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School of Computer Science, Physics and Mathematics

Licentiate Thesis

Propagation of singularities for pseudo-differential operators and generalized Schrödinger propagators

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Abstract

In this thesis we discuss different types of regularity for distributions which appear in the theory of pseudo-differential operators and partial differential equations. Partial differential equations often appear in science and technology. For example the Schrödinger equation can be used to describe the change in time of quantum states of physical systems. Pseudo-differential operators can be used to solve partial differential equations. They are also appropriate to use when modeling different types of problems within physics and engineering. For example, there is a natural connection between pseudo-differential operators and stationary and non-stationary filters in signal processing. Furthermore, the correspondence between symbols and operators when passing from classical mechanics to quantum mechanics essentially agrees with symbols and operators in the Weyl calculus of pseudo-differential operators.

In this thesis we concentrate on investigating how regularity properties for solutions of partial differential equations are affected under the mapping of pseudo-differential operators, and in particular of the free time-dependent Schrödinger operators.

The solution of the free time-dependent Schrödinger equation can be expressed as a pseudo-differential operator, with non-smooth symbol, acting on the initial condition. We generalize a result about non-tangential convergence, which was obtained by Sjögren and Sjölin in [17] for the free time-dependent Schrödinger equation.

Another way to describe regularity for a distribution is to use wave-front sets. They do not only describe where the singularities are, but also the directions in which these singularities appear. The first types of wave-front sets (analytical wave-front sets) were introduced by Sato in [15,16]. Later on Hörmander introduced "classical" wave-front sets (with respect to smoothness) and showed results in the context of pseudo-differential operators with smooth symbols, cf. [10].

In this thesis we consider wave-front sets with respect to Fourier Banach function spaces. Roughly speaking, we take $\mathscr{B}$ as a Banach space, which is invariant under translations and embedded between the space of Schwartz functions and the space of temperated distributions. Then we say that the wave-front set of a distribution contains all points (x_0, ξ_0) such that no localization of the distribution at x_0, belongs to $\mathscr{F}\mathscr{B}$ in the direction ξ_0. We prove that pseudo-differential operators with smooth symbols shrink the wave-front set and we obtain opposite embeddings by using sets of characteristic points of the operator symbols.

Keywords: Banach spaces, Fourier, Generalized time-dependent Schrödinger equation, Micro-local, Modulation spaces, Non-tangential convergence, Pseudo-differential operators, Regularity, Wave-front sets.

Sammanfattning

I denna avhandling diskuterar vi olika typer av regularitet för distributioner som uppkommer i teorin för pseudodifferentialoperatorer och partiella differentialekvationer. Partiella differentialekvationer förekommer inom naturvetenskap och teknik. Exempelvis kan Schrödingerekvationen användas för att beskriva förändringen med tiden av kvanttillstånd i fysikaliska system. Pseudodifferentialoperatorer kan användas för att lösa partiella differentialekvationer. De användas också för att modellera olika typer av problem inom fysik och teknik. Det finns till exempel en naturlig koppling mellan pseudodifferentialoperatorer och stationära och icke-stationära filter i signalbehandling. Vidare gäller att relationen mellan symboler och operatorer vid övergången från klassisk mekanik till kvantmekanik i huvudsak överensstämmer med symboler och operatorer inom Weylkalkylen för pseudodifferentialoperatorer.

I den här avhandlingen koncentrerar vi oss på att undersöka hur regularitetsegenskaper för lösningar till partiella differentialekvationer påverkas under verkan av pseudodifferentialoperatorer, och speciellt för de fria tidsberoende Schrödingeroperatorerna.

Lösningen av den fria tidsberoende Schrödingerekvationen kan uttryckas som en pseudodifferentialoperator, med icke-slät symbol, verkande på begynnelsevillkoret. Vi generaliserar ett resultat om icke-tangentiell konvergens av Sjögren och Sjölin i [17] för den fria tidsberoende Schrödingerekvationen.

Ett annat sätt att beskriva regularitet hos en distribution är med hjälp av vågfrontsmängder. De beskriver inte bara var singulariteterna finns, utan också i vilka riktningar dessa singulariteter förekommer. De första typerna av vågfrontsmängder (analytiska vågfrontsmängder) introducerades av Sato i [15, 16]. Senare introducerade Hörmander "klassiska" vågfrontsmängder (med avseende på släthet) och visade resultat för verkan av pseudodifferentialoperatorer med släta symboler, se [10].

I denna avhandling betraktar vi vågfrontsmängder med avseende på Fourier Banach funktionsrum. Detta kan ses som att vi låter $\mathscr{B}$ vara ett Banachrum, som är invariant under translationer och är inbäddat mellan rummet av Schwartzfunktioner och rummet av tempererade distributioner. Vågfrontsmängden av en distribution innehåller alla punkter (x_0, ξ_0) så att ingen lokalisering av distributionen kring x_0, tillhör $\mathscr{F}\mathscr{B}$ i riktningen ξ_0. Vi visar att pseudodifferentialoperatorer med släta symboler krymper vågfrontsmängden och vi får motsatta inbäddningar med hjälp mängder av karakteristiska punkter till operatorernas symboler.

Nyckelord: Banachrum, Fourier, Generaliserad tidsberoende Schrödinger ekvation, Icke-tangentiell konvergens, Mikrolokal, Modulationsrum, Pseudodifferentialoperatorer, Regularitet, Vågfrontsmängder.

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List of papers

I A counter example on nontangential convergence for oscillatory integrals, Karoline Johansson, Accepted for publication in *Publications de l'institute mathématique*

II Wave-front sets of Banach function types, Sandro Coriasco, Karoline Johansson and Joachim Toft, Preprint in ArXiv arXiv:0911.1867

Contents

Chapter 1

Introduction

In this thesis we consider regularity properties in the context of pseudo-differential operators. There are different ways to study regularity of distributions. For example, we may study such questions with respect to classical regularity (derivatives and their regularity) and regularity with respect to Sobolev spaces and modulation spaces. Regularity properties can be used to describe the behavior of the solution of an equation.

It was recognized around 1970 that singularities should be classified according to their spectrum. Thereby, wave-front sets were introduced to study propagation of singularities. The first types of wave-front sets (analytical wave-front sets) were introduced by Sato [15, 16]. Later on Hörmander introduced "classical" wave-front sets with respect to smoothness (cf. Sections 8.1 and 8.2 in [10]) and wave-front sets of Sobolev type in [11]. Hörmander contributed with results about properties of wave-front sets in the context of pseudo-differential operators. More precisely, it was proved that pseudo-differential operators with smooth symbols do not increase the wave-front sets. Opposite embeddings were obtained by including sets of characteristic points of the operator symbols.

In view of harmonic analysis, the theory of pseudo-differential operator contains all linear and continuous operators from $\mathscr{S}$ to $\mathscr{S}'$. For example the generalized time-dependent Schrödinger propagator can be viewed as a pseudo-differential operator. Besides using wave-front sets, propagation of singularities can be studied using non-tangential convergence. In the first paper we study non-tangential convergence for the generalized free time-dependent Schrödinger propagator. More precisely, the first paper concerns a counter-example of non-tangential convergence for the solutions u to a generalized free time-dependent Schrödinger equation, of the form

$$(\varphi(D_x) + i\partial_t)u = 0, \tag{1.1}$$

with the initial condition $u(x, 0) = f(x)$. For these equations, u converges along almost every straight line as the time t goes to zero. Here $D_x = -i\partial_x$, the function φ should satisfy some growth properties and f should belong to the Sobolev space $H^{d/2}$. The function φ should for example be real-valued and the first radial derivative should tend to infinity as the radius turns to infinity. Furthermore, the second derivative should not grow to fast compared to the first derivative. The exact conditions for φ are explicitly specified in the first paper. Some examples of functions that satisfies these conditions are $\varphi(\xi) = |\xi|^a$ where $a > 1$ and any finite linear combination of terms $|\xi|^{a_i}$ where $\max_i a_i > 1$. In particular, if $\varphi(\xi) = |\xi|^2$, then (1.1) is the free time-dependent Schrödinger equation. In this case, the result about non-existence of non-tangential convergence in the first paper was proved by Sjögren and Sjölin in [17].

More precisely, the counter-example in the first paper says that there exists f in $H^{d/2}$ and a sequence $\{(x_m, t_m)\}$ where t_m goes to zero, such that for every point x there exists a subsequence $\{x_{n_j}\}$ of $\{x_m\}$ such that x_{n_j} goes to x as j turns to infinity, but where the solution to (1.1) does not converge to the initial condition along this sequence. In particular, this subsequence $\{x_{n_j}\}$ can be chosen such that (x_{n_j}, t_{n_j}) is arbitrarily close to the vertical line through $(x, 0)$. Therefore, we can not consider regions of convergence for the solution of equation (1.1) with the specific initial conditions given in the first paper.

Regularity properties for the solution to the free time-dependent Schrödinger equation and generalizations of the Schrödinger equation have been investigated in several papers, see for example Bourgain [1], Kenig, Ponce and Vega [12] and Sjölin [18].

For appropriate f the solution of equation (1.1) with initial condition $u(x, 0) = f(x)$ can be written as

$$u(x,t) = S^{\varphi} f(x,t) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} e^{i\langle x,\xi\rangle} e^{it\varphi(\xi)} \widehat{f}(\xi)\, d\xi. \tag{1.2}$$

We may rewrite this in the following way

$$\begin{aligned} S^{\varphi} f(x,t) = (2\pi)^{-d/2} & \int e^{it\varphi(\xi)} \widehat{f}(\xi) e^{i\langle x,\xi\rangle}\, d\xi \\ &= (2\pi)^{-d} \iint e^{it\varphi(\xi)} f(y) e^{i\langle x-y,\xi\rangle}\, dy d\xi. \end{aligned} \tag{1.3}$$

Let $a(x, \xi) = e^{it\varphi(\xi)}$. Then, by (1.3) it follows that

$$S^{\varphi} f(x,t) = (2\pi)^{-d} \iint a(x,\xi) f(y) e^{i\langle x-y,\xi\rangle}\, dy d\xi. \tag{1.4}$$

The right-hand side of equation (1.4), is called a pseudo-differential operator with symbol $a(x, \xi)$ and is denoted $\mathrm{Op}(a)$ or $a(x, D)$.

In next section we give a more rigorous description of pseudo-differential operators.

1.1 Pseudo-differential operators

The calculus of pseudo-differential operators was founded by Kohn and Nirenberg [13] and Hörmander [10]. These operators were originally designed to study elliptic problems, and developed from the theory of singular integral operators.

Let $a \in \mathscr{S}(\mathbf{R}^{2d})$, and $\theta \in \mathbf{R}$ be fixed. Then the pseudo-differential operator $\operatorname{Op}_\theta(a)$, is the linear and continuous operator on $\mathscr{S}(\mathbf{R}^d)$, defined by the formula

$$(\operatorname{Op}_\theta(a)f)(x) = (2\pi)^{-d} \iint a((1-\theta)x + \theta y, \xi) f(y) e^{i\langle x-y,\xi\rangle}\, dy d\xi. \tag{1.5}$$

For general $a \in \mathscr{S}'(\mathbf{R}^{2d})$, the pseudo-differential operator $\operatorname{Op}_\theta(a)$ is defined as the continuous operator from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$ with distribution kernel

$$K_{\theta,a}(x,y) = (2\pi)^{-d/2}(\mathscr{F}_2^{-1}a)((1-\theta)x + \theta y, x-y).$$

Here $\mathscr{F}_2 F$ is the partial Fourier transform of $F(x,y) \in \mathscr{S}'(\mathbf{R}^{2d})$ with respect to the y-variable. This definition makes sense, since the mappings $\mathscr{F}_2$ and

$$F(x,y) \mapsto F((1-\theta)x + \theta y, x-y)$$

are homeomorphisms on $\mathscr{S}'(\mathbf{R}^{2d})$. We also note that the latter definition of $\operatorname{Op}_\theta(a)$ agrees with the operator in (1.5) when $a \in \mathscr{S}(\mathbf{R}^{2d})$, since

$$\begin{aligned}
(2\pi)^{-d} &\iint a((1-\theta)x + \theta y, \xi) f(y) e^{i\langle x-y,\xi\rangle}\, dy d\xi \\
&= (2\pi)^{-d} \int \Big(\int a((1-\theta)x + \theta y, \xi) e^{i\langle x-y,\xi\rangle}\, d\xi \Big) f(y)\, dy \\
&= (2\pi)^{-d/2} \int (\mathscr{F}_2^{-1}a)((1-\theta)x + \theta y, x-y) f(y)\, dy = (\operatorname{Op}_\theta(a)f)(x).
\end{aligned}$$

If $\theta = 0$, then $\operatorname{Op}_\theta(a)$ agrees with the Kohn-Nirenberg representation $\operatorname{Op}(a) = a(x,D)$ (see also the right-hand side of (1.4)) and if $\theta = 1/2$, then $\operatorname{Op}_\theta(a)$ agrees with the Weyl quantization $\operatorname{Op}_{1/2}(a) = a^w(x,D)$

Next we discuss an appropriate class of smooth symbols. Let $r, \rho, \delta \in \mathbf{R}$ be fixed and $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. Then we recall from [10] that $S^r_{\rho,\delta}(\mathbf{R}^{2d})$ is the set of all $a \in C^\infty(\mathbf{R}^{2d})$ such that for each pair of multi-indices α and β, there is a constant $C_{\alpha,\beta}$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta} \langle \xi \rangle^{r - \rho|\beta| + \delta|\alpha|}.$$

Usually we assume that $0 \le \delta < \rho \le 1$.

More generally, let ω be a polynomially moderated weight. That is,

$$\omega(x+y) \leq C\omega(x)v(y) \tag{1.6}$$

for every $x, y \in \mathbf{R}^d$, some polynomial v and some positive constant C which is independent of x and y. Then $S^{(\omega)}_{\rho,\delta}(\mathbf{R}^{2d})$ consists of all $a \in C^\infty(\mathbf{R}^{2d})$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta}\, \omega(x,\xi)\langle\xi\rangle^{-\rho|\beta|+\delta|\alpha|}. \tag{1.7}$$

A different type of regularity properties is considered in the second paper by use of pseudo-differential operators with smooth symbols.

1.2 Wave-front sets

Wave-front sets can be defined with respect to different spaces. The classical wave-front set $\mathrm{WF}(f)$ of a distribution f consists of all (x_0, ξ_0) such that there is no localisation of f at x_0 which belongs to C^∞ at the ξ_0 direction. (Cf. Hörmander [10].)

Wave-front sets with respect to Sobolev spaces were introduced by Hörmander in [11] and wave-front sets with respect to Fourier Lebesgue spaces were introduced by Pilipovic, Teofanov and Toft in [14]. In the second paper we introduce wave-front sets with respect to Fourier translation invariant Banach function spaces, in a similar way. Other generalizations of the wave-front sets given by Hörmander in [10] are the global wave-front sets with respect to $\mathscr{S}$ introduced by Coriasco and Manicia in [2]. In [8] Garello and Morando consider micro-local properties with respect to Sobolev and Hölder spaces.

The wave-front sets of Fourier Banach function types, introduced in the second paper, are a family of wave-front sets which contains the wave-front sets of Sobolev type (cf. Hörmander [11]) as well as the classical wave-front sets with respect to smoothness (cf. Sections 8.1 and 8.2 in [10]) and wave-front sets with respect to Fourier Lebesgue spaces (cf. [14]) as special cases. Roughly speaking, the wave-front set $\mathrm{WF}_{\mathscr{F}\!\mathscr{B}(\omega)}(f)$ of a distribution f on $\mathbf{R}^d$ with respect to the (weighted) Fourier Banach space

$$\mathscr{F}\!\mathscr{B}(\omega) = \{\, f \in \mathscr{S}'(\mathbf{R}^d)\, ;\, \widehat{f}(\xi)\omega(\xi) \in \mathscr{B}\,\}$$

with $\mathscr{B}$ as a translation invariant Banach function space and weight function $\omega \in L^\infty_{loc}(\mathbf{R}^d)$, consists of all pairs (x_0, ξ_0) in $\mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$ such that no localizations of the distribution at x_0 belongs to $\mathscr{F}\!\mathscr{B}(\omega)$ in the direction ξ_0.

By straight-forward calculations we see that

$$\mathrm{WF}_{\mathscr{F}\!\mathscr{B}}(f) \subseteq \mathrm{WF}(f), \tag{1.8}$$

for the Banach function spaces that we define in the next section.

Next we give a more rigorous description of the Banach function spaces and some of their properties.

Banach function spaces

We consider translation invariant Banach function spaces (cf. [3–6]), which occur frequently in Paper II. We let $\mathscr{P}(\mathbf{R}^d)$ be the set of all polynomially moderated weights. (See (1.6) in Section 1.1.) If (1.6) holds with $\omega = v$, then v is called submultiplicative. For convenience we assume that every submultiplicative weight is even. Furthermore, $\mathscr{S}$ and $\mathscr{S}'$ are the Schwartz space and the space of temperated distributions respectively. (For definitions see e.g. [10].)

Definition 1.2.1 *Assume that $\mathscr{B}$ is a Banach space of complex-valued measurable functions on $\mathbf{R}^d$ and that $v \in \mathscr{P}(\mathbf{R}^d)$ is submultiplicative. Then $\mathscr{B}$ is called a (translation) invariant Banach function space (BF-space) on $\mathbf{R}^d$ (with respect to v), if there is a constant C such that the following conditions are fulfilled:*

1. *$\mathscr{S}(\mathbf{R}^d) \subseteq \mathscr{B} \subseteq \mathscr{S}'(\mathbf{R}^d)$ (continuous embeddings).*

2. *if $x \in \mathbf{R}^d$ and $f \in \mathscr{B}$, then $f(\cdot - x) \in \mathscr{B}$, and*

$$\|f(\cdot - x)\|_{\mathscr{B}} \le Cv(x)\|f\|_{\mathscr{B}}.$$

3. *if $f, g \in L^1_{loc}(\mathbf{R}^d)$ satisfy $g \in \mathscr{B}$ and $|f| \le |g|$ almost everywhere, then $f \in \mathscr{B}$ and*

$$\|f\|_{\mathscr{B}} \le C\|g\|_{\mathscr{B}}.$$

In particular, we see that any Lebesgue space L^p, is a translation invariant Banach function space, since the conditions in Definition 1.2.1 are satisfied for $v(x) \equiv 1$ and $C = 1$. In fact, we note that any mixed Lebesgue space $L^{p,q}$ is a translation invariant Banach function space.

