' - I)^{-1} = (S - I)^{-1}(SS' - S')(SS' - I)^{-1} = S'(SS' - I)^{-1}
\]
we get
\[
M_S + M = J\left(\frac{1}{2}I - (S' - I)(SS' - I)^{-1} + S'(SS' - I)^{-1}\right)
= J\left(\frac{1}{2}I + (SS' - I)^{-1}\right)
= M_{SS'}
\]
which we set out to prove. (ii) Formula (3.16) follows from the sequence of equalities
\[
M_{S^{-1}} = \frac{1}{2}J + J(S^{-1} - I)^{-1}
= \frac{1}{2}J - JS(S - I)^{-1}
= \frac{1}{2}J - J(S - I + I)(S - I)^{-1}
= -\frac{1}{2}J - J(S - I)^{-1}
= -M_S.
\]

Definition and properties of \(\nu(S_{\infty})\)
We define on \(\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}\) a symplectic form \(\sigma^\oplus\) by
\[
\sigma^\oplus(z_1, z_2; z'_1, z'_2) = \sigma(z_1, z'_1) - \sigma(z_2, z'_2)
\]
and denote by \(\text{Sp}^\oplus(2n)\) and \(\text{Lag}^\oplus(2n)\) the corresponding symplectic group and Lagrangian Grassmannian. Let \(\mu^\oplus\) be the Leray index on \(\text{Lag}^\oplus_{\infty}(2n)\) and \(\mu^\oplus_L\) the Maslov index on \(\text{Sp}^\oplus_{\infty}(2n)\) relative to \(L \in \text{Lag}^\oplus(2n)\).

For \(S_{\infty} \in \text{Sp}_{\infty}(n)\) we define
\[
\nu(S_{\infty}) = \frac{1}{2} \mu^\oplus((I \oplus S)_{\infty} \Delta_{\infty}, \Delta_{\infty})
\]
where \((I \oplus S)_{\infty}\) is the homotopy class in \(\text{Sp}^\oplus(2n)\) of the path
\[
t \mapsto \{(z, Stz) : z \in \mathbb{R}^{2n}\}, \quad 0 \leq t \leq 1
\]
and \(\Delta = \{(z, z) : z \in \mathbb{R}^{2n}\}\) the diagonal of \(\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}\). Setting \(S_t^\oplus = I \oplus S_t\) we have \(S_t^\oplus \in \text{Sp}^\oplus(2n)\) hence formulae (3.19) is equivalent to
\[
\nu(S_{\infty}) = \frac{1}{2} \mu^\oplus_{\Delta}(S^\oplus_{\infty})
\]
where \(\mu^\oplus_{\Delta}\) is the relative Maslov index on \(\text{Sp}^\oplus_{\infty}(2n)\) corresponding to the choice \(\Delta \in \text{Lag}^\oplus(2n)\).
Note that replacing \( n \) by \( 2n \) in the congruence (2.43) (Proposition 74) we have

\[
\mu(\Theta((I \oplus S)_{\infty} \Delta_{\infty}, \Delta_{\infty}) \equiv \dim((I \oplus S)\Delta \cap \Delta) \mod 2
\equiv \dim \ker(S - I) \mod 2
\]

and hence

\[
\nu(S_{\infty}) \equiv \frac{1}{2} \dim \ker(S - I) \mod 1.
\]

Since the eigenvalue 1 of \( S \) has even multiplicity \( \nu(S_{\infty}) \) is thus always an integer.

The index \( \nu \) has the following three important properties. The third is essential for the calculation of the index of repeated periodic orbits (it clearly shows that \( \nu \) is not in general additive):

**Proposition 105**

(i) For all \( S_{\infty} \in \text{Sp}_{\infty}(n) \) we have

\[
\nu(S_{\infty}^{-1}) = -\nu(S_{\infty}) , \quad \nu(I_{\infty}) = 0
\]

(\( I_{\infty} \) the identity of the group \( \text{Sp}_{\infty}(n) \)).

(ii) For all \( r \in \mathbb{Z} \) we have

\[
\nu(\alpha r S_{\infty}) = \nu(S_{\infty}) + 2r , \quad \nu(\alpha^r) = 2r
\]

(iii) Let \( S_{\infty} \) be the homotopy class of a path \( \Sigma \) in \( \text{Sp}(n) \) joining the identity to \( S \in \text{Sp}^+(n) \), and let \( S' \in \text{Sp}(n) \) be in the same connected component \( \text{Sp}^+(n) \) as \( S \). Then \( \nu(S'_{\infty}) = \nu(S_{\infty}) \) where \( S'_{\infty} \) is the homotopy class in \( \text{Sp}(n) \) of the concatenation of \( \Sigma \) and a path joining \( S \) to \( S' \) in \( \text{Sp}_0(n) \).

