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Second-Degree Equations as Object of Learning

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Abstract

The purpose of this paper is to report aspects focused by teachers in classroom practice when teaching the solving of second-degree equations ($ax^2 + bx + c = 0$ with a, b and c parameters and $a \neq 0$) by help of the formula

$$x_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q} \text{ (p, q parameters)}$$

and the students' ways of discerning particular aspects. The presentation is based on data collected in an upper secondary school in Sweden. Data consists of video-recordings of lessons, individual sessions, interviews and the teachers'/researcher's review of the individual sessions. Test results also constitute an important part of the data. The study includes two teachers and 45 students in two different classes.

In the analysis, concepts relating to variation theory have been used as analytical tools. Data have been analysed with respect to aspects focused on by the teachers during the lessons, aspects that are ignored, and patterns of dimensions of variations that are constituted. Data have also been analysed with respect to the students' focus when solving different problems in tests. The results show that the teachers focused on the parameters and the unknown quantity of an equation in different ways and this implicated that the students had the possibility to discern different aspects referring to the relation between a second-degree equation and the p-q-formula. Furthermore, some of these aspects are identified as critical aspects in the students' learning.

Background

Various documents, such as the most recent course syllabi in mathematics in upper secondary school in Sweden (Skolverket, 2000), specify that students should be able to solve second-degree equations and apply this knowledge in problem solving. Despite this, several investigations at different universities point out that a big part of the students are unable to solve this type of equations (see, e.g., Högskoleverket, 1999; Pettersson, 2003, 2005; Thunberg & Filipsson, 2005). Although the solving of second-degree equations is crucial for solving a number of problems, it is surprising that research into the teaching and learning of this topic is so scarce (Olteanu, 2007; Vaiyavutjamai, 2004; Vaiyavutjamai & Clements, 2006).

Reviews of research in algebra education have so far generally been silent about the teaching and learning of second-degree equations (e.g., Kieran, 1992; Kieran & Chalouh, 1993; Wagner & Parker, 1993). The chapter on algebra in the *International Handbook of Mathematics Education* (Filloy & Sutherland, 1996) does not refer to second-degree equations. Vaiyavutjamai (2004) reported that immediately after the students had participated in lessons on second-degree equations, 70 % of their responses to standard second-degree equation tasks were incorrect. Zaslavsky (1997) investigated misconceptions with respect to quadratic functions in 25 different schools in Israel. Zaslavsky's research emphasis was quadratic *functions* however, and her report touched only incidentally on students' responses to second-degree equations. In the chapters on algebra in the last two four-yearly research summary publications of the Mathematics Education Research Group of Australasia (Warren, 2000; Warren & Pierce, 2004), the word "quadratic" was used when Warren and Pierce (2004) referred to a small study by Gray and Thomas (2001) about the use of a graphics calculator and multiple representations to explore second-degree equations. The results of Gray and Thomas' study indicated that the students did not improve their ability to solve second-degree equations. Hoch and Dreyfus (2004) argued that whereas $30x^2 - 28x + 6$, for example, is equal to $(5x - 3)(6x - 2)$, students with a poor sense of structure may not realise that the quadratic trinomial and its factorised equivalent are "different interpretations of the same structure" (p. 51). Lithner (2006) argues that the finding of solutions to second-degree equations builds on an algorithmic reasoning. This type of reasoning depends on the steps that should be effectuated when solving an equation.

If second-degree equations are to remain an important component of mathematics curricula around the world, research about guiding teachers to improve their students' ways of understanding how to identify the solution for this type of equation is needed. To do this, it is necessary to understand where the problem is, and in which way the contents treated in lessons influence the students' learning. Researchers point out that there are few empirical studies analysing how teachers treat mathematical contents in the classroom, both nationally and internationally (Löwing, 2004; Olteanu, 2007; Runesson, 1999; Vaiyavutjamai, 2004). We do not know what is presented to the students in a teaching situation and what is critical in their learning of them. In this article, the emphasis is on how two teachers teach students to solve second-degree equations and what the students are learning from it. A central question in this context is which aspects the teachers focus on when they treat the content, which aspects are discerned by the students and which of these aspects that are critical for their learning.

Is it possible to find an explanation to this problem by increasing our knowledge about the relations between what the students can be aware of in a content that is presented in the classroom and how the students perform when they solve different equations? The answer to these questions can give important information concerning how to improve students' learning to solve second-degree equations.

Theoretical framework

The theoretical perspective in this study is the variation theory as it is described by Marton and Booth (1997), Bowden and Marton (1998) and Marton, Runesson and Tsui (2004). One of the main points of the theory is that learning is a way of experiencing or coming to experience the world in a certain way, and that different ways of experiencing will lead to different learning outcomes. A way of experiencing something can be defined by the aspects discerned. That is:

> An aspect of a thing corresponds to the way in which that thing might differ from, or be similar to, any other thing, that is, the way it is perceived to be, or the way that it is experienced by someone as different from, or similar to something else. (Marton et al., 2004, p. 9)

If the thing is a second-degree equation (an equation in the form $ax^2 + bx + c = 0$, where a, b and c are parameters, with $a \neq 0$), the aspects are the unknown quantity x, the parameters (a, b, c), multiplication and addition as operations and the equality between the two sides of the equation. To discern certain aspects of the equation, variations must be experienced in these aspects. An aspect is therefore a value in a dimension of variation. The problem is to know in which aspects to create dimensions of variation in order to constitute the meaning of second-degree equations. These aspects are called critical aspects or critical features. If we now think that the meaning of a second-degree equation is to find its solutions by help of the formula

$x_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$ (called p-q-formula in this paper), the presumptive critical aspects could be the unknown x, the parameters p and q, the operation between different parts of the formula ($\cdot$, $\pm$, $\sqrt{\ }$) and the opposite numbers of $\frac{p}{2}$. The critical aspects prescribe that the equation and/or the p-q-formula can be seen in a certain way (for example as $x^2 + px + q = 0$) or in different ways (for example as $ax^2 + bx + c = 0$, a, b and c parameters and $a \neq 0$). Marton et al. (2004) specify that the critical aspects must be found empirically. The totality of critical aspects defines an object of learning. Variation theory takes the object of learning as the point of departure and claims that the way in which learners experience the object of learning depends on which aspects of the object of learning that are focused upon and discerned as critical. The only way to discern the critical aspects is to experience how they vary, either at the same point in time, or by remembering earlier related experiences.