Next we present some useful properties. Let $\mathscr{B}$ be a translation invariant BF-space. If $f \in \mathscr{B}$ and $h \in L^\infty$, then it follows from (3) in Definition 1.2.1 that $f \cdot h \in \mathscr{B}$ and

$$\|f \cdot h\|_{\mathscr{B}} \le C\|f\|_{\mathscr{B}}\|h\|_{L^\infty}.$$

The weighted Lebesgue space $L^p_{(v)}$ consists of all $f \in \mathscr{S}'(\mathbf{R}^d)$ such that

$$\|f\|_{L^p_{(v)}} = \|fv\|_{L^p}$$

is finite.

Let $\mathscr{B}$ be a translation invariant BF-space with respect to the submultiplicative weight v on $\mathbf{R}^d$. Then the convolution map $*$ on $\mathscr{S}(\mathbf{R}^d)$ extends to a continuous mapping from $\mathscr{B} \times L^1_{(v)}(\mathbf{R}^d)$ to $\mathscr{B}$, and for some constant C it holds

$$\|\varphi * f\|_{\mathscr{B}} \le C\|\varphi\|_{L^1_{(v)}}\|f\|_{\mathscr{B}}, \tag{1.9}$$

when $\varphi \in L^1_{(v)}(\mathbf{R}^d)$ and $f \in \mathscr{B}$. In fact, if $f, g \in \mathscr{S}$, then $f * g \in \mathscr{S} \subseteq \mathscr{B}$ in view of the definitions, and Minkowski's inequality gives

$$\begin{aligned}\|f * g\|_{\mathscr{B}} &= \Big\| \int f(\,\cdot\, - y)g(y)\, dy \Big\|_{\mathscr{B}} \\ &\le \int \|f(\,\cdot\, - y)\|_{\mathscr{B}} |g(y)|\, dy \le C \int \|f\|_{\mathscr{B}} |g(y)v(y)|\, dy = C\|f\|_{\mathscr{B}} \|g\|_{L^1_{(v)}}.\end{aligned}$$

Since $\mathscr{S}$ is dense in $L^1_{(v)}$, it follows that $\varphi * f \in \mathscr{B}$ when $\varphi \in L^1_{(v)}$ and $f \in \mathscr{S}$, and that (1.9) holds in this case. The result is now a consequence of Hahn-Banach's theorem.

From now on we assume that each translation invariant BF-space $\mathscr{B}$ is such that the convolution map $*$ on $\mathscr{S}(\mathbf{R}^d)$ is *uniquely* extendable to a continuous mapping from $\mathscr{B} \times L^1_{(v)}(\mathbf{R}^d)$ to $\mathscr{B}$, and that (1.9) holds when $\varphi \in L^1_{(v)}(\mathbf{R}^d)$ and $f \in \mathscr{B}$. We note that $\mathscr{B}$ can be any mixed and weighted Lebesgue space.

Assume that $\mathscr{B}$ is a translation invariant BF-space and that $\omega \in \mathscr{P}(\mathbf{R}^d)$. Let the Fourier Banach space $\mathscr{F}\mathscr{B}(\omega)$ be the set of all $f \in \mathscr{S}'(\mathbf{R}^d)$ such that $\xi \mapsto \widehat{f}(\xi)\omega(\xi)$ belongs to $\mathscr{B}$. It follows that $\mathscr{F}\mathscr{B}(\omega)$ is a Banach space with the norm

$$\|f\|_{\mathscr{F}\mathscr{B}(\omega)} \equiv \|\widehat{f}\,\omega\|_{\mathscr{B}}.$$

Here $\widehat{f}$ denotes the Fourier transform.

We remark that Fourier Lebesgue spaces are special cases of the Fourier BF-spaces. Also the Sobolev space, used in the first paper can be expressed as a (weighted) Fourier BF-space. More precisely, let $\mathscr{B} = L^2$ and $\omega = \langle \xi \rangle^s$ where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. Then

$$\begin{aligned}\|f\|^2_{H^s} &= \int_{\mathbf{R}^d} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2\, d\xi \\ &= \int_{\mathbf{R}^d} (\langle \xi \rangle^s |\widehat{f}(\xi)|)^2\, d\xi = \|f\|^2_{\mathscr{F}L^2_{(\omega)}} = \|f\|^2_{\mathscr{F}\mathscr{B}(\omega)}.\end{aligned}$$

Wave-front sets and pseudo-differential operators with smooth symbols

As mentioned before, there are well-known mapping properties for pseudo-differential operators on classical wave-front sets (cf. Chapters VIII and

XVIII in [10]) and for the wave-front sets of Fourier Lebesgue types (cf. [14]). For example such operators shrink the wave-front sets and opposite embeddings can be obtained by using sets of characteristic points of the operator symbols. More precisely, we have that

$$\mathrm{WF}(\mathrm{Op}(a)f) \subseteq \mathrm{WF}(f) \subseteq \mathrm{WF}(\mathrm{Op}(a)f) \cup \mathrm{Char}_{(\omega_0)}(a).$$

Here $\mathrm{Char}_{(\omega_0)}(a)$ is the set of characteristic points of the symbol a with respect to the weight function ω_0. We have that $(x_0,\xi_0) \notin \mathrm{Char}_{(\omega_0)}(a)$, if and only if, there exist a neighbourhood X of x_0, an open conical neighbourhood Γ_{ξ_0} of ξ_0 and positive constants R and c such that

$$|a(x,\xi)| \geq c\,\omega_0(x,\xi),$$

for every $(x,\xi) \in X \times \Gamma_{\xi_0}$, $|\xi| \geq R$.

The set of characteristic points, which we consider here, can be carefully adjusted to the symbol classes for the pseudo-differential operators, i. e. the set of characteristic points depend on the choices of ρ, δ and ω_0 in $S^{(\omega_0)}_{\rho,\delta}$. This set of points is the same as in [14], however in contrast to Section 18.1 in [10], it is defined for symbols which are not polyhomogenous. In the case of polyhomogenous symbols, $\mathrm{Char}_{(\omega_0)}(a)$ is smaller than the sets of characteristic points in [10] (see Example 3.9 in [14]).

In the second paper we prove that the same properties hold for wave-front sets of Fourier BF-space type, i.e. we prove that

$$\begin{aligned}\mathrm{WF}_{\mathscr{FB}(\omega/\omega_0)}(\mathrm{Op}_t(a)f) &\subseteq \mathrm{WF}_{\mathscr{FB}(\omega)}(f)\\ &\subseteq \mathrm{WF}_{\mathscr{FB}(\omega/\omega_0)}(\mathrm{Op}_t(a)f) \cup \mathrm{Char}_{(\omega_0)}(a). \qquad (1.10)\end{aligned}$$

In the following example we construct embeddings of the wave-front sets of the solution to the equation (1.1), for which we considered non-tangential convergence in the first paper, by using (1.10).

Example

It follows by straight-forward calculations that general a in (1.4) does not satisfy the conditions of smooth symbols given above. Therefore, we can not use the results obtained in the second paper on such pseudo-differential operator in the most general setting. On the other hand, we may combine techniques and results in the second paper with the choice of φ such that the left hand-side of (1.1) is a pseudo-differential operator with symbol in $S^{(\omega_0)}_{\rho,\delta}$ to obtain results concerning the solution of equation (1.1). By letting

$$a(x,t,\xi,\tau) = \varphi(\xi) - \tau \qquad (1.11)$$

it follows that

$$a(x,t,D_x,D_t) = \varphi(D_x) + i\partial_t.$$

Hence the equation (1.1) can be written as follows

$$\mathrm{Op}(a)u = 0.$$

First we consider the free time-dependent Schrödinger equation, namely, we assume that $\varphi(\xi) = |\xi|^2$. For such choice of φ we let

$$\omega_0(x,t,\xi,\tau) = (1 + |\xi|^2 + |\tau|).$$

We notice that a is a smooth symbol and that $a \in S^{(\omega_0)}_{\rho,0}(\mathbf{R}^{2d+2})$, where $0 < \rho < 1/2$. This follows by straight-forward derivations, and the fact that every derivative of a can be estimated by (1.7), where ω is replaced by ω_0.

The set of characteristic points of a consists of all points (x_0, ξ_0) such that there is no open neighbourhood X of (x_0, t_0), no open conical neighbourhood Γ of (ξ_0, τ_0) and positive constants R and c such that

$$|a(x,t,\xi,\tau)| \geq c\,\omega_0(x,t,\xi,\tau), \qquad \text{for all } (x,t,\xi,\tau) \in X \times \Gamma,\ |(\xi,\tau)| \geq R. \tag{1.12}$$

We see that $a(x,t,\xi,\tau) = 0$ for $|\xi|^2 = \tau$. This gives

$$\mathrm{Char}_{(\omega_0)}(a) = \{(x,t,0,\tau) \in \mathbf{R}^{2d+2};\ (x,t) \in \mathbf{R}^{d+1},\ \tau \in \mathbf{R} \setminus 0\}.$$

For a solution u to the free time-dependent Schrödinger equation we have that

$$\mathrm{WF}_{\mathscr{F}\mathscr{B}(\omega/\omega_0)}(\mathrm{Op}(a)u) = \mathrm{WF}_{\mathscr{F}\mathscr{B}(\omega/\omega_0)}(0) = \emptyset$$

and in view of the results given in the second paper we get

$$\mathrm{WF}_{\mathscr{F}\mathscr{B}(\omega)}(u) \subseteq \{(x,t,0,\tau) \in \mathbf{R}^{2d+2};\ (x,t) \in \mathbf{R}^{d+1},\ \tau \in \mathbf{R} \setminus 0\}.$$

For more general symbols of the form in (1.11) we see that similar calculations as above would hold by replacing $|\xi|^2$ by $\varphi(\xi)$. However, the inclusions of wave-front sets in the context of pseudo-differential operators, which we establish in the second paper, only hold for pseudo-differential operators with symbols in $S^{(\omega_0)}_{\rho,\delta}$. Therefore, the results above concerning the wave-front sets of the solution of the free time-dependent Schrödinger equation can only be generalized to solutions of equation (1.1) in the case were $a \in S^{(\omega_0)}_{\rho,0}(\mathbf{R}^{2d+2})$ for some $\rho \in (0,1]$ and

$$\omega_0(x,t,\xi,\tau) = (1 + |\varphi(\xi)| + |\tau|)$$

is a polynomially moderated weight. We note that for example every choice of the function φ as $\varphi(\xi) = |\xi|^s$, where $s \in \mathbf{N}$ and any finite linear combination of such functions satisfies these conditions, by choosing ρ sufficiently small. It then follows that

$$\mathrm{WF}_{\mathscr{F}\mathscr{B}(\omega)}(u) \subseteq \mathrm{Char}_{(\omega_0)}(a) \subseteq \{(x,t,0,\tau) \in \mathbf{R}^{2d+2};\ (x,t) \in \mathbf{R}^{d+1},\ \tau \in \mathbf{R}\backslash 0\}.$$

By replacing the Fourier BF-space with modulation spaces we can define wave-front sets of modulation space types. In the second paper we prove that the families of wave-front sets of Fourier BF-space types and wave-front sets of modulation space types coincide. Therefore, the embedding properties described above also hold for these wave-front sets.

Modulation spaces

The modulation spaces were introduced by Feichtinger in [4], and the theory was developed further and generalized in [5–7, 9]. The modulation space $M(\omega,\mathscr{B})$, where ω denotes a weight function on phase (or time-frequency shift) space $\mathbf{R}^{2d}$, appears as the set of temperated (ultra-) distributions whose short-time Fourier transform belong to the weighted BF-space $\mathscr{B}(\omega)$.

Roughly speaking, the modulation space norm gives information about the distributions time-frequency content, since the short-time Fourier transform contains local properties on both times and frequencies.

Let (the window) $\phi \in \mathscr{S}'(\mathbf{R}^d)\setminus 0$ be fixed, and let $f \in \mathscr{S}'(\mathbf{R}^d)$. Then the short-time Fourier transform $V_\phi f$ is the element in $\mathscr{S}'(\mathbf{R}^{2d})$, defined by the formula

$$(V_\phi f)(x,\xi) \equiv \mathscr{F}(f\cdot\overline{\phi(\cdot - x)})(\xi).$$

We usually assume that $\phi \in \mathscr{S}(\mathbf{R}^d)$, and in this case the short-time Fourier transform $(V_\phi f)$ takes the form

$$(V_\phi f)(x,\xi) = (2\pi)^{-d/2}\int_{\mathbf{R}^d} f(y)\overline{\phi(y-x)}e^{-i\langle y,\xi\rangle}\,dy,$$

when $f \in \mathscr{S}(\mathbf{R}^d)$.

Now let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d}$, with respect to $v \in \mathscr{P}(\mathbf{R}^{2d})$. Also let $\phi \in \mathscr{S}(\mathbf{R}^d)\setminus 0$ and $\omega \in \mathscr{P}(\mathbf{R}^{2d})$ be such that ω is v-moderate. The modulation space $M(\omega,\mathscr{B})$ consists of all $f \in \mathscr{S}'(\mathbf{R}^d)$ such that $V_\phi f\cdot\omega \in \mathscr{B}$. We note that $M(\omega,\mathscr{B})$ is a Banach space with the norm

$$\|f\|_{M(\omega,\mathscr{B})} \equiv \|V_\phi f\omega\|_{\mathscr{B}}$$

(cf. [5]).

These types of modulation spaces contains the (classical) modulation spaces $M^{p,q}_{(\omega)}$ as well as the space $W^{p,q}_{(\omega)}$ related to the Wiener amalgam spaces, by choosing $\mathscr{B} = L_1^{p,q}(\mathbf{R}^{2d})$ and $\mathscr{B} = L_2^{p,q}(\mathbf{R}^{2d})$ respectively. Here $L_1^{p,q}$ ($L_2^{p,q}$), where $p, q \in [1, \infty]$ consists of all $F \in L^1_{\mathrm{loc}}(\mathbf{R}^{2d})$ such that

$$\|F\|_{L_1^{p,q}} \equiv \Big(\int \Big(\int |F(x,\xi)|^p \, dx \Big)^{q/p} \, d\xi \Big)^{1/q} < \infty$$

$$\Big(\|F\|_{L_2^{p,q}} \equiv \Big(\int \Big(\int |F(x,\xi)|^p \, d\xi \Big)^{q/p} \, dx \Big)^{1/q} < \infty \Big).$$

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Chapter 2

Papers

- I A counter example on nontangential convergence for oscillatory integrals
- II Wave-front sets of Banach function types

Paper I

2.1 A counter example on nontangential convergence for oscillatory integrals

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A counter example on nontangential convergence for oscillatory integrals

Karoline Johansson

Abstract. Consider the solution of the time-dependent Schrödinger equation with initial data f. It is shown by Sjögren and Sjölin (1989) that there exists f in the Sobolev space $H^s(\mathbf{R}^n)$, $s = n/2$ such that tangential convergence can not be widened to convergence regions. In this paper we show that the corresponding result holds when $-\Delta_x$ is replaced by an operator $\varphi(D)$, with special conditions on φ.

1. Introduction

In this paper we establish non-existence results of non-tangential convergence for the solution $u = S^\varphi f$ to the generalized time-dependent Schrödinger equation

$$(\varphi(D) + i\partial_t)u = 0, \tag{1.1}$$

with the initial condition

$$u(x, 0) = f(x).$$

Here φ is real-valued, and its radial derivatives of first and second orders ($\varphi' = \varphi'_r$ and $\varphi'' = \varphi''_{rr}$) are continuous, outside a compact set containing origin, and fulfill appropriate growth conditions. In particular $\varphi(\xi) = |\xi|^a$ will satisfy these conditions, for $a > 1$.

For $\varphi(\xi) = |\xi|^2$ we recover Theorem 3 in [**3**] where Sjögren and Sjölin proved that there exists a function f in the Sobolev space $H^{n/2}$ such that near the vertical line $t \mapsto (x, t)$ through an arbitrary point $(x, 0)$ there are points accumulating at $(x, 0)$ such that the solution of equation (1.1) takes values far from f. This means that the solution of the time-dependent Schrödinger equation with initial condition $u(x, 0) = f(x)$ does not converge non-tangentially to f. Therefore we can not consider regions of convergence. As a consequence of our results, it follows that Theorem 3 in [**3**] holds for any $a > 1$.

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In this paper, we prove that the property which holds for $\varphi(\xi) = |\xi|^2$ also holds for more general functions $\varphi(\xi)$ of the type described above. In the proof we use some ideas by Sjögren and Sjölin in [**3**] in combination with new estimates, to construct a counter example. Some ideas can also be found in Sjölin [**5, 6**] and Walther [**9, 10**], and some related results are given in Bourgain [**1**], Kenig, Ponce and Vega [**2**], and Sjölin [**4, 7**].

Existence of regions of convergence has been studied before for other equations. For example, Stein and Weiss consider in [**8**, Chapter II Theorem 3.16] Poisson integrals acting on Lebesgue spaces. These operators are related to the operator S^φ.

For an appropriate function φ on $\mathbf{R}^n$, let S^φ be the operator acting on functions f defined by

$$f \mapsto \mathcal{F}^{-1}(\exp(it\varphi(\xi))\mathcal{F}f), \tag{1.2}$$

where $\mathcal{F}f$ is the Fourier transform of f, which takes the form

$$\widehat{f}(\xi) = \mathcal{F}f(\xi) \equiv \int_{\mathbf{R}^n} e^{-ix\cdot\xi} f(x)\, dx,$$

when $f \in L^1(\mathbf{R}^n)$. This means that, if $\widehat{f}$ is an integrable function, then S^φ in (1.2) takes the form

$$S^\varphi f(x,t) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{ix\cdot\xi} e^{it\varphi(\xi)} \widehat{f}(\xi)\, d\xi, \quad x \in \mathbf{R}^n, \quad t \in \mathbf{R}.$$

If $\varphi(\xi) = |\xi|^2$ and f belongs to the Schwartz class $\mathcal{S}(\mathbf{R}^n)$, then $S^\varphi f$ is the solution to the time-dependent Schrödinger equation $(-\Delta_x + i\partial_t)u = 0$ with the initial condition $u(x,0) = f(x)$.