**Proof.**

(i) Formulae (3.21) immediately follows from the equality \( (S_{\infty}^{\Theta})^{-1} = (I \oplus S^{-1})_{\infty} \) and the antisymmetry of \( \mu_{\Delta}^{\Theta} \). (ii) The second formula (3.22) follows from the first using (3.21). To prove the first formula (3.22) it suffices to observe that to the generator \( \alpha \) of \( \pi_1[\text{Sp}(n)] \) corresponds the generator \( I_{\infty} \oplus \alpha \) of \( \pi_1[\text{Sp}(2n)] \). In view of property (2.56) of the Maslov indices it follows that

\[
\nu(\alpha^r S_{\infty}) = \frac{1}{2} \mu_{\Delta}^{\Theta}((I_{\infty} \oplus \alpha)^r S_{\infty}^{\Theta})
= \frac{1}{2}(\mu_{\Delta}^{\Theta}(S_{\infty}^{\Theta}) + 4r)
= \nu(S_{\infty}) + 2r.
\]

(iii) Assume in fact that \( S \) and \( S' \) belong to, say, \( \text{Sp}^+(n) \). Let \( S_{\infty} \) be the homotopy class of the path \( \Sigma \), and \( \Sigma' \) a path joining \( S \) to \( S' \) in \( \text{Sp}^+(n) \) (we parametrize both paths by \( t \in [0,1] \)). Let \( \Sigma'_t \) be the restriction of \( \Sigma' \) to the interval \( [0,t'], t' \leq t \) and \( S_{\infty}(t') \) the homotopy class of the concatenation \( \Sigma \ast \Sigma'_t \). We have \( \det(S(t) - I) > 0 \) for all \( t \in [0,t'] \) hence \( S_{\infty}^{\Theta}(t)\Delta \cap \Delta \neq 0 \) as \( t \) varies from 0 to 1. It follows from the fact that the \( \mu_{\Delta}^{\Theta} \) is locally constant.
on the set \( \{ S_\infty^\ominus : S_\infty^\ominus \Delta \cap \Delta = 0 \} \) (property \((ii)\) in Proposition 82) that the function \( t \mapsto \mu^\ominus_\Delta(S_\infty^\ominus(t)) \) is constant, and hence
\[
\mu^\ominus_\Delta(S_\infty^\ominus) = \mu^\ominus_\Delta(S_\infty^\ominus(0)) = \mu^\ominus_\Delta(S_\infty^\ominus(1)) = \mu^\ominus_\Delta(S_\infty')
\]
which was to be proven. \( \blacksquare \)

The following consequence of the result above shows that the indices \( \nu \) and \( i_{CZ} \) coincide on their common domain of definition:

**Corollary 106** The restriction of the index \( \nu \) to \( Sp^*(n) \) is the Conley–Zehnder index:

\[
\nu(S_\infty) = i_{CZ}(S_\infty) \quad \text{if} \quad \det(S - I) \neq 0.
\]

**Proof.** The restriction of \( \nu \) to \( Sp^*(n) \) satisfies the properties \((CZ_1)\), \((CZ_2)\), and \((CZ_3)\) of the Conley–Zehnder index listed in \( \S 3.3 \). We showed that these properties uniquely characterize \( i_{CZ} \). \( \blacksquare \)

Let us prove a formula for the index of the product of two paths:

**Proposition 107** If \( S_\infty, S'_\infty, \) and \( S_\infty S'_\infty \) are such that \( \det(S - I) \neq 0, \) \( \det(S' - I) \neq 0, \) and \( \det(SS' - I) \neq 0 \) then
\[
\nu(S_\infty S'_\infty) = \nu(S_\infty) + \nu(S'_\infty) + \frac{1}{2} \text{sign}(M_S + M_{S'})
\]
where \( M_S \) is the symplectic Cayley transform of \( S \). In particular
\[
\nu(S'^r_\infty) = r \nu(S_\infty) + \frac{1}{2} (r - 1) \text{sign} M_S
\]
for every integer \( r \).