The concept of experience has guided the analysis of teaching and learning in this study. Learning, which in this perspective is sine qua non with the experience of variation, has been studied from two perspectives, namely the enacted and lived object of learning. The enacted object of learning is what appears in the classroom and the teacher and the students constitute it jointly. This is what it is possible for the student to experience and learn in a specific setting from the point of view of what is intended to be learned. The way in which students experience the object of learning is the lived object of learning.

The analysis of the empirical data had as point of departure that to experience something is to discern parts and the whole, aspects and relations. To experience how to solve second-degree equations is to experience the meaning and the structure of them and these two mutually constitute each other. So neither structure nor meaning can be said to precede or succeed the other. In structuring an experience, it is important for the teacher to be able to focus the students' attention on the critical aspects of that experience, to distinguish this experience from any other experience, and to make them understand the relationship between the critical aspects.

The origins of this study were in the first author's (Olteanu, 2007) investigation of the teaching and learning of second-degree equations and functions. In this paper we investigate the learning that occurs in the classroom when the students solve equations in the form:

$$ax^2 + bx + c = 0 \quad \text{(a, b and c parameters and } a \neq 0\text{)}$$

To experience the p-q-formula in solving second-degree equations, it is necessary for students to relate p and q in the formula to given numbers, that is, to see symbols as generalized numbers. In other words the students need to discern the parts in an equation and in the formula, relate them to each other and to the equation and formula as a whole. If the equation's x^2-coefficient equals 1, this relationship is direct. P and q in the formula are then the same as the x-coefficient and constant term of the equation. If the equation's x^2-coefficient is different from 1, p and q can be identified by dividing the equation with this coefficient (see equations 2 and 3 in Table 1). This means that the use of the p-q-formula needs rewriting the equation until the x^2-coefficient equals 1 and the other side of the equation is zero. In the process of rewriting second-degree equations, it is necessary for the students to discern that the parts in an equation can be related to each other in different ways while the solutions remain invariant. In the presented equation, the unknown quantity x is invariant. That means that x is invariant in the p-q-formula since the structure of this formula is invariant. In Figure 1, it can be seen that the parameters vary through different rewritings or through adopting negative and/or positive numbers. It is the identification of the parameters with p and q in the formula that makes it possible to find out the unknown quantity.

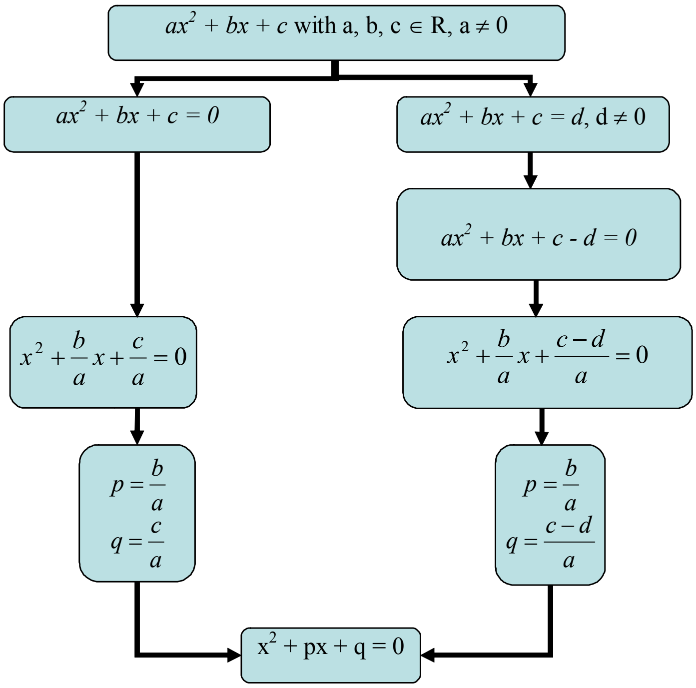

Figure 1. Variation of parameters.

To analyse the students' learning in relation to the variations opened in the classroom, our point of departure was that the students learn to solve second-degree equations by applying the p-q-formula and that this requires them to experience certain patterns of variation. This experienced variation makes it possible for them to discern some aspects that are necessary for developing abilities to solve this type of equations. For this reason, in analysing the data the aspects that students focused on when they solved different equations in test situations were first identified and thereafter the aspects that are critical in the students' learning. After that, the opportunities that the students were presented to experience in the classroom when the teachers were teaching how to solve second-degree equations were investigated.

Method and collection of data

The study was performed in an upper secondary school in Sweden during the spring term 2003. During this period, two teachers treating the same topic in mathematics were observed, together with the resulting learning of the students. In both classes, the same textbook was used. Students were selected from the Natural Science Programme. A total of 45 students (25 male, 20 female) and two teachers (Anna and Maria) participated in the study.

The present paper focuses how students in the two classes experienced the solving of second-degree equations. This means that the lessons in which the mentioned content was treated in the classroom have been selected from the main study. 12 consecutive lessons in each class, eight individual sessions, eight interviews and sequences where the teachers and the researcher looked at some video sequences together were selected from the main study for the analysis. Also, the teacher's planning of the course contents, researcher field notes and two tests taken by the students are included in the analysis. Thus, a combination of data was used, the most important, however, were video-taped lessons and the tests.

Since both teachers taught the same content and used the same textbook, it was possible to identify and describe differences between the teachers' teaching in relationship to the content they had taught. Before, after and during the observed lessons, all students took two tests. Since the students solved the same exercises in different tests, it was possible to identify and describe differences in their experiencing the contents.

Results

The results presented in this paper are divided into two parts. The lived object of learning is described in the first part and the enacted object of learning is described in the second part.

Description of the lived object of learning

The analysis of the tests shows that there are differences between the two classes and that these are based on the way in which the students discerned the equations' parameters (the coefficients and the constant term) and the unknown quantity. The coefficients and constant term refer to numbers that can be positive, zero or negative, and the unknown quantity refers to real numbers that satisfied the equation.

Table 1. Variations in the students' ways of solving second-degree equations.