For more general appropriate φ, for which the equation (1.1) is well-defined, the expression $S^\varphi f$ is the solution to the generalized time-dependent Schrödinger equation (1.1) with the initial condition $u(x,0) = f(x)$. Note here that $S^\varphi f$ is well-defined for any real-valued measurable φ and $f \in \mathcal{S}$. On the other hand, it might be difficult to interpret (1.1) if for example $\varphi \notin L^1_{loc}$.

In order to state the main result we need to specify the conditions on φ and give some definitions. The function φ should satisfy the conditions

$$\liminf_{r\to\infty} \Big(\inf_{|\omega|=1} |\varphi'(r,\omega)| \Big) = \infty, \tag{1.3}$$

and

$$\sup_{r \geqslant R} \Big(\sup_{|\omega|=1} \frac{r^\beta |\varphi''(r,\omega)|}{|\varphi'(r,\omega)|^2} \Big) < C, \tag{1.4}$$

for some $\beta > 0$ and some constant C. Here $\varphi'(r\omega) = \varphi'(r,\omega)$ denotes the derivative of $\varphi(r,\omega)$ with respect to r, and similarly for higher orders of derivatives.

We let $H^s(\mathbf{R}^n)$ be the Sobolev space of distributions with $s \in \mathbf{R}$ derivatives in L^2. That is $H^s(\mathbf{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbf{R}^n)$ such that

$$\|f\|_{H^s(\mathbf{R}^n)} \equiv \Big(\int_{\mathbf{R}^n} (1+|\xi|^2)^s |\widehat{f}(\xi)|^2 \, d\xi \Big)^{1/2} < \infty. \tag{1.5}$$

Theorem 1.1. *Assume that the function $\gamma : \mathbf{R}_+ \to \mathbf{R}_+$ is strictly increasing and continuous such that $\gamma(0) = 0$. Let $R > 0$, and let φ be real-valued functions on $\mathbf{R}^n$ such that $\varphi'(r,\omega)$ and $\varphi''(r,\omega)$ are continuous and satisfy* (1.3) *and* (1.4) *when $r > R$. Then there exists a function $f \in H^{n/2}(\mathbf{R}^n)$ such that $S^\varphi f$ is continuous in $\{(x,t); t > 0\}$ and*

$$\limsup_{(y,t)\to(x,0)} |S^\varphi f(y,t)| = +\infty \tag{1.6}$$

for all $x \in \mathbf{R}^n$, where the limit superior is taken over those (y,t) for which $|y-x| < \gamma(t)$ and $t > 0$.

Here we recall that $\varphi' = \varphi'_r$ and $\varphi' = \varphi''_{rr}$ are the first and second orders radial derivatives of φ. When $s > n/2$ no counter example of the form in Theorem 1.1 can be provided, since $S^\varphi f(y,t)$ converges to $f(x)$ as (y,t) approaches $(x,0)$ non-tangentially when $f \in H^s(\mathbf{R}^n)$. In fact, Hölder's inequality gives

$$(2\pi)^n |S^\varphi f(x,t)| \leqslant \int_{\mathbf{R}^n} |\widehat{f}(\xi)| \, d\xi \leqslant \Big(\int_{\mathbf{R}^n} (1+|\xi|^2)^{-s} \, d\xi \Big) \|f\|_{H^s(\mathbf{R}^n)},$$

which is finite when $f \in H^s(\mathbf{R}^n)$, $s > n/2$. Therefore convergence along vertical lines can be extended to convergence regions when $s > n/2$ and f belongs to $H^s(\mathbf{R}^n)$.

2. Examples and remarks

In this section we give some examples of functions φ for which Theorem 1.1 holds. In the first example we let φ be a positively homogeneous function of order $a > 1$.

Example 2.1. Let $a > 1$ and $\varphi(\xi) = |\xi|^a$, then $S^\varphi f(x,t)$ is the solution to the generalized time-dependent Schrödinger equation $((-\Delta_x)^{a/2} + i\partial_t)u = 0$. By change of variables to polar coordinates and derivate with respect to r we see that $\varphi(r,\omega) = r^a$, $\varphi'(r,\omega) = ar^{a-1}$ and $\varphi''(r,\omega) = a(a-1)r^{a-2}$. We can see that these derivatives satisfy (1.3) and (1.4). In particular for $a = 2$ this is the solution to the time-dependent Schrödinger equation $(-\Delta_x + i\partial_t)u = 0$ and this case is treated in Sjögren and Sjölin [**3**].

In the following example we let φ be a sum of positively homogeneous functions where $a > 1$ denote the term of highest order.

Example 2.2. For $a > 1$, let

$$\varphi(\xi) = \sum_{i=1}^{d} |\xi|^{a_i} \varphi_{a,i}(1,\omega), \quad a_1 < \cdots < a_d = a,$$

where

$$\inf_{\omega} |\varphi_{a,d}(1,\omega)| = h > 0 \qquad \text{and} \qquad \|\varphi_{a,i}(1,\cdot)\|_{L^\infty(S^{n-1})} < \infty$$

for each $i \in \{1, 2, \dots, d\}$. Here S^{n-1} is the $n-1$-dimensional unit sphere. By rewriting this into polar coordinates and differentiate with respect to r, we see that in the first derivative the term $\varphi_{a,i}(1,\omega)r^{a-1}$ dominates the sum and that the second derivative can be estimated by Cr^{a-2}, for some constant C. These derivatives satisfy (1.3) and (1.4).

In the examples at the above we have used functions φ such that the modulus of the radial derivative is bounded from below by a positive homogeneous function of order $a-1$ for some $a > 1$. This condition is not necessary. The hypothesis in the theorem permits a broader class of functions φ. The following example shows that there are functions, which do not grow as fast as a positive homogeneous function of order $a-1$ for any $a > 1$, but satisfy the conditions (1.3) and (1.4).

Example 2.3. Let $\varphi(\xi) = |\xi| \log |\xi|$, then $\varphi'(r,\omega) = \log r + 1$ and $\varphi''(r,\omega) = r^{-1}$ and (1.3) and (1.4) are satisfied.

We also allow the dominant part of the derivative to grow faster than any positively homogeneous function as long as we have some restrictions on the second derivative. The conditions are given explicitly in (1.3) and (1.4). The following example contains such functions.

Example 2.4. Let $\varphi(\xi) = \varphi(r,\omega) = e^{\mu(\omega)r^b}$, where $0 < b \leqslant 2$ and $\inf_{|\omega|=1} \mu(\omega) = c > 0$. Here one should note that $\varphi \notin \mathcal{S}'(\mathbf{R}^n)$. These functions grow faster than r^a for all a and the same is true for the absolute value of the first and second derivative with respect to r. This can be used to show that (1.3) and (1.4) are satisfied.

3. Notations for the proof

In order to prove Theorem 1.1 we introduce some notations. Let $B_r(x)$ be the open ball in $\mathbf{R}^n$ with center at x and radius r. Numbers denoted by C, c or C' may be different at each occurrence. We let

$$\delta_k = \delta_{k,n} \equiv \gamma(1/(k+1))/\sqrt{n}, \qquad k \in \mathbf{N},$$

where γ is the same as in Theorem 1.1. Since γ is strictly increasing it is clear that $(\delta_k)_{k\in\mathbf{N}}$ is strictly decreasing. We also let $(x_j)_{j=1}^\infty \subset \mathbf{R}^n$ be chosen such that $x_1, x_2, \dots, x_{m_1}$ denotes all points in $B_1(0) \cap \delta_1 \mathbf{Z}^n$, $x_{m_1+1}, \dots, x_{m_2}$ denotes all points in $B_2(0) \cap \delta_2 \mathbf{Z}^n$ and generally

$$\{x_{m_k+1}, \dots, x_{m_{k+1}}\} = B_{k+1}(0) \cap \delta_{k+1} \mathbf{Z}^n, \qquad \text{for } k \geqslant 1.$$

Furthermore we choose a strictly decreasing sequence $(t_j)_1^\infty$ such that $1 > t_1 > t_2 > \cdots > 0$ and

$$\frac{1}{k+2} < t_j < \frac{1}{k+1}, \qquad k \in \mathbf{N},$$

for $m_k + 1 \leqslant j \leqslant m_{k+1}$.

In the proof of Theorem 1.1 we consider the function f_φ, which is defined by the formula

$$\widehat{f}_\varphi(\xi) = |\xi|^{-n}(\log|\xi|)^{-3/4}\sum_{j=1}^{\infty}\chi_j(\xi)e^{-i(x_j\cdot\xi+t_j\varphi(\xi))}, \tag{3.1}$$

where χ_j is the characteristic function of

$$\Omega_j = \{\xi \in \mathbf{R}^n; R_j < |\xi| < R'_j\}.$$

Here $(R_j)_1^\infty$ and $(R'_j)_1^\infty$ are sequences in $\mathbf{R}$ which fulfill the following conditions:

(1) $R_1 \geqslant 2 + R$, $R'_1 \geqslant R_1 + 1$, with R given by Theorem 1.1;

(2) $R'_j = R_j^N$ when $j \geqslant 2$, where N is a large positive number and independent of j, which is specified later on;

(3) $R_j < R'_j < R_{j+1}$, when $j \geqslant 1$;

(4)

$$|\varphi'(r,\omega)| > 1 \qquad \text{when} \qquad r \geqslant R; \tag{3.2}$$

(5) for $j \geqslant 2$

$$R_j^{\min(\beta,1)} > \max_{l<j}\frac{2^j}{t_l - t_j}, \tag{3.3}$$

where $\beta > 0$ is the same constant as in (1.4) and

$$\inf_{R_j \leqslant r \leqslant R'_j}(\inf_{|\omega|=1}|\varphi'(r,\omega)|) > \max_{l<j}\frac{2|x_l - x_j|}{t_l - t_j}; \tag{3.4}$$

Remark 3.1. The sequences $(R_j)_1^\infty$ and $(R'_j)_1^\infty$ can be chosen since φ satisfies condition (1.3).

Furthermore, in order to get convenient approximations of the operator S^φ, we let

$$S_m^\varphi f(x,t) = \frac{1}{(2\pi)^n}\int_{|\xi|<R'_m} e^{ix\cdot\xi}e^{it\varphi(\xi)}\widehat{f}(\xi)\,d\xi. \tag{3.5}$$

Then

$$S_m^\varphi f_\varphi(x,t) = \sum_{j=1}^{m} A_j^\varphi(x,t), \tag{3.6}$$

where

$$A_j^{\varphi}(x,t) = \frac{1}{(2\pi)^n}\int_{\Omega_j} e^{i(x-x_j)\cdot\xi}e^{i(t-t_j)\varphi(\xi)}|\xi|^{-n}(\log|\xi|)^{-3/4}\,d\xi. \tag{3.7}$$

By using polar coordinates we get

$$A_j^{\varphi}(x_k,t_k) = \frac{1}{(2\pi)^n}\int_{|\omega|=1}\Big\{\int_{R_j}^{R_j'}\frac{1}{r(\log r)^{3/4}}e^{iF_{\varphi}(r,\omega)}\,dr\Big\}\,d\sigma(\omega), \tag{3.8}$$

where

$$F_{\varphi}(r,\omega) = r(x_k-x_j)\cdot\omega + (t_k-t_j)\varphi(r,\omega),$$

and $d\sigma(\omega)$ is the euclidean surface measure on the $n-1$-dimensional unit sphere. By differentiation we get

$$F_{\varphi}'(r,\omega) = (x_k-x_j)\cdot\omega + (t_k-t_j)\varphi'(r,\omega) \tag{3.9}$$

and

$$F_{\varphi}''(r,\omega) = (t_k-t_j)\varphi''(r,\omega). \tag{3.10}$$

Here recall that $F_{\varphi}'(r\omega) = F_{\varphi}'(r,\omega)$ and $F_{\varphi}''(r,\omega)$ denote the first and second orders of derivatives of $F_{\varphi}(r,\omega)$ with respect to the r-variable.

By integration by parts in the inner integral of (3.8) we get

$$\int_{R_j}^{R_j'}\frac{1}{r(\log r)^{3/4}}e^{iF_{\varphi}(r,\omega)}\,dr = A_{\varphi} - B_{\varphi}, \tag{3.11}$$

where

$$A_{\varphi} = \Big[\frac{e^{iF_{\varphi}(r,\omega)}}{r(\log r)^{3/4}iF_{\varphi}'(r,\omega)}\Big]_{R_j}^{R_j'} \tag{3.12}$$

and

$$B_{\varphi} = \int_{R_j}^{R_j'}\frac{d}{dr}\Big(\frac{1}{r(\log r)^{3/4}iF_{\varphi}'(r,\omega)}\Big)e^{iF_{\varphi}(r,\omega)}\,dr \tag{3.13}$$

4. Proofs

In this section we prove Theorem 1.1. We need some preparing lemmas for the proof. In the following lemma we prove that for fixed $x \in B_k(0)$ there exists sequences $(x_{n_j})_1^{\infty}$ and $(t_{n_j})_1^{\infty}$ such that

$$x_{n_j} \in \{x_{m_k+1},\dots,x_{m_{k+1}}\}, \qquad \text{and} \qquad t_{n_j} \in \{t_{m_k+1},\dots,t_{m_{k+1}}\}$$

and $|x_{n_j}-x| < \gamma(t_{n_j})$.

Lemma 4.1. *Let $x \in \mathbf{R}^n$ be fixed. Then for each $k \geqslant |x|$ there exists $x_{n_j} \in \{x_{m_k+1},\dots,x_{m_{k+1}}\}$ and $t_{n_j} \in \{t_{m_k+1},\dots,t_{m_{k+1}}\}$ such that $|x_{n_j}-x| < \gamma(t_{n_j})$. In particular $(x_{n_j},t_{n_j}) \to (x,0)$ as j turns to infinity.*

PROOF. For each $k \geqslant |x|$, x belongs to a cube with vertices in $T_k = B_{k+1}(0) \cap \delta_{k+1}\mathbf{Z}^n$ and side lengths $\gamma(1/(k+2))/\sqrt{n}$. Take a vertex x' in the cube and its diagonal $\gamma(1/(k+2))$ as center and radius of a ball respectively. This ball $B_{\gamma(1/(k+2))}(x')$ contains the whole cube and hence also x. Therefore there exists x_{n_j} for every $k \geqslant |x|$ such that $x \in B_{\gamma(1/(k+2))}(x_{n_j}) \subset B_{\gamma(t_{n_j})}(x_{n_j})$. This proves the first part of the assertion, and the second statement follows from the fact that $\gamma(0) = 0$ and γ is continuous and strictly increasing. □

We want to prove that f_φ in (3.1) belongs to $H^{n/2}(\mathbf{R}^n)$ and fulfill (1.6). The former relation is a consequence of Lemma 4.2 below, which concerns Sobolev space properties for functions of the form

$$\widehat{g}(\xi) = |\xi|^{-n}(\log|\xi|)^{-\rho/2}\sum_{j=1}^{\infty}\chi_j(\xi)b_j(\xi), \tag{4.1}$$

where χ_j is the characteristic function on disjoint sets Ω_j.

LEMMA 4.2. *Assume that $\rho > 1$, Ω_j for $j \in \mathbf{N}$ are disjoint open subsets of $\mathbf{R}^n \backslash B_\rho(0)$, $b_j \in L^1_{loc}(\mathbf{R}^n)$ for $j \in \mathbf{N}$ satisfies*

$$\sup_{j\in\mathbf{N}} \|b_j\|_{L^\infty(\Omega_j)} < \infty,$$

and let χ_j be the characteristic function for Ω_j. If g is given by (4.1), then $g \in H^{n/2}(\mathbf{R}^n)$.

PROOF. By estimating (1.5) for the function g we get that

$$\begin{aligned}\int_{\mathbf{R}^n} |\widehat{g}(\xi)|^2(1+|\xi|^2)^{n/2}\,d\xi &\leqslant C\int_{\mathbf{R}^n\smallsetminus B_\rho(0)} |\xi|^{-2n}(\log|\xi|)^{-\rho}(1+|\xi|^2)^{n/2}\,d\xi \\ &\leqslant 2^{n/2}C\int_\rho^\infty \frac{1}{r(\log r)^\rho}\,dr < \infty.\end{aligned}$$

The second inequality holds since $(1+r^2)^{n/2} < (r^2+r^2)^{n/2} = 2^{n/2}r^n$ for $r > 1$. □

In the following lemma we give estimates of the expression A_j^φ.

LEMMA 4.3. *Let $A_j^\varphi(x,t)$ be given by (3.7). Then the following is true:*

(1) $\displaystyle\sum_{j=1}^{k-1}|A_j^\varphi(x,t)| \leqslant C(\log R'_{k-1})^{1/4}$, *with C independent of k;*

(2) $A_k^\varphi(x_k,t_k) > c(\log R'_k)^{1/4}$, *with $c > 0$ independent of k.*

PROOF. (1) By triangle inequality and the fact that $|\xi| > 2$, when $\xi \in \Omega_j$, we get

$$\sum_{j=1}^{k-1} |A_j^\varphi(x,t)| \leqslant \frac{1}{(2\pi)^n} \int_{2\leqslant|\xi|\leqslant R'_{k-1}} |\xi|^{-n}(\log|\xi|)^{-3/4}\, d\xi$$
$$= C\int_2^{R'_{k-1}} \frac{1}{r(\log r)^{3/4}}\, dr \leqslant C(\log R'_{k-1})^{1/4},$$

where C is independent of k. In the last equality we have taken polar coordinates as new variables of integration.