**Proof.** In view of (3.20) and the product property (2.57) of the Maslov index (Proposition 82) we have
\[
\nu(S_\infty S'_\infty) = \nu(S_\infty) + \nu(S'_\infty) + \frac{1}{2} \tau(\Delta, S^\ominus \Delta, S^\ominus S'^\ominus \Delta)
\]
\[
= \nu(S_\infty) + \nu(S'_\infty) - \frac{1}{2} \tau(S^\ominus S'^\ominus \Delta, S^\ominus \Delta, \Delta)
\]
where \( S^\ominus = I \oplus S, S'^\ominus = I \oplus S' \) and \( \tau^\ominus \) is the signature on the symplectic space \( (\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}, \sigma^\ominus) \). The condition \( \det(SS' - I) \neq 0 \) is equivalent to \( S^\ominus S'^\ominus \Delta \cap \Delta = 0 \) hence we can apply Proposition 50 with \( \ell = S^\ominus S'^\ominus \Delta, \ell' = S^\ominus \Delta, \) and \( \ell'' = \Delta \). The projection operator onto \( S^\ominus S'^\ominus \Delta \) along \( \Delta \) is easily seen to be
\[
\operatorname{Pr}_{S^\ominus S'^\ominus \Delta, \Delta} = \begin{bmatrix}
(I - SS')^{-1} & -(I - SS')^{-1} \\
SS'(I - SS')^{-1} & -SS'(I - SS')^{-1}
\end{bmatrix}
\]

hence \( \tau^\ominus(S^\ominus S'^\ominus \Delta, S^\ominus \Delta, \Delta) \) is the signature of the quadratic form
\[
Q(z) = \sigma^\ominus(\operatorname{Pr}_{S^\ominus S'^\ominus \Delta, \Delta}(z, Sz); (z, Sz))
\]

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that is, since $\sigma^\oplus = \sigma \oplus \sigma$:

$$Q(z) = \sigma((I - SS')^{-1}(I - S)z, z)) - \sigma(SS'(I - SS')^{-1}(I - S)z, Sz))$$

$$= \sigma((I - SS')^{-1}(I - S)z, z)) - \sigma(S'(I - SS')^{-1}(I - S)z, z))$$

$$= \sigma((I - S')(I - SS')^{-1}(I - S)z, z)).$$

In view of formula (3.14) in Lemma 104 we have

$$(I - S')(SS' - I)^{-1}(I - S) = (M_S + M_{S'})^{-1}J$$

hence

$$Q(z) = -\langle (M_S + M_{S'})^{-1}Jz, Jz \rangle$$

and the signature of $Q$ is thus the same as that of

$$Q'(z) = -\langle (M_S + M_{S'})^{-1}z, z \rangle$$

that is $-\text{sign}(M_S + M_{S'})$. This proves formula (3.23). Formula (3.24) follows from (3.23) by induction on $r$. \hfill \blacksquare

It is not immediately obvious that the index $\mu_\gamma$ of the periodic orbit $\gamma$ is independent on the choice of the origin of the orbit. Let us prove that this property always holds:

**Proposition 108** Let $(f_t)$ be the flow determined by a (time-independent) Hamiltonian function on $\mathbb{R}^{2n}$ and $z \neq 0$ such that $f_T(z) = z$ for some $T > 0$. Let $z' = f_{t'}(z)$ for some $t'$ and denote by $S_T(z)$, $Df_T(z)$ and $S_T(z')$, $Df_T(z')$ the corresponding monodromy matrices. Let $S_T(z)_\infty$ and $S_T(z')_\infty$ be the homotopy classes of the paths $t \mapsto S_t(z) = Df_t(z)$ and $t \mapsto S_t(z') = Df_t(z')$, $0 \leq t \leq T$. We have $\nu(S_T(z)_\infty) = \nu(S_T(z')_\infty)$.

**Proof.** We have proven in Lemma 47 that monodromy matrices $S_T(z)$ and $S_T(z')$ are conjugate of each other. Since we will need to let $t'$ vary we write $S_T(z') = S_T(z', t')$ so that

$$S_T(z', t') = S_{t'}(z)S_T(z)S_{t'}(z')^{-1}.$$  

The paths $t \mapsto S_t(z')$ and $t \mapsto S_{t'}(z)S_t(z)S_{t'}(z')^{-1}$ being homotopic with fixed endpoints $S_T(z', t')_\infty$ is also the homotopy class of the path $t \mapsto S_{t'}(z')S_t(z)S_{t'}(z')^{-1}$. We thus have, by definition (3.19) of $\nu$,

$$\nu(S_T(z', t')_\infty) = \frac{1}{2} \mu_{\Delta'}(S_T^\oplus(z)_\infty)$$

where we have set

$$\Delta' = (I \oplus S_{t'}(z')^{-1}) \Delta \quad \text{and} \quad S_T^\oplus(z)_\infty = I_\infty \oplus S_T(z)_\infty.$$  