No.	Second-degree equations	Variations in the students' answers	Maria's class	Anna's class
1.	$x^2 + 6x + 5 = 0$	Solve the equation by using the formula	100%	55%
		Don't distinguish the constant term	0%	20%
		Don't solve the equation	0%	25%
2.	$- 2x^2 + 12x + 14 = 0$	Identified p with – 6 and q with – 7	91%	65%
		Identified p and q with other numbers	0%	13%
		Identified p with 12 and q with 14	0%	9%
		Don't solve the equation	9%	13%
3.	$10 = 8,5 + 9,8t – 4,9t^2$	Solve the equation by using the formula	55%	35%
		Identified p with – 2	50%	35%
		Identified p with 2	5%	0%
		Identified q with 1,5/4,9	45%	30%
		Identified q with – 1,5/4,9	10%	5%
		Have on side different from zero and divide only the other side by – 4,9	5%	10%
		Solve other equations or exercises	15%	15%
		Only answer	5%	15%
		Don't solve the equation	20%	25%

Table 1 shows the percentages of the ways in which students solved different equations and the aspects they discerned in this process. The equations presented differ in subtle but mathematically important ways. The first difference refers to *the unknown quantity*, namely it appears twice in an equation and therefore it forms the equation's whole with the help of addition as operation. The second difference refers to *the parameters* (x^2-coefficient, x-coefficient and constant term). The parameters appear as explicit in the first equation (p and q in the formula can bee identified directly) and implicit in the second and third equations (p and q in the formula can be identified after some rewriting of the equation).

The results show that in both classes a large part of the students preferred to apply the p-q-formula to find the solutions of second-degree equations. Apart from this there are students that use the null product law "if $a \cdot b = 0$, then $a = 0$ or $b = 0$" to solve an equation written as $x^2 + px + q = 0$. The two different ways in which students identify the unknown quantity show that they discern some aspects of the equation through separating the whole into different parts.

A closer look at the test results shows that all the students in Maria's class could discern the parameters in the first equation, but only 55 % in Anna's class. Furthermore, the students in Anna's class developed an incorrect way of solving an equation in which the x^2-coefficient equals 1 (se Equation 1 in Table 1). We can see an example of this in the following figure:

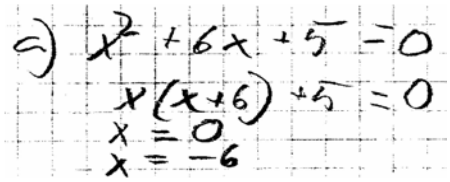

Figure 2. Håkan in Anna's class.

These students did not understand how the null factor law should be applied in the context of a general second-degree equation. They did not think about the fact that the two values of x give the result that 5 is equal to zero, which is not possible. The students did not discern the importance of the constant term to factorise quadratic trinomials, and did not check whether any of the solutions that they obtained were correct. This way of solving equations could not be identified when the students solved second-degree equations having the x^2-coefficient different from 1 (see Equations 3 and 4 in Table 1).

The second equation in Table 1 has, for instance, the coefficients – 2 and + 12, and the constant term is + 14. Since the x^2-coefficient no longer equals 1, the use of the p-q-formula demands rewriting of the equation to an equivalent equation. For this, the students have to understand that the parts constituting the whole in the following equations

$$-2x^2 + 12x + 14 = 0$$
$$x^2 - 6x - 7 = 0$$

can be related to each other in different ways, but, despite this, the equations have the same solutions. If the students do not understand this relationship, it leads to giving p the value + 12 and q the value + 14 in the formula. This means that these students have a hard time to apprehend in which ways the parameters in an equation of type $ax^2 + bx + c = 0$ (with $a \neq 0$) relate to an equation of type $x^2 + px + q = 0$ (normalised quadratic equations). In the normalised state, the coefficient in the quadratic term is 1 and one side equals 0. In order to make this relation it is important for the students to discern the x^2-coefficient. The results in Table 1 show that the students who understand the relation between the structures of the two equations also understand the structure of the p-q-formula. If the x-coefficient and the constant term become negative numbers after rewriting, the addition is used as an implicit operation in the equation $x^2 - 6x - 7 = 0$ (the equation is really $x^2 + (-6)\cdot x + (-7) = 0$). In this case, there are students that discern ways in which the parts relate to each other, that is, these students discern that the minus sign is used in order to highlight a negative number and that for instance x^2 and $-6x$ are related to each other through addition. The way in which the parts are related to each other are discerned by all students solving the equations in Maria's class, in comparison with only 65 % in Anna's class.

If the constant term in a second-degree equation appears on both sides of the equal sign (see Equation 4 in Table 1), that is, the constant appears as two parts, there are students that first discern the constant term as a whole. Thereafter, these students discern the equation's x^2-coefficient and divide the equation by this coefficient. In this way, it is possible for them to discern p and q in the p-q-formula. The part of students that could identify p and q in the formula with p and q in the equations having their x^2-coefficient different from 1 and the constant term on both sides of the equal sign, decreases in both classes but more in Anna's class. Only 45 % of the students in Maria's class and 35 % in Anna's class could distinguish the equation's parameters and relate them to the p-q-formula in the last equation presented in

Table 1. Because the normalised form of the equation is obtained from a general formulation in which the constant term appears on both sides of the equation, it is necessary for the students to discern the equation's constant term. The results presented in Table 1 show that there were students in both classes that did not discern all the parts of the constant term. Also, there were students that only divided by – 4,9 on one side of the equation, and thereafter used the formula as the following example shows:

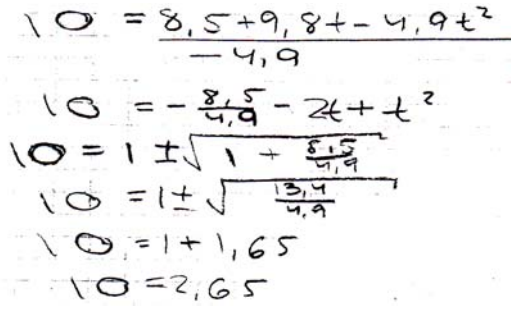

Figure 3. Ludvig in Maria's class.

This far we can establish that the parameters forming the whole in second-degree equations are critical aspects in students' learning. This is because the students have difficulty in seeing the equations in a certain way, namely as $x^2 + px + q = 0$ and thereafter be able to identify the parameters of the equations with p and q in the p-q-formula.

The enacted object of learning

One possible way to account for these differences in learning is the structural difference observed in the two teachers' ways of handling the object of learning. In order to develop the students' ability to solve some second-degree equations, several equations were chosen for presentation in the classroom and in textbooks (see Table 2).

Table 2. Different forms of second-degree equations applied in the classroom and in the textbook.