(2) Since $R_j^N = R'_j$ for sufficiently large N, we get

$$A_k^\varphi(x_k,t_k) = C\int_{R_k}^{R'_k} \frac{1}{r(\log r)^{3/4}}\, dr$$
$$= C\Big((\log R'_k)^{1/4} - (\log(R'_k)^{1/N})^{1/4}\Big)$$
$$= C\Big(1 - \frac{1}{N^{1/4}}\Big)(\log R'_k)^{1/4} > c(\log R'_k)^{1/4},$$

for some constant $c > 0$, which is independent of k. □

LEMMA 4.4. *Assume that $S_m^\varphi f_\varphi$ is given by* (3.5). *Then $S_m^\varphi f_\varphi$ is continuous on $\{(x,t); t > 0, x \in \mathbf{R}^n\}$.*

PROOF. The continuity for each $S_m^\varphi f_\varphi$ follows from the facts, that for almost every $\xi \in \mathbf{R}^n$, the map

$$(x,t) \mapsto e^{ix\cdot\xi}e^{it\varphi(\xi)}\widehat{f}_\varphi(\xi)$$

is continuous, and that

$$\int_{|\xi|<R'_m} |e^{ix\cdot\xi}e^{it\varphi(\xi)}\widehat{f}_\varphi(\xi)|\, d\xi = \int_{|\xi|<R'_m} |\widehat{f}_\varphi(\xi)|\, d\xi < C.$$

□

When proving Theorem 1.1, we first prove that the modulus of $S_m^\varphi f_\varphi(x_k,t_k)$ turns to infinity as k goes to infinity. For this reason we note that the triangle inequality and (3.6) implies that

$$|S_m^\varphi f_\varphi(x_k,t_k)| \geqslant |A_k^\varphi(x_k,t_k)| - \Big|\sum_{j=1}^{k-1} A_j^\varphi(x_k,t_k)\Big| - \Big|\sum_{j=k+1}^{m} A_j^\varphi(x_k,t_k)\Big|, \tag{4.2}$$

where $m > k$. We want to estimate the terms in (4.2). From Lemma 4.3 we get estimates for the first two terms. It remains to estimate the last term.

PROOF OF THEOREM 1.1.

Step 1. For $j > k \geqslant 2$ we shall estimate $|A_j^{\varphi}(x_k, t_k)|$ in (3.8). We have to find appropriate estimates for A_φ and B_φ in (3.11)-(3.13). By using $t_k - t_j > 0$ and $R_j < r < R'_j$ it follows from (3.4), (3.9), triangle inequality and Cauchy-Schwarz inequality that

$$
\begin{aligned}
(4.3) \quad |F'_\varphi(r,\omega)| &\geqslant (t_k - t_j)|\varphi'(r,\omega)| - |x_k - x_j| \\
&> (t_k - t_j)|\varphi'(r,\omega)| - (t_k - t_j)\frac{|\varphi'(r,\omega)|}{2} \\
&= \frac{|\varphi'(r,\omega)|}{2}(t_k - t_j).
\end{aligned}
$$

From (3.2), (3.3) and (4.3) it follows that

$$
\begin{aligned}
|A_\varphi| &= \Big|\Big[\frac{1}{r(\log r)^{3/4} i F'_\varphi(r,\omega)} e^{iF_\varphi(r,\omega)}\Big]_{R_j}^{R'_j}\Big| \\
&\leqslant \frac{C}{R_j}\Big(\frac{1}{|F'_\varphi(R_j,\omega)|} + \frac{1}{|F'_\varphi(R'_j,\omega)|}\Big) \leqslant \frac{C}{(t_k - t_j)R_j} \leqslant C2^{-j}.
\end{aligned}
$$

In order to estimate B_φ, using (1.4), (3.10) and (4.3), we have

$$
\begin{aligned}
&\Big|\frac{d}{dr}\Big(\frac{1}{r(\log r)^{3/4} i F'_\varphi(r,\omega)}\Big) e^{iF_\varphi(r,\omega)}\Big| \\
&\qquad \leqslant \frac{C}{r^2|F'_\varphi(r,\omega)|} + \frac{C|F''_\varphi(r,\omega)|}{r|F'_\varphi(r,\omega)|^2(\log r)^{3/4}} < \frac{C}{r^{1+\min(1,\beta)}(t_k - t_j)}.
\end{aligned}
$$

This together with (3.3) gives us

$$
\begin{aligned}
|B_\varphi| &= \Big|\int_{R_j}^{R'_j} \frac{d}{dr}\Big(\frac{1}{r(\log r)^{3/4} i F'_\varphi(r,\omega)}\Big) e^{iF_\varphi(r,\omega)}\, dr\Big| \\
&\leqslant \int_{R_j}^{R'_j} \frac{C}{r^{1+\min(1,\beta)}(t_k - t_j)}\, dr \leqslant \frac{C}{R_j^{\min(1,\beta)}(t_k - t_j)} \leqslant C2^{-j}.
\end{aligned}
$$

From the estimates above and the triangle inequality we get

$$
(4.4) \qquad |A_j^{\varphi}(x_k, t_k)| \leqslant C(|A_\varphi| + |B_\varphi|) < C2^{-j}, \qquad j > k \geqslant 2.
$$

Here C is independent of j and k.

Using the results from (4.2), (4.4), in combination with Lemma 4.3, and recalling that $R'_j = R_j^N$, gives us

$$
\begin{aligned}
(4.5) \quad |S_m^{\varphi} f_\varphi(x_k, t_k)| &\geqslant c(\log R'_k)^{1/4} - C'(\log R_k)^{1/4} - C\sum_{k+1}^{m} 2^{-j} \\
&\geqslant c(\log(R'_k))^{1/4} - \frac{C'}{N^{1/4}}(\log(R'_k))^{1/4} - C \geqslant c(\log R'_k)^{1/4},
\end{aligned}
$$

when $m > k$ and N is chosen sufficiently large. Here $c > 0$ is independent of k.

Step 2. Now it remains to show that $S^\varphi f_\varphi$ is continuous when $t > 0$, and then it suffices to prove this continuity on a compact subset L of

$$\{(x,t);\, t > 0,\, x \in \mathbf{R}^n\}.$$

We want to replace (x_l, t_l) with $(x,t) \in L$ in (3.3) and (3.4). Since we have maximum over all l less than j, we can choose $j_0 < \infty$ large enough such that for all $j > l > j_0$ we have that $t_j < t_l < t$. Hence we may replace (x_l, t_l) with $(x,t) \in L$ on the right-hand sides in (3.3) and (3.4) for all $j > j_0$. This in turn implies that (4.4) holds when (x_k, t_k) is replaced by $(x,t) \in L$ and $j > j_0$. We use (4.4) to conclude that

$$\begin{aligned}
&|S^\varphi_m f_\varphi(x,t) - S^\varphi f_\varphi(x,t)| \\
&\quad = \left|(2\pi)^{-n} \int_{|\xi|<R'_m} e^{ix\cdot\xi} e^{it\varphi(\xi)} \widehat{f}_\varphi(\xi)\, d\xi - (2\pi)^{-n} \int_{\mathbf{R}^n} e^{ix\cdot\xi} e^{it\varphi(\xi)} \widehat{f}_\varphi(\xi)\, d\xi\right| \\
&\qquad = (2\pi)^{-n} \left| \int_{|\xi|>R'_m} e^{ix\cdot\xi} e^{it\varphi(\xi)} \widehat{f}_\varphi(\xi)\, d\xi \right| \leqslant C \sum_{i=m+1}^{\infty} 2^{-i} = C2^{-m},
\end{aligned}$$

when $m > j_0$. Hence $S^\varphi_m f_\varphi$ converge uniformly to $S^\varphi f_\varphi$ on every compact set.

We have now showed that $S^\varphi_m f_\varphi$ converge uniformly to $S^\varphi f_\varphi$ on every compact set and from Lemma 4.4 it follows that each $S^\varphi_m f_\varphi$ is a continuous function. Therefore it follows that $S^\varphi f_\varphi$ is continuous on $\{(x,t); t > 0\}$. In particular there is an $N \in \mathbf{N}$ such that

$$|S^\varphi_m f_\varphi(x_k,t_k) - S^\varphi f_\varphi(x_k,t_k)| < 1,$$

when $m > N$. Using (4.5) and the triangle inequality we get

$$\begin{aligned}
c(\log R'_k)^{1/4} &\leqslant |S^\varphi_m f_\varphi(x_k,t_k)| \\
&\leqslant |S^\varphi_m f_\varphi(x_k,t_k) - S^\varphi f_\varphi(x_k,t_k)| + |S^\varphi f_\varphi(x_k,t_k)| < \\
&\qquad\qquad 1 + |S^\varphi f_\varphi(x_k,t_k)|.
\end{aligned}$$

This gives us

$$|S^\varphi f_\varphi(x_k,t_k)| > c(\log R'_k)^{1/4} - 1 \to +\infty \quad \text{as} \quad k \to +\infty.$$

For any fixed $x \in \mathbf{R}^n$ we can by Lemma 4.1 choose a subsequence (x_{n_j}, t_{n_j}) of (x_k, t_k) that goes to $(x,0)$ as j turns to infinity. This gives the result. □

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Paper II

2.2 Wave-front sets of Banach function types

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WAVE-FRONT SETS OF BANACH FUNCTION TYPES

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Abstract. Let ω, ω_0 be appropriate weight functions and $\mathscr{B}$ be an invariant BF-space. We introduce the wave-front set $\mathrm{WF}_{\mathscr{F}\!\mathscr{B}(\omega)}(f)$ with respect to weighted Fourier Banach space $\mathscr{F}\!\mathscr{B}(\omega)$. We prove that usual mapping properties for pseudo-differential operators $\mathrm{Op}_t(a)$ with symbols a in $S^{(\omega_0)}_{\rho,0}$ hold for such wave-front sets. In particular we prove $\mathrm{WF}_{\mathscr{F}\!\mathscr{B}(\omega/\omega_0)}(\mathrm{Op}_t(a)f) \subseteq \mathrm{WF}_{\mathscr{F}\!\mathscr{B}(\omega)}(f)$ and $\mathrm{WF}_{\mathscr{F}\!\mathscr{B}(\omega)}(f) \subseteq \mathrm{WF}_{\mathscr{F}\!\mathscr{B}(\omega/\omega_0)}(\mathrm{Op}_t(a)f) \bigcup \mathrm{Char}(a)$. Here $\mathrm{Char}(a)$ is the set of characteristic points of a.

0. Introduction

In this paper we introduce wave-front sets with respect to Fourier images of translation invariant Banach function spaces (BF-spaces). The family of such wave-front sets contains the wave-front sets of Sobolev type, introduced by Hörmander in [25], the classical wave-front sets (cf. Sections 8.1 and 8.2 in [24]), and wave-front sets of Fourier Lebesgue types, introduced in [31]. Roughly speaking, for any given distribution f and for appropriate Banach (or Frechét) space $\mathcal{B}$ of temperate distributions, the wave-front set $\mathrm{WF}_{\mathcal{B}}(f)$ of f consists of all pairs (x_0, ξ_0) in $\mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$ such that no localizations of the distribution at x_0 belongs to $\mathcal{B}$ in the direction ξ_0.

We also establish mapping properties for a quite general class of pseudo-differential operators on such wave-front sets, and show that the micro-local analysis in [31] in background of Fourier Lebesgue spaces can be further generalized. It follows that our approach gives rise to flexible micro-local analysis tools which fit well to the most common approach developed in e.g. [24, 25]. In particular, we prove that usual mapping properties, which are valid for classical wave-front sets (cf. Chapters VIII and XVIII in [24]), also hold for wave-front sets of Fourier BF-types. For example, we show

$$\begin{aligned}\mathrm{WF}_{\mathscr{F}\!\mathscr{B}(\omega/\omega_0)}(\mathrm{Op}_t(a)f) &\subseteq \mathrm{WF}_{\mathscr{F}\!\mathscr{B}(\omega)}(f)\\ &\subseteq \mathrm{WF}_{\mathscr{F}\!\mathscr{B}(\omega/\omega_0)}(\mathrm{Op}_t(a)f) \bigcup \mathrm{Char}(a).\end{aligned} \tag{0.1}$$

That is, any operator $\mathrm{Op}(a)$ shrinks the wave-front sets and opposite embeddings can be obtained by including $\mathrm{Char}(a)$, the set of characteristic points of the operator symbol a.

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The symbol classes for the pseudo-differential operators are denoted by $S^{(\omega_0)}_{\rho,\delta}(\mathbf{R}^{2d})$, the set of all smooth functions a on $\mathbf{R}^{2d}$ such that $a/\omega_0 \in S^0_{\rho,\delta}(\mathbf{R}^{2d})$. Here $\rho, \delta \in \mathbf{R}$ and ω_0 is an appropriate smooth function on $\mathbf{R}^{2d}$. We note that $S^{(\omega_0)}_{\rho,\delta}(\mathbf{R}^{2d})$ agrees with the Hörmander class $S^r_{\rho,\delta}(\mathbf{R}^{2d})$ when $\omega_0(x,\xi) = \langle \xi \rangle^r$, where $r \in \mathbf{R}$ and $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$.

The set of characteristic points $\operatorname{Char}(a)$ of $a \in S^{(\omega_0)}_{\rho,\delta}$ is the same as in [31], and depends on the choices of ρ, δ and ω_0 (see Definition 1.8 and Proposition 2.3). We recall that this set is smaller than the set of characteristic points given by [24]. It is empty when a satisfies a local ellipticity condition with respect to ω_0, which is fulfilled for any hypoelliptic partial differential operator with constant coefficients (cf. [31]). As a consequence of (0.1), it follows that such hypoelliptic operators preserve the wave-front sets, as expected (cf. Example 3.9 in [31]).

Information on regularity in background of wave-front sets of Fourier BF-types might be more detailed compared to classical wave-front sets, because of our choices of different weight functions ω and BF-spaces when defining our Fourier BF-space $\mathscr{F}\mathscr{B}(\omega)$. For example, the space $\mathscr{F}\mathscr{B}(\omega) = \mathscr{F}L^1_{(\omega)}(\mathbf{R}^d)$, with $\omega(\xi) = \langle \xi \rangle^N$ for some integer $N \geq 0$, is locally close to $C^N(\mathbf{R}^d)$ (cf. the Introduction of [31]). Consequently, the wave-front set with respect to $\mathscr{F}L^1_{(\omega)}$ can be used to investigate a sort of regularity which is close to smoothness of order N.

Furthermore, we are able to apply our results on pseudo-differential operators in context of modulation space theory, when discussing mapping properties of pseudo-differential operators with respect to wave-front sets. The modulation spaces were introduced by Feichtinger in [7], and the theory was developed in [9–11, 15]. The modulation space $M(\omega, \mathscr{B})$, where ω is an appropriate weight function (or time-frequency shift) on phase space $\mathbf{R}^{2d}$, appears as the set of temperate (ultra-)distributions whose short-time Fourier transform belong to the weighted Banach space $\mathscr{B}(\omega)$. This family of modulation spaces contains the (classical) modulation spaces $M^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ as well as the space $W^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ related to the Wiener amalgam spaces. In fact, these spaces which occur frequently in the time-frequency community are obtained by choosing $\mathscr{B} = L^{p,q}_1(\mathbf{R}^{2d})$ or $\mathscr{B} = L^{p,q}_2(\mathbf{R}^{2d})$ (see Remark 6.1).

Parallel to this development, modulation spaces have been incorporated into the calculus of pseudo-differential operators, in the sense of the study of continuity of (classical) pseudo-differential operators acting on modulation spaces (cf. [6, 29, 30, 37–39]), and the study of operators of non-classical type, where modulation spaces are used as symbol classes. We refer to [16–20, 22, 23, 29, 34, 35, 40–42, 44] for more facts about pseudo-differential operators in background of modulation space theory.

In the last part of the paper we define wave-front sets with respect to weighted modulation spaces, and prove that they coincide with the wave-front sets of Fourier BF-types.

The paper is organized as follows. In Section 1 we recall the definition and basic properties for pseudo-differential operators, translation invariant Banach function spaces (BF-spaces) and (weighted) Fourier Banach spaces. Here we also define sets of characteristic points for a broad class of pseudo-differential operators. In Section 2 we prove some properties for the sets of characteristic points, which shows that our definition coincide with the sets of characteristic points defined in [31]. These sets might be smaller than "the classical" characteristic sets in [24] (cf. [31, Example 3.9]).

In Section 3 we define wave-front sets with respect to (weighted) Fourier BF-spaces, and prove some of their main properties. Thereafter, in Section 4 we show how these wave-front sets propagate under the action of pseudo-differential operators. In particular, we prove (0.1), when ω_0 and ω are appropriate weights and a belongs to $S^{(\omega_0)}_{\rho,0}$ with $\rho > 0$.

In Section 5 we consider wave-front sets obtained from sequences of Fourier BF-spaces. These types of wave-front sets contain the classical ones (with respect to smoothness), and the mapping properties for pseudo-differential operators also hold in this context (cf. Section 18.1 in [24]).

In Section 6 we define wave-front sets with respect to modulation spaces, and prove that they can be identified with wave-front sets of Fourier BF-types. This part can also be considered as a starting point for discussions of global wave-front sets of modulation space types, which are investigated in [4]. Here we remark that a notion of global wave-front sets with respect to smoothness and weighted Sobolev spaces was introduced and investigated in [5, 27].

1. Preliminaries

In this section we recall some notation and basic results. The proofs are in general omitted. In what follows we let Γ denote an open cone in $\mathbf{R}^d \setminus 0$. An open cone which contains $\xi \in \mathbf{R}^d \setminus 0$ is sometimes denoted by Γ_ξ.

Let $\omega, v \in L^\infty_{loc}(\mathbf{R}^d)$ be positive functions. Then ω is called v-moderate if

$$\omega(x+y) \le C\omega(x)v(y) \tag{1.1}$$

for some constant C which is independent of $x, y \in \mathbf{R}^d$. If v in (1.1) can be chosen as a polynomial, then ω is called polynomially moderate. We let $\mathscr{P}(\mathbf{R}^d)$ be the set of all polynomially moderated functions on $\mathbf{R}^d$. We say that v is *submultiplicative* when (1.1) holds with $\omega = v$. Throughout we assume that the submultiplicative weights are even.