Consider now the mapping $t' \mapsto \mu_{\Delta'}(S_T^\oplus(z)_\infty)$. We have

$$S_T^\oplus(z)\Delta_{t'} \cap \Delta_{t'} = \{z : Sz = z\}$$
hence the dimension of the intersection $S^\ominus_T(z) \Delta_{t'} \cap \Delta_{t'}$ remains constant as $t'$ varies. In view of the topological property of the relative Maslov index the mapping $t' \mapsto \mu^\ominus_{\Delta_{t'}}(S^\ominus_T(z)_\infty)$ is thus constant and hence

$$\nu(S_T(z', t')_\infty) = \nu(S_T(z', 0)_\infty) = \nu(S_T(z)_\infty)$$

which concludes the proof. ■

**Relation between $\nu$ and $\mu_{\ell_P}$**

The index $\nu$ can be expressed in simple – and useful – way in terms of the Maslov index $\mu_{\ell_P}$ on $\text{Sp}_\infty(n)$. The following technical result will be helpful in establishing this relation. Recall that $S \in \text{Sp}(n)$ is free if $S \ell_P \cap \ell_P = 0$ and that this condition is equivalent to $\det B \neq 0$ when $S$ is identified with the matrix

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

(3.25)

in the canonical basis. The set of all free symplectic matrices is dense in $\text{Sp}(n)$. The quadratic form $W$ on $\mathbb{R}^n_x \times \mathbb{R}^n_x$ defined by

$$W(x, x') = \frac{1}{2} \langle Px, x \rangle - \langle Lx, x' \rangle + \frac{1}{2} \langle Qx', x' \rangle$$

where

$$P = DB^{-1}, L = B^{-1}, Q = B^{-1}A$$

(3.26)

then generates $S$ in the sense that

$$(x, p) = S(x', p') \iff p = \partial_x W(x, x'), p' = -\partial_{x'} W(x, x').$$

We have:

**Lemma 109** Let $S = S_W \in \text{Sp}(n)$ be given by (3.25). We have

$$\det(S_W - I) = (-1)^n \det B \det(B^{-1}A + DB^{-1} - B^{-1} - (B^T)^{-1})$$

(3.27)

that is:

$$\det(S_W - I) = (-1)^n \det(L^{-1}) \det(P + Q - L - L^T).$$

In particular the symmetric matrix

$$P + Q - L - L^T = DB^{-1} + B^{-1}A - B^{-1} - (B^T)^{-1}$$

is invertible.

**Proof.** Since $B$ is invertible we can factorize $S - I$ as

$$\begin{bmatrix} A - I & B \\ C & D - I \end{bmatrix} = \begin{bmatrix} 0 & B \\ I & D - I \end{bmatrix} \begin{bmatrix} C - (D - I)B^{-1}(A - I) & 0 \\ B^{-1}(A - I) & I \end{bmatrix}$$

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and hence
\[
\det(S_W - I) = \det(-B) \det(C - (D - I)B^{-1}(A - I)) \\
= (-1)^n \det B \det(C - (D - I)B^{-1}(A - I)).
\]
Since \(S\) is symplectic we have \(C - DB^{-1}A = -(B^T)^{-1}\) and hence
\[
C - (D - I)B^{-1}(A - I) = B^{-1}A + DB^{-1} - B^{-1} - (B^T)^{-1}
\]
the Lemma follows. \(\blacksquare\)

Let us now introduce the notion of index of concavity of a Hamiltonian periodic orbit \(\gamma\), defined for \(0 \leq t \leq T\), with \(\gamma(0) = \gamma(T) = z_0\). As \(t\) goes from 0 to \(T\) the linearized part \(D\gamma(t) = S_t(z_0)\) goes from the identity to \(S_T(z_0)\) (the monodromy matrix) in \(\text{Sp}(n)\). We assume that \(S_T(z_0)\) is free and that \(\det(S_T(z_0) - I) \neq 0\). Writing
\[
S_t(z_0) = \begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix}
\]
we thus have \(\det B(t) \neq 0\) in a neighborhood \([T - \varepsilon, T + \varepsilon]\) of the time \(T\).