No.	Maria's class	Anna's class	Textbooks
1.	$x^2 = 144$	$x^2 – 4 = 0$	$x^2 = 5$
2.	$5x^2 = 845$	-	$2x^2 = 50$
3.	$(x + 14)^2 = 4$	$(x – 1)^2 – 9 = 0$	$(x + 3)^2 = 16$
4.	$x^2 – 6x + 9 = 0$	$x^2 – 4x – 5 = 0$	-
5.	$x^2 + 12x + 35 = 0$	-	$x^2 + 5x + 6 = 0$
6.	$x^2 + 2x – 15 = 0$	-	$x^2 + 6x – 16 = 0$
7.	$x^2 – x – 30 = 0$	-	-
8.	$2x – 3x^2 = – 1$	$– 0,01x^2 + x + 2,3 = 0$	$4x^2 – 12x – 7 = 0$

The distinction between the selected equations is that the x^2-coefficient in Equations 1 and 4-7 equals 1, while this coefficient is different from 1 in the Equations 2 and 8. In Maria's class and in the textbook, the x-coefficient and constant term are represented by alternating positive and negative values. This indicates a variation in the way in which the parts of equations are related to each other. Furthermore, the unknown quantity appears both as a monom (Equations 1 and 2), and a binom (Equation 2). Another distinction between the presented equations in the two classes is that in Maria's class, the constant term appears on the same side as the unknown quantity in Equations 4-7 and on the opposite side of the unknown quantity in Equations 1-3 and 8. In Anna's class, this phenomenon could not be identified. In the textbook, both these aspects are focused.

In the exposé of how to solve the first and second equation presented in Table 2, Maria focused on two aspects. The first aspect refers to the square root of positive numbers being both positive and negative numbers with the students' earlier experiences as background, that is, the students have only experienced the positive square root in the geometric context. At this opportunity the difference between a first and a second-degree equation is lifted to the front. The second aspect refers to the equation's x^2-coefficient, namely that this coefficient must be 1 in order for it to be possible to extract the square root. This can be seen in the following example:

[1.2] Maria: If I then have $5x^2$, I can't begin to take the square root of the right side directly, and what should I do first?
[2.2] Leonard: Divide.
[3.2] Maria: Yes, I must divide away the five because I must have only x^2 before I, eh, take the square root of. (Lesson 24, 2003-03-31)

This focused aspect is related to the way in which it is possible to identify the unknown through focusing on the fact that the equation can only be solved if the x^2 is on one side of the equality sign and the number is on the opposite side and only if the x^2-coefficient equals 1. In Anna's exposé, it could also be identified that she focused on two aspects, but it was not directly possible to discern the importance of the x^2-coefficient for drawing the square root. This aspect is important for the students when they solve equation 8 in Table 2.

Thereafter both the teachers present that the same pattern can be used to solve a second-degree equation in which the unknown appears in a binom, namely to draw the square root of the number on the other side of the equal sign. This can be seen in the following figures:

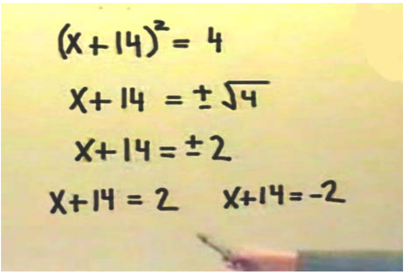

Figure 4. Implicit x^2 and explicit constant term (Maria, Lesson 24).

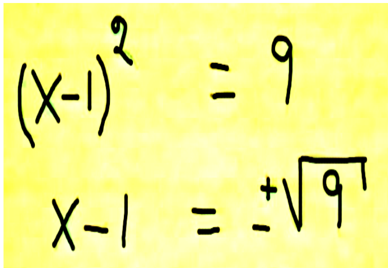

Figure 5. Implicit x^2 and explicit constant term (Anna, Lesson 25).

From Figure 4 it is apparent that Maria clearly present the two first-degree equations that result from the square root drawing. In this way she shows how the relations between the equation's parts change through square root drawing. Furthermore, she points out that the solutions of the first-degree equations are the same as the solutions of the second-degree equation. Anna solves the equation without clearly writing which the first-degree equations are and through only focusing on the answer [10.2]. This means that Anna does not discern the way in which the parts relate to each other and to the equations as a whole.

[8.2] Anna: It wants to say …
[9.2] Lydia: Mm … x minus one is equal to the square root of nine, plus minus …

[10.2] Anna: Plus, minus … square root of nine, that means x is 1 plus, minus, we can write three instead of square root of nine … (Lesson 25, 2003-03-28)

In order to solve second-degree equations that have the x-coefficient and the constant term different from zero, Maria developed the way in which the parts of an equation can be related to each other to obtain the square of a binom by introducing the completing square. She does this in the following ways:

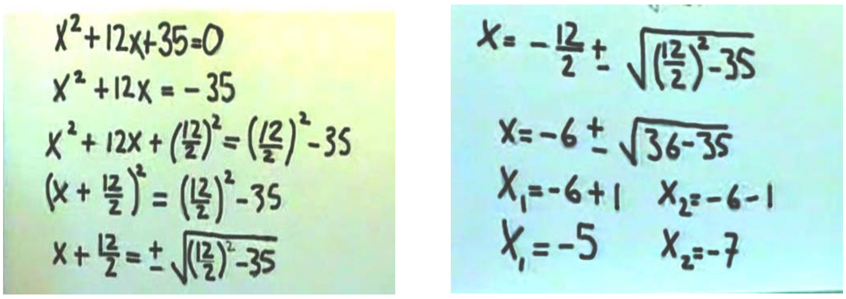

Figure 6. Completing the square (Maria, Lesson 25).

In the numerical example Maria focused on varying the relationship between the equation's parts several times and she points out that it is possible to identify the equation's unknown quantity in this way. Thereafter Maria uses generalised numbers (p and q) as x-coefficient and constant term and shows how the p-q-formula is obtained. In this way Maria focuses on the relation of one part of the equation (the unknown quantity) with the other parts of it (the x-coefficient and constant term), and obtains the p-q-formula as an entity. In order to solve different second-degree equations (see Table 2), Maria constantly used the newly introduced formula. This signifies that she accentuated the meaning of the relation between the x-coefficient and constant term of second-degree equations with p and q in the p-q-formula. From the following dialogue, taking place when Maria solves the equation $x^2 + 2x - 15 = 0$, it is clear that it sometimes is not straight-forward for the students to do this identification:

[1.10] Maria: Is it written in the form of $x^2 + px + q = 0$?
[2.10] Emilia: No …
[3.10] Maria: Is it not? Yes, it is minus, yes. But if we still compare it (the teacher writes $x^2 + px + q = 0$ on top of the given equation), I say that it is the same, but what is q equal to?
[4.10] Emilia: – 15
[5.10] Maria: Yes, that is minus 15, since the minus sign lies in q, q is equal to minus 15 and p is equal to 2, and I can't have any coefficient in front of x^2, and it should be equal to zero, so it has that form. (Lesson 25, 2003-04-03)

From the transcript, it is clear that Maria first assumed that the students could identify p and q with 2 and – 15 and thereafter use the p-q-formula [3.10]. But it shows that there are students that do not immediately see that – 15 is the same as +(– 15), which is a fundamental convention in mathematics. This leads to the fact that these students cannot see the relation between the general equation $x^2 + px + q = 0$ and the given equation (see e.g. [2.10] in the above extract). Through pointing out that the minus sign lies in q and that q in this case is – 15 and p is 2 [5.10], Maria points out that a presumption in order to identify the solutions to a given equation of the second-degree with help of the p-q-formula is to discern the equation's x-coefficient and constant term and relate them to p and q in the p-q-formula.