If $\omega(x,\xi) \in \mathscr{P}(\mathbf{R}^{2d})$ is constant with respect to the x-variable (ξ-variable), then we sometimes write $\omega(\xi)$ ($\omega(x)$) instead of $\omega(x,\xi)$. In this case we consider ω as an element in $\mathscr{P}(\mathbf{R}^{2d})$ or in $\mathscr{P}(\mathbf{R}^{d})$ depending on the situation.

We also need to consider classes of weight functions, related to $\mathscr{P}$. More precisely, we let $\mathscr{P}_0(\mathbf{R}^d)$ be the set of all $\omega \in \mathscr{P}(\mathbf{R}^d)\bigcap C^\infty(\mathbf{R}^d)$ such that $\partial^\alpha \omega/\omega \in L^\infty$ for all multi-indices α. For each $\omega \in \mathscr{P}(\mathbf{R}^d)$, there is an equivalent weight $\omega_0 \in \mathscr{P}_0(\mathbf{R}^d)$, that is, $C^{-1}\omega_0 \le \omega \le C\omega_0$ holds for some constant C (cf. [42, Lemma 1.2]).

Assume that $\rho,\delta \in \mathbf{R}$. Then we let $\mathscr{P}_{\rho,\delta}(\mathbf{R}^{2d})$ be the set of all $\omega(x,\xi)$ in $\mathscr{P}(\mathbf{R}^{2d}) \cap C^\infty(\mathbf{R}^{2d})$ such that

$$\langle \xi\rangle^{\rho|\beta|-\delta|\alpha|}\frac{(\partial_x^\alpha\partial_\xi^\beta\omega)(x,\xi)}{\omega(x,\xi)} \in L^\infty(\mathbf{R}^{2d}),$$

for every multi-indices α and β. Note that in contrast to $\mathscr{P}_0$, we do not have an equivalence between $\mathscr{P}_{\rho,\delta}$ and $\mathscr{P}$ when $\rho > 0$. On the other hand, if $s \in \mathbf{R}$ and $\rho \in [0,1]$, then $\mathscr{P}_{\rho,\delta}(\mathbf{R}^{2d})$ contains $\omega(x,\xi) = \langle\xi\rangle^s$, which are one of the most important type of weights in the applications.

For any weight ω in $\mathscr{P}(\mathbf{R}^d)$, we let $L^p_{(\omega)}(\mathbf{R}^d)$ be the set of all $f \in L^1_{loc}(\mathbf{R}^d)$ such that $f\cdot\omega \in L^p(\mathbf{R}^d)$. We also set $L^p_s(\mathbf{R}^d) = L^p_{(\omega)}$ when $\omega(x) = \langle x\rangle^s$ and $s \in \mathbf{R}$.

The Fourier transform $\mathscr{F}$ is the linear and continuous mapping on $\mathscr{S}'(\mathbf{R}^d)$ which takes the form

$$(\mathscr{F}f)(\xi) = \widehat{f}(\xi) \equiv (2\pi)^{-d/2}\int_{\mathbf{R}^d} f(x)e^{-i\langle x,\xi\rangle}\,dx$$

when $f \in L^1(\mathbf{R}^d)$. We recall that $\mathscr{F}$ is a homeomorphism on $\mathscr{S}'(\mathbf{R}^d)$ which restricts to a homeomorphism on $\mathscr{S}(\mathbf{R}^d)$ and to a unitary operator on $L^2(\mathbf{R}^d)$.

Next we recall the definition of Banach function spaces.

Definition 1.1. Assume that $\mathscr{B}$ is a Banach space of complex-valued measurable functions on $\mathbf{R}^d$ and that $v \in \mathscr{P}(\mathbf{R}^d)$ is submultiplicative. Then $\mathscr{B}$ is called a *(translation) invariant Banach function space (BF-space) on* $\mathbf{R}^d$ (with respect to v), if there is a constant C such that the following conditions are fulfilled:

(1) $\mathscr{S}(\mathbf{R}^d) \subseteq \mathscr{B} \subseteq \mathscr{S}'(\mathbf{R}^d)$ (continuous embeddings).

(2) If $x \in \mathbf{R}^d$ and $f \in \mathscr{B}$, then $f(\cdot - x) \in \mathscr{B}$, and

$$\|f(\cdot - x)\|_{\mathscr{B}} \le Cv(x)\|f\|_{\mathscr{B}}. \tag{1.2}$$

(3) if $f,g \in L^1_{loc}(\mathbf{R}^d)$ satisfy $g \in \mathscr{B}$ and $|f| \le |g|$ almost everywhere, then $f \in \mathscr{B}$ and

$$\|f\|_{\mathscr{B}} \le C\|g\|_{\mathscr{B}}.$$

Assume that $\mathscr{B}$ is a translation invariant BF-space. If $f \in \mathscr{B}$ and $h \in L^\infty$, then it follows from (3) in Definition 1.1 that $f \cdot h \in \mathscr{B}$ and

$$\|f \cdot h\|_{\mathscr{B}} \le C\|f\|_{\mathscr{B}}\|h\|_{L^\infty}. \tag{1.3}$$

Remark 1.2. Assume that $\omega_0, v, v_0 \in \mathscr{P}(\mathbf{R}^d)$ are such v and v_0 are submultiplicative, ω_0 is v_0-moderate, and assume that $\mathscr{B}$ is a translation-invariant BF-space on $\mathbf{R}^d$ with respect to v. Also let $\mathscr{B}_0$ be the Banach space which consists of all $f \in L^1_{loc}(\mathbf{R}^d)$ such that $\|f\|_{\mathscr{B}_0} \equiv \|f\,\omega_0\|_{\mathscr{B}}$ is finite. Then $\mathscr{B}_0$ is a translation invariant BF-space with respect to $v_0 v$.

Remark 1.3. Let $\mathscr{B}$ be an invariant BF-space. Then it is easy to find Sobolev type spaces which are continuously embedded in $\mathscr{B}$. In fact, for each $p \in [1,\infty]$ and integer $N \ge 0$, let $Q^p_N(\mathbf{R}^d)$ be the set of all $f \in L^p(\mathbf{R}^d)$ such that $\|f\|_{Q^p_N} < \infty$, where

$$\|f\|_{Q^p_N} \equiv \sum_{|\alpha+\beta|\le N} \|x^\alpha D^\beta f\|_{L^p}.$$

Then for each p fixed, the topology for $\mathscr{S}(\mathbf{R}^d)$ can be defined by the semi-norms $f \mapsto \|f\|_{Q^p_N}$, for $N = 0,1,\dots$. A combination of this fact and (1) and (3) in Definition 1.1 now shows that for each $p \in [1,\infty]$ and each translation invariant BF-space $\mathscr{B}$, there is an integer $N \ge 0$ such that $Q^p_N(\mathbf{R}^d) \subseteq \mathscr{B}$. This proves the assertion.

In particular it follows that $\langle\,\cdot\,\rangle^{-N} \in \mathscr{B}$, provided $N \ge 0$ is chosen large enough. This gives

$$\|f\|_{\mathscr{B}} = \|\langle\,\cdot\,\rangle^{-N}(f\langle\,\cdot\,\rangle^{N})\|_{\mathscr{B}} \le C_1\|\langle\,\cdot\,\rangle^{-N}\|_{\mathscr{B}}\|f\langle\,\cdot\,\rangle^{N}\|_{L^\infty} = C_2\|f\|_{L^\infty_N},$$

for some costants C_1 and C_2. Hence $L^\infty_N \subseteq \mathscr{B}$ for some $N \ge 0$.

For future references we note that if $\mathscr{B}$ is a translation invariant BF-space with respect to the submultiplicative weight v on $\mathbf{R}^d$, then the convolution map $*$ on $\mathscr{S}(\mathbf{R}^d)$ extends to a continuous mapping from $\mathscr{B} \times L^1_{(v)}(\mathbf{R}^d)$ to $\mathscr{B}$, and for some constant C it holds

$$\|\varphi * f\|_{\mathscr{B}} \le C\|\varphi\|_{L^1_{(v)}}\|f\|_{\mathscr{B}}, \tag{1.4}$$

when $\varphi \in L^1_{(v)}(\mathbf{R}^d)$ and $f \in \mathscr{B}$. In fact, if $f, g \in \mathscr{S}$, then $f*g \in \mathscr{S} \subseteq \mathscr{B}$ in view of the definitions, and Minkowski's inequality gives

$$\|f * g\|_{\mathscr{B}} = \Big\| \int f(\,\cdot\, - y)g(y)\,dy \Big\|_{\mathscr{B}}$$

$$\le \int \|f(\,\cdot - y)\|_{\mathscr{B}}|g(y)|\,dy \le C\int \|f\|_{\mathscr{B}}|g(y)v(y)|\,dy = C\|f\|_{\mathscr{B}}\|g\|_{L^1_{(v)}}.$$

Since $\mathscr{S}$ is dense in $L^1_{(v)}$, it follows that $\varphi * f \in \mathscr{B}$ when $\varphi \in L^1_{(v)}$ and $f \in \mathscr{S}$, and that (1.4) holds in this case. The result is now a consequence of Hahn-Banach's theorem.

From now on we assume that each translation invariant BF-space $\mathscr{B}$ is such that the convolution map $*$ on $\mathscr{S}(\mathbf{R}^d)$ is *uniquely* extendable to a continuous mapping from $\mathscr{B} \times L^1_{(v)}(\mathbf{R}^d)$ to $\mathscr{B}$, and that (1.4) holds when $\varphi \in L^1_{(v)}(\mathbf{R}^d)$ and $f \in \mathscr{B}$. We note that $\mathscr{B}$ can be any mixed and weighted Lebesgue space.

For each translation invariant BF-space $\mathscr{B}$ on $\mathbf{R}^d$, and each pair of vector spaces (V_1, V_2) such that $V_1 \oplus V_2 = \mathbf{R}^d$, we define the projection spaces $\mathscr{B}_1$ and $\mathscr{B}_2$ of $\mathscr{B}$ by the formulae

$$\mathscr{B}_1 \equiv \{\, f \in \mathscr{S}'(V_1)\,;\, f \otimes \varphi \in \mathscr{B} \text{ for every } \varphi \in \mathscr{S}(V_2)\,\} \tag{1.5}$$

and

$$\mathscr{B}_2 \equiv \{\, f \in \mathscr{S}'(V_2)\,;\, \varphi \otimes f \in \mathscr{B} \text{ for every } \varphi \in \mathscr{S}(V_1)\,\}. \tag{1.6}$$

Proposition 1.4. *Assume that $f \in \mathscr{S}'$, $\mathscr{B}$ is a translation invariant BF-space on $\mathbf{R}^d$, and let $\mathscr{B}_1$ and $\mathscr{B}_2$ be the same as in* (1.5) *and* (1.6). *Then*

$$\mathscr{B}_1 = \{\, f \in \mathscr{S}'(V_1)\,;\, f \otimes \varphi \in \mathscr{B} \textit{ for some } \varphi \in \mathscr{S}(V_2) \setminus 0\,\} \tag{1.5$'$}$$

and

$$\mathscr{B}_2 = \{\, f \in \mathscr{S}'(V_2)\,;\, \varphi \otimes f \in \mathscr{B} \textit{ for some } \varphi \in \mathscr{S}(V_1) \setminus 0\,\}. \tag{1.6$'$}$$

In particular, if $\varphi_j \in \mathscr{S}(V_j) \setminus 0$ for $j = 1, 2$ are fixed, then $\mathscr{B}_1$ and $\mathscr{B}_2$ are translation invariant BF-spaces under the norms

$$\|f\|_{\mathscr{B}_1} \equiv \|f \otimes \varphi_1\|_{\mathscr{B}} \quad \textit{and} \quad \|f\|_{\mathscr{B}_2} \equiv \|\varphi_2 \otimes f\|_{\mathscr{B}}$$

respectively.

Proof. We only prove (1.6)$'$. The equality (1.5)$'$ follows by similar arguments and is left for the reader. We may assume that $V_j = \mathbf{R}^{d_j}$ with $d_1 + d_2 = d$.

Let $\mathscr{B}_0$ be the right-hand side of (1.6)$'$. Then it is obvious that $\mathscr{B}_2 \subseteq \mathscr{B}_0$. We have to prove the opposite inclusion.

Therefore, assume that $f \in \mathscr{B}_0$, and choose $\varphi_0 \in \mathscr{S}(\mathbf{R}^{d_1}) \setminus 0$ such that $\varphi_0 \otimes f \in \mathscr{B}$. Also let $\varphi \in \mathscr{S}(\mathbf{R}^{d_1})$ be arbitrary. We shall prove that $\varphi \otimes f \in \mathscr{B}$.

First we assume that $\varphi \in C_0^\infty(\mathbf{R}^d)$. Let $Q \subseteq \mathbf{R}^{d_1}$ be an open ball and $c > 0$ be chosen such that $|\varphi_0(x)| > c$ when $x \in Q$. Also let the lattice $\Lambda \subseteq \mathbf{R}^{d_1}$ and $\varphi_1 \in C_0^\infty(Q)$ be such that $0 \le \varphi_1 \le 1$ and

$$\sum_{\{x_j\} \in \Lambda} \varphi_1(\,\cdot\, - x_j) = 1,$$

and let J be a finite set such that $\sum_{j \in J} \varphi_1(\,\cdot\, - x_j) = 1$ on $\operatorname{supp} \varphi$. Then $|\varphi_1 \varphi(\,\cdot\, + x_j)| \le C|\varphi_0|$, for some constant $C > 0$, which gives $(\varphi_1(\,\cdot\, - x_j)\varphi) \otimes f \in \mathscr{B}$ and

$$\|(\varphi_1(\,\cdot\, - x_j)\varphi) \otimes f\|_{\mathscr{B}} \le C_1 \|\varphi_0(\,\cdot\, - x_j) \otimes f\|_{\mathscr{B}} \le C_2 v(x_j, 0) \|\varphi_0 \otimes f\|_{\mathscr{B}},$$

for some constants C_1 and C_2. From this fact together with the formula

$$\varphi \otimes f = \sum_{j\in J} (\varphi_1(\,\cdot\, - x_j)\varphi) \otimes f,$$

with finite sum on the right-hand side, it follows that $\varphi \otimes f \in \mathscr B$, and

$$\begin{aligned}\|\varphi \otimes f\|_{\mathscr B} &\le \sum \|(\varphi_1(\,\cdot\, - x_j)\varphi) \otimes f\|_{\mathscr B} \\ &\le \sum v(x_j,0)\|(\varphi_1\varphi(\,\cdot\, + x_j)) \otimes f\|_{\mathscr B} \\ &\le C\Big(\sum v(x_j,0)\|\varphi(\,\cdot\, + x_j)\|_{L^\infty(Q)}\Big)\|\varphi_1 \otimes f\|_{\mathscr B}, \qquad (1.7)\end{aligned}$$

where the sums are taken over all $j \in J$. Since $v \in \mathscr P$ and $\varphi \in \mathscr S$, it follows that the sum in the right-hand side of (1.7) is finite. Hence $f \in \mathscr B_2$, and we have proved the assertion in the case $\varphi \in C_0^\infty$. The result now follows for general $\varphi \in \mathscr S$ from (1.7) and the fact that C_0^∞ is dense in $\mathscr S$. The proof is complete. □

Remark 1.5. We note that the last sum in (1.7) is the norm

$$\|\varphi\|_{W_{(v)}} \equiv \sum v(x_j,0)\|\varphi(\,\cdot\, + x_j)\|_{L^\infty(Q)}$$

for the weighted Wiener space

$$W_{(v)}(\mathbf R^d) = \{\, f \in L^\infty_{loc}(\mathbf R^d)\,;\, \|f\|_{W_{(v)}} < \infty \,\}$$

(cf. [16]). The results in Proposition 1.4 can therefore be improved in such way that we may replace $\mathscr S$ by $W_{(v)}$ in (1.5), (1.6), (1.5)′ and (1.6)′.

Assume that $\mathscr B$ is a translation invariant BF-space on $\mathbf R^d$, and that $\omega \in \mathscr P(\mathbf R^d)$. Then we let $\mathscr F\!\mathscr B(\omega)$ be the set of all $f \in \mathscr S'(\mathbf R^d)$ such that $\xi \mapsto \widehat f(\xi)\omega(\xi)$ belongs to $\mathscr B$. It follows that $\mathscr F\!\mathscr B(\omega)$ is a Banach space under the norm

$$\|f\|_{\mathscr F\!\mathscr B(\omega)} \equiv \|\widehat f\,\omega\|_{\mathscr B}. \qquad (1.8)$$

Remark 1.6. In many situations it is convenient to permit an x dependency for the weight ω in the definition of Fourier BF-spaces. More precisely, for each $\omega \in \mathscr P(\mathbf R^{2d})$ and each translation invariant BF-space $\mathscr B$ on $\mathbf R^d$, we let $\mathscr F\!\mathscr B(\omega)$ be the set of all $f \in \mathscr S'(\mathbf R^d)$ such that

$$\|f\|_{\mathscr F\!\mathscr B(\omega)} = \|f\|_{\mathscr F\!\mathscr B(\omega),x} \equiv \|\widehat f\,\omega(x,\,\cdot\,)\|_{\mathscr B}$$

is finite. Since ω is v-moderate for some $v \in \mathscr P(\mathbf R^{2d})$ it follows that different choices of x give rise to equivalent norms. Therefore the condition $\|f\|_{\mathscr F\!\mathscr B(\omega)} < \infty$ is independent of x, and it follows that $\mathscr F\!\mathscr B(\omega)$ is independent of x although $\|\cdot\|_{\mathscr F\!\mathscr B(\omega)}$ might depend on x.