The generating function
\[
W(x, x', t) = \frac{1}{2} \langle P(t)x, x \rangle - \langle L(t)x, x' \rangle + \frac{1}{2} \langle Q(t)x', x' \rangle
\]
(with \(P(t), Q(t), L(t)\) defined by (3.26) thus exists for \(T - \varepsilon \leq t \leq T + \varepsilon\). By definition Morse’s index of concavity [60] of the periodic orbit \(\gamma\) is the index of inertia
\[
\text{Inert} W_{xx}'' = \text{Inert}(P + Q - L - L^T)
\]
of \(W_{xx}''\), the matrix of second derivatives of the function \(x \mapsto W(x, x; T)\) (we have set \(P = P(T), Q = Q(T), L = L(T)\)).

Let us now prove the following essential result. Recall that \(m_\ell\) denotes the reduced Maslov index associated to \(\mu_\ell\):

**Proposition 110** Let \(t \mapsto S_t\) be a symplectic path, \(0 \leq t \leq 1\). Let \(S_\infty \in \text{Sp}_\infty(n)\) be the homotopy class of that path and set \(S = S_1\). If \(\det(S - I) \neq 0\) and \(S\ell_{\ell_P} \cap \ell_{\ell_P} = 0\) then
\[
\nu(S_\infty) = \frac{1}{2} (\mu_{\ell_P}(S_\infty) + \text{sign} W_{xx}''') = m_{\ell_P}(S_\infty) - \text{Inert} W_{xx}''
\]
where \(\text{Inert} W_{xx}'''\) is the index of concavity corresponding to the endpoint \(S\) of the path \(t \mapsto S_t\).

**Proof.** We will divide the proof in three steps. Step 1. Let \(L \in \text{Lag}^{\ominus}(4n)\). Using successively formulae (3.20) and (2.59) we have
\[
\nu(S_\infty) = \frac{1}{2} (\mu_{\ell_L}(S_{\ominus}) + \tau(\ominus(S_{\ominus} \Delta, \Delta, L) - \tau(\ominus(S_{\ominus} \Delta, S_{\ominus} L, L))).
\]
Choosing in particular $L = L_0 = \ell_P \oplus \ell_P$ we get
\[
\mu_{L_0}(S_\infty^{\ominus}) = \mu(\ell_P \oplus \ell_P) = \mu(\ell_P, \ell_P) - \mu(\ell_P, S_\infty \ell_P) = \mu(\ell_P, S_\infty \ell_P)
\]
so that there remains to prove that
\[
\tau^{\ominus}(S^{\ominus} \Delta, \Delta, L_0) - \tau^{\ominus}(S^{\ominus} \Delta, S^{\ominus} L_0, L_0) = -2 \text{sign } W''_{xx}.
\]

**Step 2.** We are going to show that
\[
\tau^{\ominus}(S^{\ominus} \Delta, S^{\ominus} L_0, L_0) = 0.
\]

In view of the symplectic invariance and the antisymmetry of $\tau^{\ominus}$ this is equivalent to
\[
\tau^{\ominus}(L_0, \Delta, (S^{\ominus})^{-1} L_0) = 0. \tag{3.30}
\]

We have
\[
\Delta \cap L_0 = \{(0, p; 0, p) : p \in \mathbb{R}^n\}
\]
and $(S^{\ominus})^{-1} L_0 \cap L_0$ consists of all $(0, p', S^{-1}(0, p''))$ with $S^{-1}(0, p'') = (0, p')$. Since $S$ (and hence $S^{-1}$) is free we must have $p' = p'' = 0$ so that
\[
(S^{\ominus})^{-1} L_0 \cap L_0 = \{(0, p; 0, 0) : p \in \mathbb{R}^n\}.
\]

It follows that we have
\[
L_0 = \Delta \cap L_0 + (S^{\ominus})^{-1} L_0 \cap L_0
\]
hence (3.30) in view of Proposition 51. **Step 3.** Let us finally prove that
\[
\tau^{\ominus}(S^{\ominus} \Delta, \Delta, L_0) = -2 \text{sign } W''_{xx}.
\]