Thereafter Maria used a positive value for the equation's x-coefficient (p) to keep it invariant, while the constant term (q) could have both positive and negative values. This means that an opportunity to discern the value of q in the p-q-formula was created. In the next step Maria kept q invariant with the help of a negative number and varied the value of p, with both negative and positive values, which made it simpler to discern p in the p-q-formula. After this, Maria varied the equation's x^2-coefficient and constant term. She made this variation through contrast, that is, the x^2-coefficient equals 1 or is different from 1 and the constant term appears on the same side as the unknown quantity or on both sides of the equation.

Maria generalised the importance of the x^2-coefficient when she for example solved the equation $2x – 3x^2 = – 1$.

[1.12] Maria: What do you say about this then? Is it ready for the formula? Is it written in the form $x^2 + px + q = 0$?
[2.12] Sune: No …
[3.12] Maria: No, it's not. Firstly, this is the x^2 term. It should be positive (points at x^2 in $x^2 + px + q = 0$) and it's not. Then I'll begin by making it positive and I move it over (points at $– 3x^2$ and the right side) and it becomes $3x^2$, and then it shall always be gathered on one side in order to make it equal to zero …
[…]
[6.12] Maria: Is it ready for the formula now?
[7.12] Josefina: No …
[8.12] Maria: No, it's not, for there can't be a 3 in front of x^2, only x^2 is allowed, so what are we going to do with the 3?
[9.12] Viveka: Divide.
[10.12] Maria: Yes, divide, and then I divide all the terms. (Lesson 25, 2003-04-03)

Maria rewrites the equation in order to use the p-q-formula [1.12]. In her rewritings, she simultaneously varies it one side from being different from zero to being equal to zero [3.12]. This means that she focused on the fact that the constant term must be on the same side as the x^2- and x-coefficient. Thereafter Maria simultaneously varies the x^2-coefficient from being different from 1 to equalling 1 by dividing the equation with 3 [8.12]-[9.12]. She accentuated that all terms must be divided with the x^2-coefficient, which means that she also implicitly focused on the variation in the equation's x-coefficient and constant term [10.12]. By doing this Maria shows that the parts that constitute these equations can be related to each other in different ways, but that they despite this have the same solutions. In order to solve the last equation, Maria uses the p-q-formula by focusing on identifying the x-coefficient and the constant term in the equation with p and q in the p-q-formula.

The dimensions of variation appearing in Maria's class and referring to the equation's x^2-coefficient, x-coefficient and constant term open up a strong relationship between these and p and q in the p-q-formula. To constitute this relationship, the focus was on varying the parts that constitute the equations' whole, but to allow the structure of the p-q-formula still to remain invariant. This leads to the possibility of generalizing this pattern of solving second-degree equations and the relations that appear in between in the use of the p-q-formula and these equations. This generalization makes it possible for the students to constitute a confident manipulation of different types of second-degree equations, to learn the meaning of these, relate the second-degree equations' coefficients and constant terms to the p-q-formula, and use these relations for new manipulations.

In Anna's exposé of the same contents, the p-q-formula was presented in symbolic form without connection to the students' experiences.

[37.4] Anna: And now you will learn a little formula …
[…]

[49.4] Anna: And so if you look into your formulae book, and you don't have to, because I'll write it up for you, you will see that there is written something like this: $x^2 + px + q = 0$ (the teacher writes on the blackboard and talks out loud).

[50.4] Anna: And then, under it, you'll see that it is written like this … (the teacher writes on the blackboard: $x = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$) (Lesson 26, 2003-04-01)

From the transcript, it is clear that Anna begins to solve the new equations, which the students now encounter for the first time, by formulating the p-q-formula. Furthermore, she presents this formula only in symbolic form, that is, Anna does neither explain what different symbols stand for, nor in which way these symbols relate to a given second-degree equation. Apart from this she does not connect the introduced formula to the students' earlier experiences of rewriting a second-degree polynom. This leads to discussions and commotions among the students.

Thereafter the p-q-formula is used in numerical form, but Anna neither focuses on the fact that p and q can be positive as well as negative numbers, nor on the fact that one side of the equation can be different from zero, which limits the students' possibility to discern the equation's x-coefficient and constant term in relation to p and q in the formula. The only variation that appears in Anna's class refers to the x^2-coefficient that varies from being equal to 1 to being different from 1. This is presented with the help of a transparency in the following way:

[29.5] Anna: How do I get this? How do I get this step?

The teacher points at $x^2 - 10x - 2{,}3 = 0$, as the following figure shows:

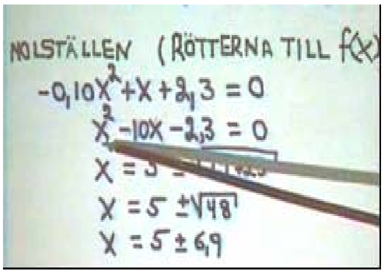

Figure 7. Focused x^2-coefficient (Anna, Lesson 31).

[30.5] Eve: You multiplied by 10.
[31.5] Anna: Why? And not just with …
[32.5] Eve: – 10.
[33.5] Anna: Why?
[34.5] Eve: In order to get x^2.
[35.5] Anna: This is very important information. (Lesson 31, 200304-10)

From the transcript, it is clear that Anna focused on the fact that an equivalent second-degree equation can be obtained through multiplying the equation with – 10, but she does not explain why this is ”very important information”[1] [35.5]. In other words, Anna does not clearly accentuate that the p-q-formula can only be used if the x^2-coefficient is 1 and that this leads to an alteration in the equation's x-coefficient and constant term.

[1] In addition, we can notice that the first and the second step are not equivalent, that is, the constant term, despite the multiplication with – 10, still is 2,3 (compare step 1 and 2 in Figure 7).