Recall that a topological vector space $V \subseteq \mathscr{D}'(X)$ is called *local* if $V \subseteq V_{loc}$. Here $X \subseteq \mathbf{R}^d$ is open, and V_{loc} consists of all $f \in \mathscr{D}'(X)$ such that $\varphi f \in V$ for every $\varphi \in C_0^\infty(X)$. For future references we note that if $\mathscr{B}$ is a translation invariant BF-space on $\mathbf{R}^d$ and $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, then it follows from (1.4) that $\mathscr{F}\mathscr{B}(\omega)$ is a local space, i.e.

$$\mathscr{F}\mathscr{B}(\omega) \subseteq \mathscr{F}\mathscr{B}(\omega)_{loc} \equiv (\mathscr{F}\mathscr{B}(\omega))_{loc}. \tag{1.9}$$

Next we recall some facts from Chapter XVIII in [24] concerning pseudo-differential operators. Let $a \in \mathscr{S}(\mathbf{R}^{2d})$, and $t \in \mathbf{R}$ be fixed. Then the pseudo-differential operator $\operatorname{Op}_t(a)$ is the linear and continuous operator on $\mathscr{S}(\mathbf{R}^d)$, defined by the formula

$$(\operatorname{Op}_t(a)f)(x)(2\pi)^{-d} \iint a((1-t)x+ty,\xi)f(y)e^{i\langle x-y,\xi\rangle}\, dyd\xi. \tag{1.10}$$

For general $a \in \mathscr{S}'(\mathbf{R}^{2d})$, the pseudo-differential operator $\operatorname{Op}_t(a)$ is defined as the continuous operator from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$ with distribution kernel

$$K_{t,a}(x,y) = (2\pi)^{-d/2}(\mathscr{F}_2^{-1}a)((1-t)x+ty, x-y). \tag{1.11}$$

Here $\mathscr{F}_2 F$ is the partial Fourier transform of $F(x,y) \in \mathscr{S}'(\mathbf{R}^{2d})$ with respect to the y-variable. This definition makes sense, since the mappings $\mathscr{F}_2$ and

$$F(x,y) \mapsto F((1-t)x+ty, x-y)$$

are homeomorphisms on $\mathscr{S}'(\mathbf{R}^{2d})$. We also note that the latter definition of $\operatorname{Op}_t(a)$ agrees with the operator in (1.10) when $a \in \mathscr{S}(\mathbf{R}^{2d})$. If $t=0$, then $\operatorname{Op}_t(a)$ agrees with the Kohn-Nirenberg representation $\operatorname{Op}(a) = a(x,D)$.

If $a \in \mathscr{S}'(\mathbf{R}^{2d})$ and $s,t \in \mathbf{R}$, then there is a unique $b \in \mathscr{S}'(\mathbf{R}^{2d})$ such that $\operatorname{Op}_s(a) = \operatorname{Op}_t(b)$. By straight-forward applications of Fourier's inversion formula, it follows that

$$\operatorname{Op}_s(a) = \operatorname{Op}_t(b) \quad \Longleftrightarrow \quad b(x,\xi) = e^{i(t-s)\langle D_x, D_\xi\rangle}a(x,\xi). \tag{1.12}$$

(Cf. Section 18.5 in [24].)

Next we discuss symbol classes which we use. Let $r,\rho,\delta \in \mathbf{R}$ be fixed. Then we recall from [24] that $S^r_{\rho,\delta}(\mathbf{R}^{2d})$ is the set of all $a \in C^\infty(\mathbf{R}^{2d})$ such that for each pair of multi-indices α and β, there is a constant $C_{\alpha,\beta}$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta}\langle \xi\rangle^{r-\rho|\beta|+\delta|\alpha|}.$$

Usually we assume that $0 \le \delta \le \rho \le 1$, $0 < \rho$ and $\delta < 1$.

More generally, assume that $\omega \in \mathscr{P}_{\rho,\delta}(\mathbf{R}^{2d})$. Then we recall from the introduction that $S^{(\omega)}_{\rho,\delta}(\mathbf{R}^{2d})$ consists of all $a \in C^\infty(\mathbf{R}^{2d})$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta}\omega(x,\xi)\langle \xi\rangle^{-\rho|\beta|+\delta|\alpha|}. \tag{1.13}$$

We note that $S^{(\omega)}_{\rho,\delta}(\mathbf{R}^{2d}) = S(\omega, g_{\rho,\delta})$, when $g = g_{\rho,\delta}$ is the Riemannian metric on $\mathbf{R}^{2d}$, defined by the formula

$$\big(g_{\rho,\delta}\big)_{(y,\eta)}(x,\xi) = \langle \eta\rangle^{2\delta}|x|^2 + \langle \eta\rangle^{-2\rho}|\xi|^2$$

(cf. Section 18.4–18.6 in [24]). Furthermore, $S^{(\omega)}_{\rho,\delta} = S^{r}_{\rho,\delta}$ when $\omega(x,\xi) = \langle \xi\rangle^r$, as remarked in the introduction.

The following result shows that pseudo-differential operators with symbols in $S^{(\omega)}_{\rho,\delta}$ behave well. We refer to [24] or [31] for the proof.

Proposition 1.7. *Let $\rho,\delta \in [0,1]$ be such that $0 \le \delta \le \rho \le 1$ and $\delta < 1$, and let $\omega \in \mathscr{P}_{\rho,\delta}(\mathbf{R}^{2d})$. If $a \in S^{(\omega)}_{\rho,\delta}(\mathbf{R}^{2d})$, then $\operatorname{Op}_t(a)$ is continuous on $\mathscr{S}(\mathbf{R}^d)$ and extends uniquely to a continuous operator on $\mathscr{S}'(\mathbf{R}^d)$.*

We also need to define the set of characteristic points of a symbol $a \in S^{(\omega)}_{\rho,\delta}(\mathbf{R}^{2d})$, when $\omega \in \mathscr{P}_{\rho,\delta}(\mathbf{R}^{2d})$ and $0 \le \delta < \rho \le 1$. In Section 2 we show that this definition is equivalent to Definition 1.3 in [31]. We remark that our sets of characteristic points are smaller than the corresponding sets in [24]. (Cf. [24, Definition 18.1.5] and Remark 2.4 in Section 2).

Definition 1.8. Assume that $0 \le \delta < \rho \le 1$, $\omega_0 \in \mathscr{P}_{\rho,\delta}(\mathbf{R}^{2d})$ and $a \in S^{(\omega_0)}_{\rho,\delta}(\mathbf{R}^{2d})$. Then a is called *ψ-invertible* with respect to ω_0 at the point $(x_0,\xi_0) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$, if there exist a neighbourhood X of x_0, an open conical neighbourhood Γ of ξ_0 and positive constants R and C such that

$$|a(x,\xi)| \ge C\omega_0(x,\xi), \tag{1.14}$$

for $x \in X$, $\xi \in \Gamma$ and $|\xi| \ge R$.

The point (x_0,ξ_0) is called *characteristic* for a with respect to ω_0 if a is *not* ψ-invertible with respect to ω_0 at (x_0,ξ_0). The set of characteristic points (the characteristic set), for a with respect to ω_0 is denoted $\operatorname{Char}(a) = \operatorname{Char}_{(\omega_0)}(a)$.

We note that $(x_0,\xi_0) \notin \operatorname{Char}_{(\omega_0)}(a)$ means that a is elliptic near x_0 in the direction ξ_0. Since the case $\omega_0 = 1$ in Definition 1.8 is especially important, we omit the phrase "with respect to ω_0" in this case. That is, the element $c \in S^{0}_{\rho,\delta}(\mathbf{R}^{2d})$ is *ψ-invertible* at $(x_0,\xi_0) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$, if $(x_0,\xi_0) \notin \operatorname{Char}_{(\omega_0)}(c)$ with $\omega_0 = 1$.

It will also be convenient to have the following definition of different types of cutoff functions.

Definition 1.9. Let $X \subseteq \mathbf{R}^d$ be open, $\Gamma \subseteq \mathbf{R}^d \setminus 0$ be an open cone, $x_0 \in X$ and let $\xi_0 \in \Gamma$.

(1) A smooth function φ on $\mathbf{R}^d$ is called a cutoff function with respect to x_0 and X, if $0 \le \varphi \le 1$, $\varphi \in C_0^\infty(X)$ and $\varphi = 1$ in an open neighbourhood of x_0. The set of cutoff functions with respect to x_0 and X is denoted by $\mathscr{C}_{x_0}(X)$;

(2) A smooth function ψ on $\mathbf{R}^d$ is called a directional cutoff function with respect to ξ_0 and Γ, if there is a constant $R > 0$ and open conical neighbourhood Γ_1 of ξ_0 such that the following is true:

- $0 \le \psi \le 1$ and $\operatorname{supp} \psi \subseteq \Gamma$;
- $\psi(t\xi) = \psi(\xi)$ when $t \ge 1$ and $|\xi| \ge R$;
- $\psi(\xi) = 1$ when $\xi \in \Gamma_1$ and $|\xi| \ge R$.

The set of directional cutoff functions with respect to ξ_0 and Γ is denoted by $\mathscr{C}^{\mathrm{dir}}_{\xi_0}(\Gamma)$.

Remark 1.10. We note that if $\varphi \in \mathscr{C}_{x_0}(X)$ and $\psi \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}(\Gamma)$ for some $(x_0, \xi_0) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$, then $c \equiv \varphi \otimes \psi$ belongs to $S^0_{1,0}(\mathbf{R}^{2d})$ and is ψ-invertible at (x_0, ξ_0).

2. Pseudo-differential calculus with symbols in $S^{(\omega)}_{\rho,\delta}$

In this section we make a review of basic results for pseudo-differential operators with symbols in classes of the form $S^{(\omega)}_{\rho,\delta}(\mathbf{R}^{2d})$, when $0 \le \delta < \rho \le 1$ and $\omega \in \mathscr{P}_{\rho,\delta}(\mathbf{R}^{2d})$. For the standard properties in the pseudo-differential calculus we only state the results and refer to [24] for the proofs. Though there are similar stated and proved properties concerning sets of characteristic points, we include proofs of these properties in order to be more self-contained.

We start with the following result concerning compositions and invariance properties for pseudo-differential operators. Here we set

$$\sigma_s(x, \xi) = \langle \xi \rangle^s,$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$ as before. We also recall that

$$S^{-\infty}_{\rho,\delta}(\mathbf{R}^{2d}) = S^{-\infty}_{1,0}(\mathbf{R}^{2d}) = S^{-\infty}(\mathbf{R}^{2d})$$

consists of all $a \in C^\infty(\mathbf{R}^{2d})$ such that for each $N \in \mathbf{R}$ and multi-index α, there is a constant $C_{N,\alpha}$ such that

$$|\partial^\alpha a(x, \xi)| \le C_{N,\alpha} \langle \xi \rangle^{-N}.$$

Proposition 2.1. *Let $0 \le \delta < \rho \le 1$, $\mu = \rho - \delta > 0$ and $\omega, \omega_1, \omega_2 \in \mathscr{P}_{\rho,\delta}(\mathbf{R}^{2d})$. Also let $\{m_j\}_{j=0}^\infty$ be a sequence of real numbers such that $m_j \to -\infty$ as $j \to \infty$. Then the following is true:*

(1) *if $a_1 \in S^{(\omega_1)}_{\rho,\delta}(\mathbf{R}^{2d})$ and $a_2 \in S^{(\omega_2)}_{\rho,\delta}(\mathbf{R}^{2d})$, then $\operatorname{Op}(a_1) \circ \operatorname{Op}(a_2) = \operatorname{Op}(c)$, for some $c \in S^{(\omega_1\omega_2)}_{\rho,\delta}(\mathbf{R}^{2d})$. Furthermore,*

$$c(x, \xi) - \sum_{|\alpha|<N} \frac{i^{|\alpha|}(D^\alpha_\xi a_1)(x, \xi)(D^\alpha_x a_2)(x, \xi)}{\alpha!} \in S^{(\omega_1\omega_2\sigma_{-N\mu})}_{\rho,\delta}(\mathbf{R}^{2d}) \quad (2.1)$$

for every $N \ge 0$;

(2) *if* $M=\sup_{k\ge 0}(m_k)$, $M_j=\sup_{k\ge j}(m_k)$ *and* $a_j\in S^{(\omega\sigma_{m_j})}_{\rho,\delta}(\mathbf R^{2d})$, *then it exists* $a\in S^{(\omega\sigma_M)}_{\rho,\delta}(\mathbf R^{2d})$ *such that*

$$a(x,\xi)-\sum_{j<N}a_j(x,\xi)\in S^{(\omega\sigma_{M_N})}_{\rho,\delta}(\mathbf R^{2d}); \tag{2.2}$$

for every $N\ge 0$*;*

(3) *if* $a,b\in \mathscr S'(\mathbf R^{2d})$ *and* $s,t\in\mathbf R$ *are such that* $\operatorname{Op}_s(a)=\operatorname{Op}_t(b)$, *then* $a\in S^{(\omega)}_{\rho,\delta}(\mathbf R^{2d})$, *if and only if* $b\in S^{(\omega)}_{\rho,\delta}(\mathbf R^{2d})$, *and*

$$b(x,\xi)-\sum_{k<N}\frac{(i(t-s)\langle D_x,D_\xi\rangle)^k a(x,\xi)}{k!}\in S^{(\omega\sigma_{-N\mu})}_{\rho,\delta}(\mathbf R^{2d}) \tag{2.3}$$

for every $N\ge 0$.

As usual we write

$$a\sim\sum a_j \tag{2.2$'$}$$

when (2.2) is fulfilled for every $N\ge 0$. In particular it follows from (2.1) and (2.3) that

$$c\sim\sum\frac{i^{|\alpha|}(D_\xi^\alpha a_1)(D_x^\alpha a_2)}{\alpha!} \tag{2.1$'$}$$

when $\operatorname{Op}(a_1)\circ\operatorname{Op}(a_2)=\operatorname{Op}(c)$, and

$$b\sim\sum\frac{(i(t-s)\langle D_x,D_\xi\rangle)^k a}{k!} \tag{2.3$'$}$$

when $\operatorname{Op}_s(a)=\operatorname{Op}_t(b)$.

In the following proposition we show that the set of characteristic points for a pseudo-differential operator is independent of the choice of pseudo-differential calculus.

Proposition 2.2. *Assume that* $s,t\in\mathbf R$, $0\le\delta<\rho\le 1$, $\omega_0\in\mathscr P_{\rho,\delta}$ *and that* $a,b\in S^{(\omega_0)}_{\rho,\delta}(\mathbf R^{2d})$ *satisfy* $\operatorname{Op}_s(a)=\operatorname{Op}_t(b)$. *Then*

$$\operatorname{Char}_{(\omega_0)}(a)=\operatorname{Char}_{(\omega_0)}(b). \tag{2.4}$$

Proof. Let μ and σ_s be the same as in Proposition 2.1. By Proposition 2.1 (3) we have

$$b=a+h,$$

for some $h\in S^{(\omega_0\sigma_{-\mu})}_{\rho,\delta}$.

Assume that $(x_0,\xi_0)\notin\operatorname{Char}_{(\omega_0)}(a)$. By the definitions, there is a neighbourhood X of x_0, an open conical neighbourhood Γ of ξ_0, $C>0$ and $R>0$ such that

$$|a(x,\xi)|\ge C\omega_0(x,\xi)\quad\text{and}\quad |h(x,\xi)|\le C\omega_0(x,\xi)/2,$$

as $x\in X$, $\xi\in\Gamma$ and $|\xi|\ge R$. This gives

$$|b(x,\xi)|\ge C\omega_0(x,\xi)/2,\quad\text{when}\quad x\in X,\ \xi\in\Gamma,\ |\xi|\ge R,$$

and it follows that $(x_0,\xi_0)\notin \operatorname{Char}_{(\omega_0)}(b)$. Hence $\operatorname{Char}_{(\omega_0)}(b)\subseteq \operatorname{Char}_{(\omega_0)}(a)$. By symmetry, the opposite inclusion also holds. Hence $\operatorname{Char}_{(\omega_0)}(a) = \operatorname{Char}_{(\omega_0)}(b)$, and the proof is complete. □

The following proposition shows different aspects of set of characteristic points, and is important when investigating wave-front properties for pseudo-differential operators. In particular it shows that $\operatorname{Op}(a)$ satisfy certain invertibility properties outside the set of characteristic points for a. More precisely, outside $\operatorname{Char}_{(\omega_0)}(a)$, we prove that

$$\operatorname{Op}(b)\operatorname{Op}(a) = \operatorname{Op}(c) + \operatorname{Op}(h), \tag{2.5}$$

for some convenient b, c and h which take the role of inverse, identity symbol and smoothing remainder respectively.

Proposition 2.3. *Let $0\le \delta<\rho\le 1$, $\omega_0\in \mathscr{P}_{\rho,\delta}(\mathbf{R}^{2d})$, $a\in S^{(\omega_0)}_{\rho,\delta}(\mathbf{R}^{2d})$, $(x_0,\xi_0)\in \mathbf{R}^d\times(\mathbf{R}^d\setminus 0)$, and let $\mu=\rho-\delta$. Then the following conditions are equivalent:*

1. *$(x_0,\xi_0)\notin \operatorname{Char}_{(\omega_0)}(a)$;*
2. *there is an element $c\in S^0_{\rho,\delta}$ which is ψ-invertible at (x_0,ξ_0), and an element $b\in S^{(1/\omega_0)}_{\rho,\delta}$ such that $ab=c$;*
3. *there is an element $c\in S^0_{\rho,\delta}$ which is ψ-invertible at (x_0,ξ_0), and elements $h\in S^{-\mu}_{\rho,\delta}$ and $b\in S^{(1/\omega_0)}_{\rho,\delta}$ such that (2.5) holds;*
4. *for each neighbourhood X of x_0 and conical neighbourhood Γ of ξ_0, there is an element $c=\varphi\otimes\psi$ where $\varphi\in\mathscr{C}_{x_0}(X)$ and $\psi\in\mathscr{C}^{\mathrm{dir}}_{\xi_0}(\Gamma)$, and elements $h\in\mathscr{S}$ and $b\in S^{(1/\omega_0)}_{\rho,\delta}$ such that (2.5) holds. Furthermore, the supports of b and h are contained in $X\times\mathbf{R}^d$.*

For the proof we note that μ in Proposition 2.3 is positive, which in turn implies that $\cap_{j\ge 0}S^{(\omega_0\sigma_{-j\mu})}_{\rho,\delta}(\mathbf{R}^{2d})$ agrees with $S^{-\infty}(\mathbf{R}^{2d})$.