This will complete the proof of the proposition. The condition $\det(S-I) \neq 0$ is equivalent to $S^{\ominus} \Delta \cap \Delta = 0$ hence, using Proposition 50, the number
\[
\tau^{\ominus}(S^{\ominus} \Delta, L_0) = -\tau^{\ominus}(S^{\ominus} \Delta, L_0, \Delta)
\]
is the signature of the quadratic form $Q$ on $L_0$ defined by
\[
Q(0, p, 0, p') = -\sigma^{\ominus}(\text{Pr}_{S^{\ominus} \Delta, \Delta}(0, p, 0, p'); 0, p, 0, p')
\]
where
\[
\text{Pr}_{S^{\ominus} \Delta, \Delta} = \begin{bmatrix}
(S-I)^{-1} & -(S-I)^{-1} \\
S(S-I)^{-1} & -S(S-I)^{-1}
\end{bmatrix}
\]
is the projection on $S^{\ominus} \Delta$ along $\Delta$ in $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$. It follows that the quadratic form $Q$ is given by
\[
Q(0, p, 0, p') = -\sigma^{\ominus}((I - S)^{-1}(0, p''), S(I - S)^{-1}(0, p''); 0, p, 0, p')
\]
where we have set $p'' = p - p'$. By definition of $\sigma^\oplus$ this is
\[
Q(0, p, 0, p') = -\sigma((I - S)^{-1}(0, p''), (0, p)) + \sigma(S(I - S)^{-1}(0, p''), (0, p')).
\]
Let now $M_S$ be the symplectic Cayley transform (3.12) of $S$. We have
\[
(I - S)^{-1} = J M_S + \frac{1}{2} I , \quad S(I - S)^{-1} = J M_S - \frac{1}{2} I
\]
and hence
\[
Q(0, p, 0, p') = -\sigma((J M_S + \frac{1}{2} I)(0, p''), (0, p)) + \sigma((J M_S - \frac{1}{2} I)(0, p''), (0, p'))
\]
\[
= -\sigma(J M_S(0, p''), (0, p)) + \sigma(J M_S(0, p''), (0, p'))
\]
\[
= \sigma(J M_S(0, p''), (0, p''))
\]
\[
= -\langle M_S(0, p''), (0, p'') \rangle.
\]
Let us calculate explicitly $M_S$. Writing $S$ in usual block-form we have
\[
S - I = \begin{bmatrix} 0 & B \\ I & D - I \end{bmatrix} \begin{bmatrix} C - (D - I) B^{-1}(A - I) & 0 \\ B^{-1}(A - I) & I \end{bmatrix}
\]
that is
\[
S - I = \begin{bmatrix} 0 & B \\ I & D - I \end{bmatrix} \begin{bmatrix} W''_{xx} & 0 \\ B^{-1}(A - I) & I \end{bmatrix}
\]
where we have used the identity
\[
C - (D - I) B^{-1}(A - I)) = B^{-1} A + DB^{-1} - B^{-1} - (B^T)^{-1}
\]
which follows from the relation $C - DB^{-1} A = -(B^T)^{-1}$ (the latter is a rephrasing of the equalities $DTA - B^T C = I$ and $DTB = B^T D$, which follow from the fact that $STJS = STJS$ since $S \in \text{Sp}(n)$). It follows that
\[
(S - I)^{-1} = \begin{bmatrix} (W''_{xx})^{-1} & 0 \\ B^{-1}(I - A)(W''_{xx})^{-1} & I \end{bmatrix} \begin{bmatrix} (I - D) B^{-1} & I \\ B^{-1} & 0 \end{bmatrix}
\]
\[
= \begin{bmatrix} (W''_{xx})^{-1}(I - D) B^{-1} & (W''_{xx})^{-1} \\ B^{-1}(I - A)(W''_{xx})^{-1}(I - D) B^{-1} + B^{-1} & B^{-1}(I - A)(W''_{xx})^{-1} \end{bmatrix}
\]
and hence
\[
M_S = \begin{bmatrix} B^{-1}(I - A)(W''_{xx})^{-1}(I - D) B^{-1} + B^{-1} & \frac{1}{2} I + B^{-1}(I - A)(W''_{xx})^{-1} \\ -\frac{1}{2} I - (W''_{xx})^{-1}(I - D) B^{-1} & -(W''_{xx})^{-1} \end{bmatrix}
\]
from which follows that
\[
Q(0, p, 0, p') = \langle (W''_{xx})^{-1} p'', p'' \rangle
\]
\[
= \langle (W''_{xx})^{-1}(p - p'), (p - p') \rangle.
\]
The matrix of the quadratic form $Q$ is thus
\[
2 \begin{bmatrix}
(W''_{xx})^{-1} & -(W''_{xx})^{-1} \\
-(W''_{xx})^{-1} & (W''_{xx})^{-1}
\end{bmatrix}
\]
and this matrix has signature $2 \text{sign}(W''_{xx})^{-1} = 2 \text{sign}(W''_{xx})$, proving the first equality (3.28). The second equality follows because $\mu_{\ell_P}(S_\infty) = 2m_{\ell_P}(S_\infty) - n$ since $S_{\ell_P} \cap \ell_P = 0$ and the fact that $W''_{xx}$ has rank $n$ in view of Lemma 109.