Anna's exposé is characterized by not tying together the students' earlier experiences with new experiences and she mainly leaves the students to read the most important explanations in their textbook. In the textbook's exposé dimensions of variation in the above-mentioned aspects are constituted, but it is assumed that the students understand the message of the textbooks. That this is not obvious for the students can be seen in the following transcript:

> Hanna asks what she did wrong when she solved the equation $x^2 - 6x + 5 = 0$. Anna looks at what she wrote.
>
> [1] Anna: Half the coefficient (points at – 6) with opposite sign, and so it's simpler to do that, half of that (points at – 6) with opposite sign.
>
> [2] Hanna: But I did it like it's written in the textbook…
>
> [3] Anna: OK, then you have to, save yourself on fractions … quarters (looks at what the student wrote), and then you forget that there should be minus, opposite signs, that's the only error that you make …
>
> Hanna erases what she has written.
>
> [4] Hanna: Should the sign be reversed for everything or?
>
> [5] Anna: No, for it in the formula is written, it is written… you do have the formula (starts to look for the formula in the textbook). If you look here, it is written plus (points at p in $x^2 + px + q = 0$) and minus is written here (points at – 6), it means in reality then, there is a plus here (points at p in $x^2 + px + q = 0$) and it becomes a minus there (points at $-p/2$ in the p-q-formula), if there is minus in front of this (points at p in $x^2 + px + q = 0$) it becomes plus (points at $-p/2$ in p-q-formula), with opposite signs for everything, opposite, so … and then it will be simpler for you. (Lesson 27, 2003-04-03)

If the students understand the textbook's message, and it is not at all obvious that they do, it becomes possible for them to develop their ability to generalize the use of the p-q-formula. This means that the students in Anna's class have had less possibility to experience a pronounced variation in the aspects that constitute the structure of second-degree equations and the relations between these equations and the p-q-formula.

Conclusion

Two relations could be identified in the enacted and lived object of learning, namely the *relation between the focused aspects* and the *relation between the opened variations*. The first relation refers to the identified aspects that the teachers and the textbook focus on and the identified aspects that were shown to be critical in the students' learning. The second relation refers to the constituted patterns of variation in the enacted and lived object of learning. These different ways in which the students discern the parameters and the unknown quantity of equations are represented in Table 3.

The relation between focused aspects

If we look back at the way in which the students in the two observed classes experience second-degree equations, we can establish that almost all students in Maria's class can solve these equations, in comparison with about half of the students in Anna's class. Furthermore, the amount of students that solve these equations if the x^2-coefficient and constant term is written in different ways decreases in both classes (see Equation 3 in Table 1), but again, there is a greater amount of students solving them in Maria's class than in Anna's class.

The differences in the students' way of solving second-degree equations with the help of the p-q-formula mainly depend, firstly, on the way in which they discern the equations' parameters (coefficients and constant term), and secondly on the way in which they relate these parameters to the parts of the formula. Furthermore, the way in which the students discern the parameters affect the way in which the students solve different equations. This means that the students in Maria's class have developed a greater ability to solve second-degree equations than the students in Anna's class. What is the reason for this? We can see some explanations to this in the following table:

Table 3. The enacted and lived object of learning – discerned aspects and patterns of solving second-degree equations.

Second-degree equations	Aspects (enacted object of learning)		Patterns of solving equations (lived object of learning)	
	Discerned	**Not discerned**	**Discerned the unknown**	**Did not discern the unknown**
$x^2 + px + q = 0$	**p > 0**		**p-q-formula**	
	Maria's class Anna's class		Discern the two solutions correctly (Maria's and Anna's class)	
	q > 0		**p-q-formula**	
	Maria's class Anna's class	Anna's class	Discern the two solutions correctly (Maria's and Anna's class)	
				Applied the null product law wrong (Anna's class) Discern the two solutions wrong (Anna's class)
	p < 0		**p-q-formula**	
	Maria's class Anna's class	Anna's class	Discern the two solutions correctly (Maria's and Anna's class)	Discern the two solutions wrong (Anna's class)
	q < 0		**p-q-formula**	
	Maria's class Anna's class	Anna's class	Discern the two solutions correctly (Maria's and Anna's class)	Discern the two solutions wrong (Anna's class)
$ax^2 + bx + c = 0$ $(a \neq 0, a \neq 1)$	**a**		**p-q-formula**	
	Maria's class Anna's class	Anna's class	Discern the two solutions correctly (Maria's and Anna's class)	Discern the two solutions wrong (Anna's class)
$ax^2 + bx + c = d$ $(a \neq 0, a \neq 1)$	**a**		**p-q-formula**	
	Maria's class Anna's class	Anna's class	Discern the two solutions correctly (Maria's and Anna's class)	Discern the two solutions wrong (Anna's class)
	c and d		**p-q-formula**	
	Maria's class Anna's class	Anna's class Maria's class	Discern the two solutions correctly (Maria's and Anna's class)	Discern the two solutions wrong (Maria's and Anna's class)

In Maria's and the textbook's exposé of the equation $x^2 + px + q = 0$, the students had the possibility to discern parameters *as general numbers*. The parameters that constitute this form of equation, that is, the x-coefficient and constant term, are generalised with the starting point in solving specific cases of equations (see Equations 1- 4 in Table 1). Thereafter the number of parameters with x^2-coefficient increases. In this process the parts, namely the parameters, were discerned by letting the unknown quantity x remain invariant. These parts are discerned by successively varying the parameters. In Anna's class, this systematic variation (generalization) is not experienced. Anna presents only certain cases of second-degree equations and her students only have the possibility to generalize these with the help of the textbook's exposé.

The students also had the possibility to experience parameters *as variables* in different ways. The parameters are exposed as *variables* in Maria's and the textbook's exposé through the fact that that the parameters p and q in an equation were both positive and negative numbers. This variation makes it possible for the students to discern the parts that constitute an equation of the second degree and relate them to the parts that constitute the p-q-formula. In Anna's class, these variations are not experienced.

Apart from the focused aspects mentioned, it was also established that the teachers prefer to develop the students' ability to solve second-degree equations, that is, to find out the unknown quantity x, by using the p-q-formula. In this formula the parameters p and q appear as *given numbers*. These aspects are experienced in both classes and in the textbooks. There are students in both classes (see Table 1) that in order to decide these numbers simultaneously discern the parameters as generalized numbers (in the p-q-formula) and as given numbers (in an equation) and thereafter relate them to each other. In other words the students simultaneously discern the parts that form an equation of the second degree and the parts in the p-q-formula. They also discern the relations between the structure of the equations and the formula. If there are students that do not develop this ability, these students understand the p-q-formula wrongly (see Table 1).