Proof. The equivalence between (1) and (2) follows by letting $b(x,\xi)=\varphi(x)\psi(\xi)/a(x,\xi)$ for some appropriate $\varphi\in\mathscr{C}_{x_0}(\mathbf{R}^d)$ and $\psi\in\mathscr{C}^{\mathrm{dir}}_{\xi_0}(\mathbf{R}^d\setminus 0)$.

(4) ⇒ (3) is obvious in view of Remark 1.10. Assume that (3) holds. We shall prove that (1) holds, and since $|b|\le C/\omega_0$, it suffices to prove that

$$|a(x,\xi)b(x,\xi)|\ge 1/2 \tag{2.6}$$

when

$$(x,\xi)\in X\times\Gamma,\ \ |\xi|\ge R \tag{2.7}$$

holds for some conical neighbourhood Γ of ξ_0, some open neighbourhood X of x_0 and some $R>0$.

By Proposition 2.1 (1) it follows that $ab = c+h$ for some $h \in S^{-\mu}_{\rho,\delta}$. By choosing R large enough and Γ sufficiently small conical neighbourhood of ξ_0, it follows that $c(x,\xi) = 1$ and $|h(x,\xi)| \le 1/2$ when (2.7) holds. This gives (2.6), and (1) follows.

It remains to prove that (1) implies (4). Therefore assume that (1) holds, and choose an open neighbourhood X of x_0, an open conical neighbourhood Γ of ξ_0 and $R > 0$ such that (1.14) holds when $(x,\xi) \in X \times \Gamma$ and $|\xi| > R$. Also let $\varphi_j \in \mathscr{C}_{x_0}(X)$ and $\psi_j \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}(\Gamma)$ for $j = 1,2,3$ be such that $\varphi_j = 1$ on $\operatorname{supp}\varphi_{j+1}$, $\psi_j = 1$ on $\operatorname{supp}\psi_{j+1}$ when $j = 1,2$, and $\psi_j(\xi) = 0$ when $|\xi| \le R$. We also set $c_j = \varphi_j \otimes \psi_j$ when $j \le 2$ and $c_j = c_2$ when $j \ge 3$.

If $b_1(x,\xi) = \varphi_1(x)\psi_1(\xi)/a(x,\xi) \in S^{(1/\omega_0)}_{\rho,\delta}$, then the symbol of $\operatorname{Op}(b_1)\operatorname{Op}(a)$ is equal to $c_1 \mod (S^{-\mu}_{\rho,\delta})$. Hence

$$\operatorname{Op}(b_j)\operatorname{Op}(a) = \operatorname{Op}(c_j) + \operatorname{Op}(h_j) \tag{2.8}$$

holds for $j = 1$ and some $h_1 \in S^{-\mu}_{\rho,\delta}$.

For $j \ge 2$ we now define $\widetilde{b}_j \in S^{(1/\omega_0)}_{\rho,\delta}$ by the Neumann serie

$$\operatorname{Op}(\widetilde{b}_j) = \sum_{k=0}^{j-1} (-1)^k \operatorname{Op}(\widetilde{r}_k),$$

where $\operatorname{Op}(\widetilde{r}_k) = \operatorname{Op}(h_1)^k \operatorname{Op}(b_1) \in \operatorname{Op}(S^{(\sigma_{-k\mu}/\omega_0)}_{\rho,\delta})$. Then (2.8) gives

$$\operatorname{Op}(\widetilde{b}_j)\operatorname{Op}(a) = \sum_{k=0}^{j-1} (-1)^k \operatorname{Op}(h_1)^k \operatorname{Op}(b_1)\operatorname{Op}(a)$$

$$= \sum_{k=0}^{j-1} (-1)^k \operatorname{Op}(h_1)^k (\operatorname{Op}(c_1) + \operatorname{Op}(h_1)).$$

That is

$$\operatorname{Op}(\widetilde{b}_j)\operatorname{Op}(a) = \operatorname{Op}(c_1) + \operatorname{Op}(\widetilde{h}_{1,j}) + \operatorname{Op}(\widetilde{h}_{2,j}), \tag{2.9}$$

where

$$\operatorname{Op}(\widetilde{h}_{1,j}) = (-1)^{j-1} \operatorname{Op}(h_1)^j \in \operatorname{Op}(S^{-j\mu}_{\rho,\delta}) \tag{2.10}$$

and

$$\operatorname{Op}(\widetilde{h}_{2,j}) = -\sum_{k=1}^{j-1} (-1)^k \operatorname{Op}(h_1)^k \operatorname{Op}(1-c_1) \in \operatorname{Op}(S^{-\mu}_{\rho,\delta}).$$

By Proposition 2.1 (1) and asymptotic expansions it follows that

$$\operatorname{Op}(\widetilde{h}_{2,j}) = -\sum_{k=1}^{j-1} (-1)^k \operatorname{Op}(1-c_1)\operatorname{Op}(h_1)^k + \operatorname{Op}(\widetilde{h}_{3,j}) + \operatorname{Op}(\widetilde{h}_{4,j}), \tag{2.11}$$

for some $\widetilde{h}_{3,j} \in S^{-\mu}_{\rho,\delta}$ which is equal to zero in $\operatorname{supp} c_1$ and $\widetilde{h}_{4,j} \in S^{-j\mu}_{\rho,\delta}$. Now let b_j and r_k be defined by the formulae

$$\operatorname{Op}(b_j) = \operatorname{Op}(c_2)\operatorname{Op}(\widetilde{b}_j) \in \operatorname{Op}(S^{(1/\omega_0)}_{\rho,\delta})$$

$$\operatorname{Op}(r_k) = \operatorname{Op}(c_2)\operatorname{Op}(\widetilde{r}_k) \in \operatorname{Op}(S^{(\sigma_{-k\mu}/\omega_0)}_{\rho,\delta}).$$

Then

$$\operatorname{Op}(b_j) = \sum_{k=0}^{j-1}(-1)^k \operatorname{Op}(r_k)$$

and (2.9)–(2.11) give

$$\operatorname{Op}(b_j)\operatorname{Op}(a) = \operatorname{Op}(c_2)\operatorname{Op}(c_1) + \operatorname{Op}(c_2)\operatorname{Op}(\widetilde{h}_{1,j})$$

$$-\sum_{k=1}^{j-1}(-1)^k \operatorname{Op}(c_2)\operatorname{Op}(1-c_1)\operatorname{Op}(h_1)^k + \operatorname{Op}(c_2)\operatorname{Op}(\widetilde{h}_{3,j}) + \operatorname{Op}(c_2)\operatorname{Op}(\widetilde{h}_{4,j}).$$

Since $c_1 = 1$ and $\widetilde{h}_{3,j} = 0$ on $\operatorname{supp} c_2$, it follows that

$$\operatorname{Op}(c_2)\operatorname{Op}(c_1) = \operatorname{Op}(c_2) \mod \operatorname{Op}(S^{-\infty}),$$

$$\operatorname{Op}(c_2)\operatorname{Op}(\widetilde{h}_{1,j}) \in \operatorname{Op}(S^{-j\mu}_{\rho,\delta}),$$

$$\sum_{k=1}^{j-1}(-1)^k \operatorname{Op}(c_2)\operatorname{Op}(1-c_1)\operatorname{Op}(h_1)^k \in \operatorname{Op}(S^{-\infty}),$$

$$\operatorname{Op}(c_2)\operatorname{Op}(\widetilde{h}_{3,j}) \in \operatorname{Op}(S^{-\infty})$$

and

$$\operatorname{Op}(c_2)\operatorname{Op}(\widetilde{h}_{4,j}) \in \operatorname{Op}(S^{-j\mu}_{\rho,\delta}).$$

Hence, (2.8) follows for some $h_j \in S^{-j\mu}_{\rho,\delta}$.

By choosing $b_0 \in S^{(1/\omega)}_{\rho,\delta}$ such that

$$b_0 \sim \sum r_k,$$

it follows that $\operatorname{Op}(b_0)\operatorname{Op}(a) = \operatorname{Op}(c_2) + \operatorname{Op}(h_0)$, with

$$h_0 \in S^{-\infty}.$$

The assertion (4) now follows by letting

$$b(x,\xi) = \varphi_3(x)b_0(x,\xi), \quad c(x,\xi) = \varphi_3(x)c_2(x,\xi),$$

$$\text{and} \quad h(x,\xi) = \varphi_3(x)h_0(x,\xi),$$

and using the fact that if $\varphi_3 \in C_0^\infty(\mathbf{R}^d)$ and $h_0 \in S^{-\infty}(\mathbf{R}^{2d})$, then $\varphi_3(x)h_0(x,\xi) \in \mathscr{S}(\mathbf{R}^{2d})$. The proof is complete. □

Remark 2.4. By Proposition 2.3 it follows that Definition 1.3 in [31] is equivalent to Definition 1.8. We also remark that if a is an appropriate symbol, and $\mathrm{Char}'(a)$ the set of characteristic points for a in the sense of [24, Definition 18.1.5], then $\mathrm{Char}_{(\omega_0)}(a) \subseteq \mathrm{Char}'(a)$. Furthermore, strict embedding might occur, especially for symbols to hypoelliptic partial operators with constant coefficients, which are not elliptic (cf. Example 3.9 in [31]).

3. Wave front sets with respect to Fourier BF-spaces

In this section we define wave-front sets with respect to Fourier BF-spaces, and show some basic properties.

Let $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $\Gamma \subseteq \mathbf{R}^d \setminus 0$ be an open cone and let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^d$. For any $f \in \mathscr{E}'(\mathbf{R}^d)$, let

$$|f|_{\mathscr{F}\mathscr{B}(\omega,\Gamma)} = |f|_{\mathscr{F}\mathscr{B}(\omega,\Gamma)_x} \equiv \|\widehat{f}\omega(x,\,\cdot\,)\chi_\Gamma\|_{\mathscr{B}}. \tag{3.1}$$

We note that $\widehat{f}\omega(x,\,\cdot\,)\chi_\Gamma \in \mathscr{B}_{loc}$ for every $f \in \mathscr{E}'$. If $\widehat{f}\omega(x,\,\cdot\,)\chi_\Gamma \notin \mathscr{B}$, then we set $|f|_{\mathscr{F}\mathscr{B}(\omega,\Gamma)} = +\infty$. Hence $|\,\cdot\,|_{\mathscr{F}\mathscr{B}(\omega,\Gamma)}$ defines a semi-norm on $\mathscr{E}'$ which might attain the value $+\infty$. Since ω is v-moderate for some $v \in \mathscr{P}(\mathbf{R}^{2d})$, it follows that different $x \in \mathbf{R}^d$ gives rise to equivalent semi-norms. Furthermore, if $\Gamma = \mathbf{R}^d \setminus 0$ and $f \in \mathscr{F}\mathscr{B}(\omega) \cap \mathscr{E}'$, then $|f|_{\mathscr{F}\mathscr{B}(\omega,\Gamma)}$ agrees with $\|f\|_{\mathscr{F}\mathscr{B}(\omega)}$. For simplicity we write $|f|_{\mathscr{F}\mathscr{B}(\Gamma)}$ instead of $|f|_{\mathscr{F}\mathscr{B}(\omega,\Gamma)}$ when $\omega = 1$.

For the sake of notational convenience we set

$$\mathcal{B} = \mathscr{F}\mathscr{B}(\omega). \tag{3.2}$$

when

$$|\,\cdot\,|_{\mathcal{B}(\Gamma)} = |\,\cdot\,|_{\mathscr{F}\mathscr{B}(\omega,\Gamma)_x} \tag{3.3}$$

We let $\Theta_{\mathcal{B}}(f) = \Theta_{\mathscr{F}\mathscr{B}(\omega)}(f)$ be the set of all $\xi \in \mathbf{R}^d \setminus 0$ such that $|f|_{\mathcal{B}(\Gamma)} < \infty$, for some $\Gamma = \Gamma_\xi$. We also let $\Sigma_{\mathcal{B}}(f)$ be the complement of $\Theta_{\mathcal{B}}(f)$ in $\mathbf{R}^d \setminus 0$. Then $\Theta_{\mathcal{B}}(f)$ and $\Sigma_{\mathcal{B}}(f)$ are open respectively closed subsets in $\mathbf{R}^d \setminus 0$, which are independent of the choice of $x \in \mathbf{R}^d$ in (3.1).

Definition 3.1. Let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^d$, $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $\mathcal{B}$ be as in (3.2), and let X be an open subset of $\mathbf{R}^d$. The wave-front set of $f \in \mathscr{D}'(X)$, $\mathrm{WF}_{\mathcal{B}}(f) \equiv \mathrm{WF}_{\mathscr{F}\mathscr{B}(\omega)}(f)$ with respect to $\mathcal{B}$ consists of all pairs (x_0, ξ_0) in $X \times (\mathbf{R}^d \setminus 0)$ such that $\xi_0 \in \Sigma_{\mathcal{B}}(\varphi f)$ holds for each $\varphi \in C_0^\infty(X)$ such that $\varphi(x_0) \neq 0$.

We note that $\mathrm{WF}_{\mathcal{B}}(f)$ in Definition 3.1 is a closed set in $X \times (\mathbf{R}^d \setminus 0)$, since it is obvious that its complement is open. We also note that if $x \in \mathbf{R}^d$ is fixed and $\omega_0(\xi) = \omega(x,\xi)$, then $\mathrm{WF}_{\mathscr{F}\mathscr{B}(\omega)}(f) = \mathrm{WF}_{\mathscr{F}\mathscr{B}(\omega_0)}(f)$, since $\Sigma_{\mathcal{B}}$ is independent of x.

The following theorem shows that wave-front sets with respect to $\mathscr{F}\mathscr{B}(\omega)$ satisfy appropriate micro-local properties. It also shows that such wave-front sets decreases when the local Fourier BF-spaces increases, or when the weight ω decreases.

Theorem 3.2. *Let $X \subseteq \mathbf{R}^d$ be open, $\mathscr{B}_1, \mathscr{B}_2$ be translation invariant BF-spaces, $\varphi \in C^\infty(\mathbf{R}^d)$, $\omega_1, \omega_2 \in \mathscr{P}(\mathbf{R}^{2d})$ and $f \in \mathscr{D}'(X)$. If $\mathscr{F}\mathscr{B}_1(\omega_1)_{loc} \subseteq \mathscr{F}\mathscr{B}_2(\omega_2)_{loc}$, then*

$$\mathrm{WF}_{\mathscr{F}\mathscr{B}_2(\omega_2)}(\varphi f) \subseteq \mathrm{WF}_{\mathscr{F}\mathscr{B}_1(\omega_1)}(f).$$

Proof. It suffices to prove

$$\Sigma_{\mathcal{B}_2}(\varphi f) \subseteq \Sigma_{\mathcal{B}_1}(f). \tag{3.4}$$

when $\mathcal{B}_j = \mathscr{F}\mathscr{B}_j(\omega_j)$, $\varphi \in \mathscr{S}(\mathbf{R}^d)$ and $f \in \mathscr{E}'(\mathbf{R}^d)$. The local properties and Remark 1.2 also imply that it is no restriction to assume that $\omega_1 = \omega_2 = 1$.

Let $\xi_0 \in \Theta_{\mathcal{B}_2}(f)$, and choose open cones Γ_1 and Γ_2 in $\mathbf{R}^d$ such that $\overline{\Gamma_2} \subseteq \Gamma_1$. Since f has compact support, it follows that $|\widehat{f}(\xi)| \le C\langle \xi\rangle^{N_0}$ for some positive constants C and N_0. The result therefore follows if we prove that for each N, there are constants C_N such that

$$|\varphi f|_{\mathcal{B}_2(\Gamma_2)} \le C_N\Big(|f|_{\mathcal{B}_1(\Gamma_1)} + \sup_{\xi\in\mathbf{R}^d} \big(|\widehat{f}(\xi)|\langle \xi\rangle^{-N}\big)\Big)$$
$$\text{when} \quad \overline{\Gamma}_2 \subseteq \Gamma_1 \quad \text{and} \quad N = 1,2,\dots. \tag{3.5}$$

By letting $F(\xi) = |\widehat{f}(\xi)|$ and $\psi(\xi) = |\widehat{\varphi}(\xi)|$, it follows that ψ turns rapidly to zero at infinity and

$$|\varphi f|_{\mathcal{B}_2(\Gamma_2)} = |\varphi f|_{\mathscr{F}\mathscr{B}_2(\Gamma_2)} \|\mathscr{F}(\varphi f)\chi_{\Gamma_2}\|_{\mathscr{B}_2}$$
$$C\Big\| \Big(\int_{\mathbf{R}^d} \widehat{\varphi}(\,\cdot\, - \eta)\widehat{f}(\eta)\, d\eta\Big)\chi_{\Gamma_2}\Big\|_{\mathscr{B}_2} \le C(J_1 + J_2)$$

for some positive constant C, where

$$J_1 \Big\| \Big(\int_{\Gamma_1} \widehat{\varphi}(\,\cdot\, - \eta)\widehat{f}(\eta)\, d\eta\Big)\chi_{\Gamma_2}\Big\|_{\mathscr{B}_2} \tag{3.6}$$

and

$$J_2 \Big\| \Big(\int_{\complement\Gamma_1} \widehat{\varphi}(\,\cdot\, - \eta)\widehat{f}(\eta)\, d\eta\Big)\chi_{\Gamma_2}\Big\|_{\mathscr{B}_2} \tag{3.7}$$

and χ_{Γ_2} is the characteristic function of Γ_2. First we estimate J_1. By (3) in Definition 1.1 and (1.4), it follows for some constants $C_1, \dots, C_5$ that

$$J_1 \le C_1 \Big\| \int_{\Gamma_1} \widehat{\varphi}(\,\cdot\, - \eta)\widehat{f}(\eta)\, d\eta\Big\|_{\mathscr{B}_2} C_1 \|\widehat{\varphi} * (\chi_{\Gamma_1}\widehat{f})\|_{\mathscr{B}_2}$$
$$C_2 \|\varphi \mathscr{F}^{-1}(\chi_{\Gamma_1}\widehat{f})\|_{\mathscr{F}\mathscr{B}_2} \le C_3 \|\varphi \mathscr{F}^{-1}(\chi_{\Gamma_1}\widehat{f})\|_{\mathscr{F}\mathscr{B}_1}$$
$$C_4 \|\widehat{\varphi} * (\chi_{\Gamma_1}\widehat{f})\|_{\mathscr{B}_1} \le C_5 \|\widehat{\varphi}\|_{L^1_{(v)}} \|\chi_{\Gamma_1}\widehat{f}\|_{\mathscr{B}_1} C_\psi |f|_{\mathscr{F}\mathscr{B}_1(\Gamma_1)}, \tag{3.8}$$

where $C_\psi = C_5\|\widehat{\varphi}\|_{L^1_{(v)}} < \infty$, since $\widehat{\varphi}$ turns rapidly to zero at infinity. In the second inequality we have used the fact that $(\mathscr{F}\mathscr{B}_1)_{loc} \subseteq (\mathscr{F}\mathscr{B}_2)_{loc}$.