**Remark 111** Lemma 109 above shows that if $S$ is free then we have
\[
\frac{1}{\pi} \arg \det(S - I) \equiv n + \arg \det B + \arg \det W''_{xx} \mod 2 \\
\equiv n - \arg \det B + \arg \det W''_{xx} \mod 2.
\]
The reduced Maslov index $m_{\ell_P}(S_\infty)$ corresponds to a choice of $\arg \det B \mod 4$. Proposition 110 thus justifies the following definition of the argument of $\det(S - I) \mod 4$:
\[
\frac{1}{\pi} \arg \det(S - I) \equiv n - \nu(S_\infty) \mod 4.
\]

Let us finish with an example. Consider first the one-dimensional harmonic oscillator with Hamiltonian function
\[
H = \frac{\omega}{2} (p^2 + x^2).
\]
All the orbits are periodic with period $2\pi/\omega$. The monodromy matrix is simply the identity: $\Sigma_T = I$ where
\[
\Sigma_t = \begin{bmatrix}
\cos \omega t & \sin \omega t \\
-\sin \omega t & \cos \omega t
\end{bmatrix}.
\]
Let us calculate the corresponding index $\nu(\Sigma_\infty)$. The homotopy class of path $t \mapsto \Sigma_t$ as $t$ goes from 0 to $T = 2\pi/\omega$ is just the inverse of $\alpha$, the generator of $\pi_1[\text{Sp}(1)]$ hence $\nu(\Sigma_\infty) = -2$ in view of (3.22). If we had considered $r$ repetitions of the orbit we would likewise have obtained $\nu(\Sigma_\infty) = -2r$.

Consider next a two-dimensional harmonic oscillator with Hamiltonian function
\[
H = \frac{\omega_x}{2} (p_x^2 + x^2) + \frac{\omega_y}{2} (p_y^2 + y^2).
\]
We assume that the frequencies $\omega_y, \omega_x$ are incommensurate, so that the only periodic orbits are librations along the $x$ and $y$ axes. Let us focus
on the orbit $\gamma_x$ along the $x$ axis. Its prime period is $T = 2\pi/\omega_x$ and the corresponding monodromy matrix is

$$S_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \chi & 0 & \sin \chi \\ 0 & 0 & 1 & 0 \\ 0 & -\sin \chi & 0 & \cos \chi \end{bmatrix}, \quad \chi = 2\pi \frac{\omega_y}{\omega_x};$$

it is the endpoint of the symplectic path $t \mapsto S_t$, $0 \leq t \leq 1$, consisting of the matrices

$$S_t = \begin{bmatrix} \cos 2\pi t & 0 & \sin 2\pi t & 0 \\ 0 & \cos \chi_t & 0 & \sin \chi_t \\ -\sin 2\pi t & 0 & \cos 2\pi t & 0 \\ 0 & -\sin \chi_t & 0 & \cos \chi_t \end{bmatrix}.$$ 

In Gutzwiller’s trace formula [35] the sum is taken over periodic orbits, including their repetitions. We are thus led to calculate the Conley–Zehnder index of the path $t \mapsto S_t$ with $0 \leq t \leq r$ where the integer $r$ indicates the number of repetitions of the orbit. Let us calculate the Conley–Zehnder index $\nu(S_{r,\infty})$ of this path. We have

$$S_t = \Sigma_t \oplus \tilde{S}_t$$

where

$$\Sigma_t = \begin{bmatrix} \cos 2\pi t & \sin 2\pi t \\ -\sin 2\pi t & \cos 2\pi t \end{bmatrix}, \quad \tilde{S}_t = \begin{bmatrix} \cos \chi_t & \sin \chi_t \\ -\sin \chi_t & \cos \chi_t \end{bmatrix}.$$ 