The second aspect that is focused on in the enacted object of learning is the *unknown quantity.* This aspect refers to the fact that the unknown quantity x can appear in an equation in explicit (see Equations 1, 2 and 4-8 in Table 2) or implicit form (see Equation 3 in Table 2). This means that the unknown quantity can be obtained either by using the p-q-formula or by reducing an equation of the second degree to the first-degree equations. The way in which the unknown quantity appears in second-degree equations leads the teachers and the textbooks to partly present this quantity as invariant, partly to give it a new meaning by reducing second-degree equations to first-degree equations. This reduction is made with the help of different algebraic rewritings that refer to completing the square (see Figure 6). Through these rewritings, the x-exponent changed from being of the second degree to that being of the first degree. The meaning with the reduction of second-degree equations is presented in Maria's class and in the textbook but not in Anna's class. This leads the students in Anna's class to elaborate other ways of solving second-degree equations (see Figure 2).

This far, we can establish that there is a strong relationship between the aspects that have been focused on in the object of learning and the way in which the students discerned these aspects. Furthermore, there is a strong relationship between the aspects that are focused on and the students' possibility to develop the ability to solve second-degree equations. This can be seen in Maria's class when she does not focus on the importance of the constant term appearing as separate parts in equations and the use of the p-q-formula (see Table 1 and Figure 3). This makes it possible to arrive at an important conclusion, namely that the teachers unwarily used dimensions of variation in certain aspects but that a crucial role in order for the students to develop their learning is to create variations in the aspects that make it possible to understand the relations between the parts that constitute a second-degree equation and the p-q-formula, and a second-degree equations and the p-q-formula as whole.

Relation between the opened variations

The second relation that could be identified is called a relation between the opened variations and is characterised by the variations that are opened up in some aspects of the contents in the teachers' and textbook's exposé of the object of learning and the displayed variations when the students experience these aspects.

The teachers and the textbook opened up dimensions of variation in two different ways. In the first way, variations were opened up through different examples focusing on the same aspect. Thereafter they were tied to the object of learning as a whole. This kind of variation is called *convergent variation*. A convergent variation means that different aspects of the object of learning refer the parts to the relations between these parts and thereafter create the whole in the object of learning. It seems to be this variation that leads to a positive development in the students' learning. This way to open up dimensions of variation can be observed in Maria's class and in the textbook's exposé. In the second way, the object of learning is presented as a whole and thereafter the parts that constitute this object are varied without being discerned. This variation is called *divergent variation* and could be observed in Anna's class. The analysis of the students' learning indicates that it could be this variation that makes the students focus on finding alternative solutions and ideas that are not mathematically correct. If the meaning of an aspect is varied simultaneously without first discerning this aspect it seems to become problematic for the students since these aspects often refer to other mathematical objects. These two ways of creating dimensions of variation are represented in the following figures:

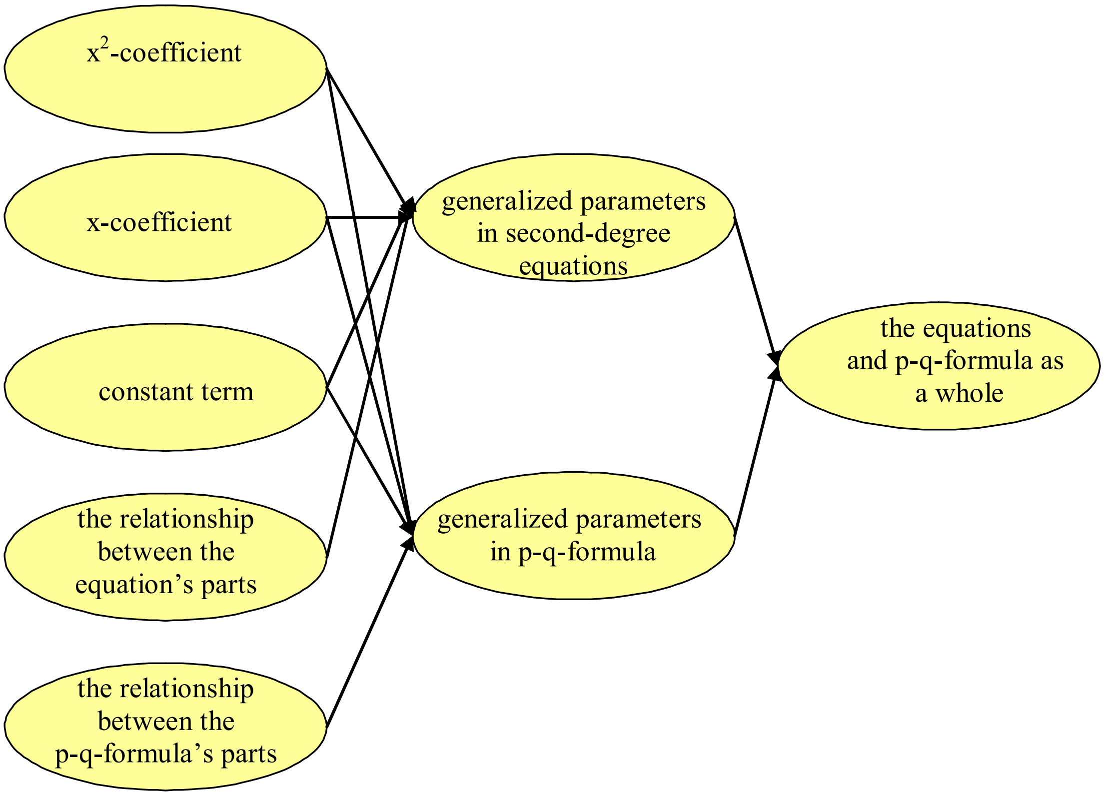

Figure 8. Focused aspects in Maria's class and in textbook.

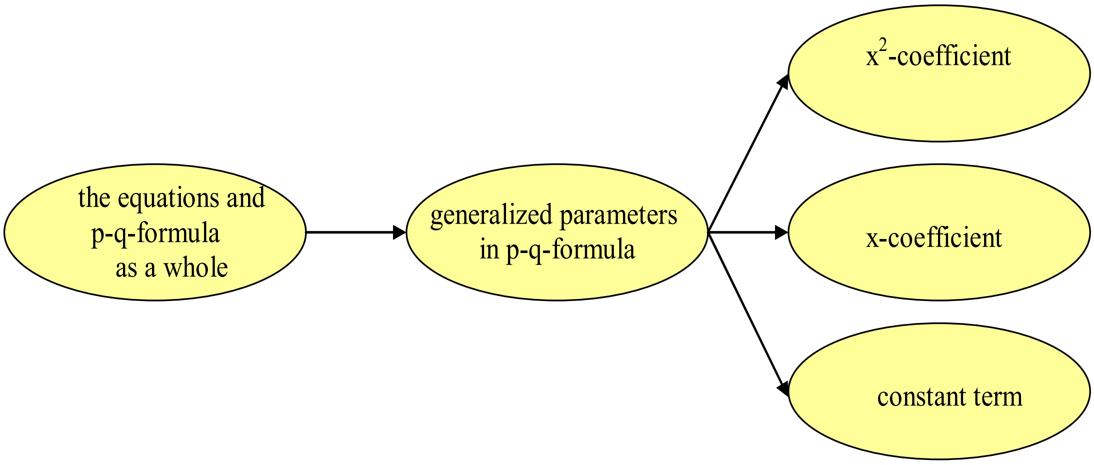

Figure 9. Focused aspects in Anna's class and in textbook.