In view of the additivity property of the relative Maslov index we thus have

$$\nu(S_{r,\infty}) = \nu(\Sigma_{r,\infty}) + \nu(\tilde{S}_{r,\infty})$$

where the first term is just

$$\nu(\Sigma_{r,\infty}) = -2r$$

in view of the calculation we made in the one-dimensional case with a different parametrization. Let us next calculate $\nu(\tilde{S}_{r,\infty})$. We will use formula (3.28) relating the index $\nu$ to the Maslov index via the index of concavity, so we begin by calculating the relative Maslov index

$$m_{\ell_P}(\tilde{S}_{r,\infty}) = m(\tilde{S}_{r,\infty}, \ell_{P,\infty}).$$

Here is a direct argument. In more complicated cases the formulas we proved in [31] are useful. When $t$ goes from 0 to $r$ the line $\tilde{S}_t\ell_P$ describes a loop in Lag(1) going from $\ell_P$ to $\tilde{S}_r\ell_P$. We have $\tilde{S}_t \in U(1)$. Its image in $U(1, \mathbb{C})$ is $e^{-i\chi t}$ hence the Souriau mapping identifies $\tilde{S}_t\ell_P$ with $e^{-2i\chi t}$. It follows, using formula (2.40), that

$$m_{\ell_P}(\tilde{S}_{r,\infty}) = \frac{1}{2\pi} \left( -2r\chi + i \text{Log}(e^{-2ir\chi}) \right) + \frac{1}{2}$$

$$= \frac{1}{2\pi} \left( -2r\chi + i \text{Log}(e^{i(2r\chi+\pi)}) \right) + \frac{1}{2}.$$
The logarithm is calculated as follows: for \( \theta \neq (2k + 1)\pi \) \((k \in \mathbb{Z})\)

\[
\log e^{i\theta} = i\theta - 2\pi i \left[ \frac{\theta + \pi}{2\pi} \right]
\]

and hence

\[
\log(e^{i(-2r\chi + \pi)}) = -i(2r\chi + \pi + 2\pi \left[ \frac{r\chi}{\pi} \right]).
\]

It follows that the Maslov index is

\[
m_{\ell_p}(\tilde{S}_r, \infty) = -\left[ \frac{r\chi}{\pi} \right]. \tag{3.31}
\]

To obtain \( \nu(\tilde{S}_r, \infty) \) we note that by (3.28)

\[
\nu(\tilde{S}_r, \infty) = m_{\ell_p}(\tilde{S}_1, \infty) - \operatorname{Inert} W_{xx}''
\]

where \( \operatorname{Inert} W_{xx}'' \) is the concavity index corresponding to the generating function of \( \tilde{S}_t \). The latter is

\[
W(x, x', t) = \frac{1}{2\sin \chi t}((x^2 + x'^2) \cos \chi t - 2xx')
\]

hence \( W_{xx}'' = -\tan(\chi t/2) \). We thus have, taking (3.31) into account,

\[
\nu(\tilde{S}_r, \infty) = -\left[ \frac{r\chi}{\pi} \right] - \operatorname{Inert} \left( -\tan \frac{r\chi}{2} \right).
\]

A straightforward induction on \( r \) shows that this can be rewritten more conveniently as

\[
\nu(\tilde{S}_r, \infty) = -1 - 2\left[ \frac{r\chi}{2\pi} \right].
\]

Summarizing, we have

\[
\nu(S_r, \infty) = \nu(\Sigma_r, \infty) + \nu(\tilde{S}_r, \infty)
\]

\[
= -2r - 1 - 2\left[ \frac{r\chi}{2\pi} \right]
\]

hence the index in Gutzwiller’s formula corresponding to the \( r \)-th repetition is

\[
\mu_{x,r} = -\nu(S_r, \infty) = 1 + 2r + 2\left[ \frac{r\chi}{2\pi} \right]
\]

that is, by definition of \( \chi \),

\[
\mu_{x,r} = 1 + 2r + 2\left[ \frac{\omega_y}{\omega_x} \right]. \tag{3.32}
\]

**Remark 112** The calculations above are valid when the frequencies are incommensurate. If, say, \( \omega_x = \omega_y \), the calculations are much simpler: in this case the homotopy class of the loop \( t \mapsto S_t, 0 \leq t \leq 1, \) is \( \alpha^{-1} \oplus \alpha^{-1} \) and by the second formula (3.22),

\[
\mu_{x,r} = -\nu(S_r, \infty) = 4r
\]

which is zero modulo 4.
Bibliography


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Nedan följer en lista på skrifter publicerade i den nuvarande Acta-serien, serie III. För förteckning av skrifter i tidigare Acta-serier, se Växjö University Press sidor på www.vxu.se

Serie III (ISSN 1404-4307)


Växjö University Press
351 95 Växjö
www.vxu.se