The way in which variations are opened up in the enacted object of learning seems to have a direct connection to the meaning in which the students experience the aspects. The relationship between the aspects focused on in the enacted and the lived objects of learning display that several focused aspects in the enacted object of learning lead to less variation in the students' experience of them, and, on the contrary, if less aspects are focused on, the students experience this object of learning in more ways (see Table 3). This means that the focused aspects and the way in which these aspects are focused makes it possible for the students to discern the meaning of the object of learning or to develop own ideas that often differ from the mathematical thinking (see Figures 2 and 3). The way in which variations are opened up can probably contribute to an explanation of why so many students think that mathematics is difficult.

References

Bowden, J., & Marton, F. (1998). *The university of learning: Beyond quality and competence in higher education*. London: Kogan Page.

Filloy, E., & Sutherland, R. (1996). Designing curricula for teaching and learning algebra. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick & C. Laborde (Eds.), *International handbook of mathematics education* (pp. 139–160). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Gray, R., & Thomas, M. O. J. (2001). Quadratic equation representations and graphic calculators: Procedural and conceptual interactions. In J. Bobis, B. Perry & M. Mitchelmore (Eds.), *Numeracy and beyond* (Proceedings of the 24th Conference for the Mathematics Education Research Group of Australasia, Sydney, pp. 257–264). Sydney: MERGA

Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390–419). New York: Macmillan.

Kieran, C., & Chalouh, L. (1993). Prealgebra: The transition from arithmetic to algebra. In P. S. Wilson (Ed.), *Research ideas for the classroom: Middle grades mathematics* (pp. 119–139). New York: Macmillan.

Hoch, M., & Dreyfus, T. (2004). Equations – a structural approach. I M. J. Høines & A. B. Fuglestad (Red.), *Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 152–155). Bergen, Norge: Bergen University College.

Högskoleverket. (1999). *Räcker kunskaperna i matematik?* Stockholm: Högskoleverket.

Lithner, J. (2006). *A framework for analysing creative and imitative mathematical reasoning*

(Department of Mathematics and Mathematical Statistics, Umeå University, Research Reports in Mathematics Education, no. 2). Umeå: Umeå universitet.

Löwing, M. (2004). *Matematikundervisningens konkreta gestaltning: En studie av kommunikationen lärare – elev och matematiklektionens didaktiska ramar* (Göteborg Studies in Educational Sciences 208). Göteborg: Acta Universitatis Gothoburgensis.

Marton, F., & Booth, S. (1997). *Learning and awareness*. Mahwah, NJ: Erlbaum.

Marton, F., Runesson, U., & Tsui, A. B. M. (2004). The space of learning. I F. Marton & A. B. M. Tsui (Red.), *Classroom discourse and the space of learning* (pp. 3–40). Mahwah, NJ: Erlbaum.

Olteanu, C. (2007). *"Vad skulle x kunna vara?" Andragradsekvation och andragradsfunktion som objekt för lärande.* (Diss., Doktorsavhandlingar inom den Nationella Forskarskolan i Pedagogiskt Arbete, 10, doktorsavhandlingar i Pedagogiskt arbete, 19 och Skrifter utgivna vid Högskolan Kristianstad, 10). Umeå: Umeå universitet, fakulteten för lärarutbildning, Nationella Forskarskolan i Pedagogiskt Arbete.

Petterson, R. (2003). *Resultat av diagnostiska prov i matematik för nyantagna teknologer vid civilingenjörslinjerna Chalmers, 1973- 2003*. Göteborg: Chalmers tekniska högskola.

Pettersson, R. (2005). *Resultat av diagnostiska prov i matematik för nyantagna teknologer vid civilingenjörslinjerna Chalmers 1973-2005*. Göteborg: Chalmers tekniska högskola.

Runesson, U. (1999). *Variationens pedagogik: Skilda sätt att behandla ett matematiskt innehåll* (Göteborg Studies in Educational Sciences 129). Göteborg: Acta Universitatis Gothoburgenesis.

Skolverket. (2000). *Naturvetenskapsprogrammet, Gy2000: Programmål, kursplaner, betygskriterier och kommentarer*. Stockholm: Fritzes.

Thunberg, H., & Filipsson, L. (2005). *Gymnasielärares syn på KTHs introduktionskurs i matematik*. www.math.kth.se/gmhf

Vaiyavutjamai, P. (2004). Correct answers do not imply understanding: Analyses of student solutions to quadratic equations. In I. P. Cheong, H. S. Dhindsa, I. J. Kyeleve & O. Chukwu (Eds.), *Globalisation trends in science, mathematics and technical education* (pp. 153–162). Gadong, Brunei Darussalam: University Brunei Darussalam.

Vaiyavutjamai, P. Clements, M. A. (2006). Effects of Classroom Instruction on Students' Understanding of Quadratic Equations. *Mathematics Education Research Journal*, (VOL 18; pp. 47-77).

Wagner, S., & Parker, S. (1993). Advancing algebra. In P. S. Wilson (Eds.), *Research ideas for the classroom: High school mathematics* (pp. 119–139). New York: Macmillan.

Warren, E. (2000). Research in teaching and learning algebra. In K. Owens & J. Mousley (Eds.), *Research in mathematics education in Australasia 1996–1999* (pp. 161–180). Sydney: Mathematics Education Research Group of Australasia.

Warren, E., & Pierce, R. (2004). Learning and teaching algebra. In B. Perry, G. Anthony & C. Diezmann (Eds.), *Research in mathematics education in Australasia 2000–2003* (pp. 291–312). Flaxton, Australia: Mathematics Education Research Group of Australasia.

Zaslavsky, O. (1997). Conceptual obstacles in the learning of quadratic functions. *Focus on learning problems in mathematics*, *19(1)*, 20-44.