In order to estimate J_2, we note that the conditions $\xi \in \Gamma_2$, $\eta \notin \Gamma_1$ and the fact that $\overline{\Gamma_2} \subseteq \Gamma_1$ imply that $|\xi - \eta| > c\max(|\xi|, |\eta|)$ for some constant $c > 0$, since this is true when $1 = |\xi| \ge |\eta|$. We also note that if N_1 is large enough, then $\langle\,\cdot\,\rangle^{-N_1} \in \mathscr{B}_2$, because $\mathscr{S}$ is continuously embedded in $\mathscr{B}_2$. Since ψ turns rapidly to zero at infinity, it follows that for each $N_0 > d + N_1$ and $N \in \mathbf{N}$ such that $N > N_0$, it holds

$$
\begin{aligned}
J_2 &\le C_1\Big\|\Big(\int_{\complement\Gamma_1}\langle\,\cdot\, - \eta\rangle^{-(2N_0+N)}F(\eta)\,d\eta\Big)\chi_{\Gamma_2}\Big\|_{\mathscr{B}_2}\\
&\le C_2\Big\|\Big(\int_{\complement\Gamma_1}\langle\,\cdot\,\rangle^{-N_0}\langle\eta\rangle^{-N_0}(\langle\eta\rangle^{-N}F(\eta))\,d\eta\Big)\chi_{\Gamma_2}\Big\|_{\mathscr{B}_2}\\
&\le C_2\int_{\complement\Gamma_1}\|\langle\,\cdot\,\rangle^{-N_0}\chi_{\Gamma_2}\|_{\mathscr{B}_2}\langle\eta\rangle^{-N_0}(|\langle\eta\rangle^{-N}F(\eta)|)\,d\eta\\
&\le C\sup_{\eta\in\mathbf{R}^d}|\langle\eta\rangle^{-N}F(\eta)|, \qquad (3.9)
\end{aligned}
$$

for some constants C_1 and $C_2 > 0$, where

$$C = C_2\|\langle\,\cdot\,\rangle^{-N_0}\|_{\mathscr{B}_2}\|\langle\,\cdot\,\rangle^{-N_0}\|_{L^1} < \infty.$$

This proves (3.5), and the result follows. □

4. Mapping properties for pseudo-differential operators on wave-front sets

In this section we establish mapping properties for pseudo-differential operators on wave-front sets of Fourier Banach types. More precisely, we prove the following result (cf. (0.1)):

Theorem 4.1. *Let $t \in \mathbf{R}$, $\rho > 0$, $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $\omega_0 \in \mathscr{P}_{\rho,0}(\mathbf{R}^{2d})$, $a \in S^{(\omega_0)}_{\rho,0}(\mathbf{R}^{2d})$, and $f \in \mathscr{S}'(\mathbf{R}^d)$. Also let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^d$. Then*

$$
\begin{aligned}
\mathrm{WF}_{\mathscr{F}\mathscr{B}(\omega/\omega_0)}(\mathrm{Op}_t(a)f) &\subseteq \mathrm{WF}_{\mathscr{F}\mathscr{B}(\omega)}(f)\\
&\subseteq \mathrm{WF}_{\mathscr{F}\mathscr{B}(\omega/\omega_0)}(\mathrm{Op}_t(a)f)\bigcup \mathrm{Char}_{(\omega_0)}(a). \qquad (4.1)
\end{aligned}
$$

We shall mainly follow the proof of Theorem 3.1 in [31]. The following restatement of Proposition 3.2 in [31] shows that $(x_0, \xi) \notin \mathrm{WF}_{\mathscr{F}\mathscr{B}(\omega/\omega_0)}(\mathrm{Op}(a)f)$ when $x_0 \notin \operatorname{supp} f$.

Proposition 4.2. *Let $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $\omega_0 \in \mathscr{P}_{\rho,\delta}(\mathbf{R}^{2d})$, $0 \le \delta < \rho \le 1$, and let $a \in S^{(\omega_0)}_{\rho,\delta}(\mathbf{R}^{2d})$. Also let $\mathscr{B}$ be a translation invariant BF-space, and let the operator L_a on $\mathscr{S}'(\mathbf{R}^d)$ be defined by the formula*

$$(L_af)(x) \equiv \varphi_1(x)(\mathrm{Op}(a)(\varphi_2 f))(x), \quad f \in \mathscr{S}'(\mathbf{R}^d), \qquad (4.2)$$

where $\varphi_1 \in C_0^\infty(\mathbf{R}^d)$ *and* $\varphi_2 \in S_{0,0}^0(\mathbf{R}^d)$ *are such that*

$$\operatorname{supp} \varphi_1 \bigcap \operatorname{supp} \varphi_2 = \emptyset.$$

Then the kernel of L_a *belongs to* $\mathscr{S}(\mathbf{R}^{2d})$*. In particular, the following is true:*

(1) $L_a = \operatorname{Op}(a_0)$ *for some* $a_0 \in \mathscr{S}(\mathbf{R}^{2d})$*;*
(2) $\operatorname{WF}_{\mathscr{F}\mathscr{B}(\omega/\omega_0)}(L_a f) = \emptyset$*.*

Next we consider properties of the wave-front set of $\operatorname{Op}(a)f$ at a fixed point when f is concentrated to that point.

Proposition 4.3. *Let* ρ*,* ω*,* ω_0*,* a *and* $\mathscr{B}$ *be as in Theorem 4.1. Also let* $f \in \mathscr{E}'(\mathbf{R}^d)$*. Then the following is true:*

(1) *if* $\Gamma_1, \Gamma_2 \subseteq \mathbf{R}^d \setminus 0$ *are open cones such that* $\overline{\Gamma_2} \subseteq \Gamma_1$*, and* $|f|_{\mathscr{F}\mathscr{B}(\omega,\Gamma_1)} < \infty$*, then* $|\operatorname{Op}(a)f|_{\mathscr{F}\mathscr{B}(\omega/\omega_0,\Gamma_2)} < \infty$*;*
(2) $\operatorname{WF}_{\mathscr{F}\mathscr{B}(\omega/\omega_0)}(\operatorname{Op}(a)f) \subseteq \operatorname{WF}_{\mathscr{F}\mathscr{B}(\omega)}(f)$*.*

We note that $\operatorname{Op}(a)f$ in Proposition 4.3 makes sense as an element in $\mathscr{S}'(\mathbf{R}^d)$, by Proposition 1.7.

Proof. We shall mainly follow the proof of Proposition 3.3 in [31]. We may assume that $\omega(x,\xi) = \omega(\xi)$, $\omega_0(x,\xi) = \omega_0(\xi)$, and that $\operatorname{supp} a \subseteq K \times \mathbf{R}^d$ for some compact set $K \subseteq \mathbf{R}^d$, since the statements only involve local assertions.

Let $F(\xi) = |\widehat{f}(\xi)\omega(\xi)|$, and let $\mathscr{F}_1 a$ denote the partial Fourier transform of $a(x,\xi)$ with respect to the x-variable. By straightforward computation, it follows that for every $N \ge 0$, there is a constant C such that

$$|\mathscr{F}(\operatorname{Op}(a)f)(\xi)\omega(\xi)/\omega_0(\xi)| \le C \int_{\mathbf{R}^d} \langle \xi - \eta \rangle^{-N} F(\eta)\, d\eta \tag{4.3}$$

(cf. (3.6) and (3.8) in [31]).

We have to estimate

$$|(\operatorname{Op}(a)f)|_{\mathscr{F}\mathscr{B}(\omega/\omega_0,\Gamma_2)} \|\mathscr{F}(\operatorname{Op}(a)f)\omega/\omega_0 \chi_{\Gamma_2}\|_{\mathscr{B}}.$$

By (4.3) we get

$$\|\mathscr{F}(\operatorname{Op}(a)f)\omega/\omega_0\chi_{\Gamma_2}\|_{\mathscr{B}} \le C \Big\| \Big(\int \langle \cdot - \eta \rangle^{-N} F(\eta)\, d\eta \Big) \chi_{\Gamma_2} \Big\|_{\mathscr{B}} \le C(J_1 + J_2),$$

where C is a constant and

$$J_1 = \Big\| \Big(\int_{\Gamma_1} \langle \cdot - \eta \rangle^{-N} F(\eta)\, d\eta \Big) \chi_{\Gamma_2} \Big\|_{\mathscr{B}}$$

and

$$J_2 = \Big\| \Big(\int_{\complement\Gamma_1} \langle \cdot - \eta\rangle^{-N} F(\eta)\, d\eta \Big) \chi_{\Gamma_2} \Big\|_{\mathscr B}.$$

In order to estimate J_1 and J_2 we argue as in the proof of (3.5). More precisely, by (1.4) we get

$$J_1 \le \Big\| \int_{\Gamma_1} \langle\, \cdot\, - \eta\rangle^{-N} F(\eta)\, d\eta \Big\|_{\mathscr B} \| \langle\, \cdot\,\rangle^{-N} * (\chi_{\Gamma_1} F)\|_{\mathscr B}$$

$$\le C \| \langle\, \cdot\,\rangle^{-N} \|_{L^1_{(v)}} \| \chi_{\Gamma_1} F\|_{\mathscr B} < \infty.$$

Next we estimate J_2. Since $\overline{\Gamma_2} \subseteq \Gamma_1$, we get

$$|\xi - \eta| \ge c \max(|\xi|, |\eta|), \quad \text{when} \quad \xi \in \Gamma_2, \ \text{and } \eta \in \complement\Gamma_1,$$

for some constant $c > 0$. (Cf. the proof of Proposition 3.3.)

Since f has compact support, it follows that $F(\eta) \le C\langle \eta\rangle^{N_0}$ for some constants C and N_0. By combining these estimates we obtain

$$J_2 \le \Big\| \Big(\int_{\complement\Gamma_1} F(\eta) \langle\, \cdot\, - \eta\rangle^{-N}\, d\eta \Big) \chi_{\Gamma_2} \Big\|_{\mathscr B}$$

$$\le C \Big\| \Big(\int_{\complement\Gamma_1} \langle \eta\rangle^{N_0} \langle\, \cdot\,\rangle^{-N/2} \langle \eta\rangle^{-N/2}\, d\eta \Big) \chi_{\Gamma_2} \Big\|_{\mathscr B}$$

$$\le C \| \langle\, \cdot\,\rangle^{-N/2} \chi_{\Gamma_2} \|_{\mathscr B} \int_{\complement\Gamma_1} \langle \eta\rangle^{-N/2+N_0}\, d\eta.$$

Hence, if we choose N sufficiently large, it follows that the right-hand side is finite. This proves (1).

The assertion (2) follows immediately from (1) and the definitions. The proof is complete. □

Proof of Theorem 4.1. By Proposition 2.1 it is no restriction to assume that $t = 0$. We start to prove the first inclusion in (4.1). Assume that $(x_0, \xi_0) \notin \mathrm{WF}_{\mathscr F\mathscr B(\omega)}(f)$, let $\chi \in C_0^\infty(\mathbf R^d)$ be such that $\chi = 1$ in a neighborhood of x_0, and set $\chi_1 = 1 - \chi$ and $a_0(x, \xi) = \chi(x) a(x, \xi)$. Then it follows from Proposition 4.2 that

$$(x_0, \xi_0) \notin \mathrm{WF}_{\mathscr F\mathscr B(\omega/\omega_0)}(\mathrm{Op}(a)(\chi_1 f)).$$

Furthermore, by Proposition 4.3 we get

$$(x_0, \xi_0) \notin \mathrm{WF}_{\mathscr F\mathscr B(\omega/\omega_0)}(\mathrm{Op}(a_0)(\chi f)),$$

which implies that

$$(x_0, \xi_0) \notin \mathrm{WF}_{\mathscr F\mathscr B(\omega/\omega_0)}(\mathrm{Op}(a)(\chi f)),$$

since $\operatorname{Op}(a)(\chi f)$ is equal to $\operatorname{Op}(a_0)(\chi f)$ near x_0. The result is now a consequence of the inclusion

$$\begin{aligned}
&\operatorname{WF}_{\mathscr{F}\mathscr{B}(\omega/\omega_0)}(\operatorname{Op}(a)f)\\
&\qquad \subseteq \operatorname{WF}_{\mathscr{F}\mathscr{B}(\omega/\omega_0)}(\operatorname{Op}(a)(\chi f)) \bigcup \operatorname{WF}_{\mathscr{F}\mathscr{B}(\omega/\omega_0)}(\operatorname{Op}(a)(\chi_1 f)).
\end{aligned}$$

It remains to prove the last inclusion in (4.1). By Proposition 4.2 it follows that it is no restriction to assume that f has compact support. Assume that

$$(x_0,\xi_0) \notin \operatorname{WF}_{\mathscr{F}\mathscr{B}(\omega/\omega_0)}(\operatorname{Op}(a)f) \bigcup \operatorname{Char}_{(\omega_0)}(a),$$

and choose b, c and h as in Proposition 2.3 (4). We shall prove that $(x_0,\xi_0) \notin \operatorname{WF}_{\mathscr{F}\mathscr{B}(\omega)}(f)$. Since

$$f = \operatorname{Op}(1-c)f + \operatorname{Op}(b)\operatorname{Op}(a)f - \operatorname{Op}(h)f,$$

the result follows if we prove

$$(x_0,\xi_0) \notin \mathsf{S}_1 \bigcup \mathsf{S}_2 \bigcup \mathsf{S}_3,$$

where

$$\begin{aligned}
&\mathsf{S}_1 = \operatorname{WF}_{\mathscr{F}\mathscr{B}(\omega)}(\operatorname{Op}(1-c)f), \quad \mathsf{S}_2 = \operatorname{WF}_{\mathscr{F}\mathscr{B}(\omega)}(\operatorname{Op}(b)\operatorname{Op}(a)f)\\
&\mathsf{S}_3 = \operatorname{WF}_{\mathscr{F}\mathscr{B}(\omega)}(\operatorname{Op}(h)f),
\end{aligned}$$

and $c_0(x,\xi) = \chi(x)(1-c(x,\xi))$.

We start to consider S_2. By the first embedding in (4.1) it follows that

$$\mathsf{S}_2 = \operatorname{WF}_{\mathscr{F}\mathscr{B}(\omega)}(\operatorname{Op}(b)\operatorname{Op}(a)f) \subseteq \operatorname{WF}_{\mathscr{F}\mathscr{B}(\omega/\omega_0)}(\operatorname{Op}(a)f).$$

Since we have assumed that $(x_0,\xi_0) \notin \operatorname{WF}_{\mathscr{F}\mathscr{B}(\omega/\omega_0)}(\operatorname{Op}(a)f)$, it follows that $(x_0,\xi_0) \notin \mathsf{S}_2$.

Next we consider S_3. Since $h \in \mathscr{S}$, it follows that $\operatorname{Op}(h)f \in \mathscr{S}$. Hence S_3 is empty.

Finally we consider S_1. By the assumptions it follows that c_0 is zero in Γ, and by replacing Γ with a smaller cone, if necessary, we may assume that $c_0 = 0$ in a conical neighborhood of Γ. Hence, if $\Gamma \equiv \Gamma_1$, Γ_2, J_1 and J_2 are the same as in the proof of Proposition 4.3, then it follows from that proof and the fact that $c_0(x,\xi) \in S^0_{\rho,0}$ is compactly supported in the x-variable, that $J_1 < +\infty$, and that for each $N \ge 0$, there are constants C_N and C'_N such that

$$\begin{aligned}
&|\operatorname{Op}(c_0)f|_{\mathscr{F}\mathscr{B}(\omega/\omega_0,\Gamma_2)} \le C_N(J_1+J_2)\\
&\qquad \le C'_N\Big(J_1 + \Big\| \int_{\complement\Gamma_1} \langle\,\cdot\,\rangle^{-N}\langle\eta\rangle^{-N}\,d\eta\,\chi_{\Gamma_2}\Big\|_{\mathscr{B}}\Big). \qquad (4.4)
\end{aligned}$$

By choosing N large enough, it follows that

$$|\operatorname{Op}(c_0)f|_{\mathscr{F}\mathscr{B}(\omega/\omega_0,\Gamma_2)} < \infty.$$

This proves that $(x_0,\xi_0) \notin \mathsf{S}_1$, and the proof is complete. □

Remark 4.4. We note that the statements in Theorems 4.1 are not true if $\omega_0 = 1$ and the assumption $\rho > 0$ is replaced by $\rho = 0$. (Cf. Remark 3.7 in [31].)

Next we apply Theorem 4.1 on operators which are elliptic with respect to $S^{(\omega_0)}_{\rho,\delta}(\mathbf R^{2d})$, where $\omega_0 \in \mathscr P_{\rho,\delta}(\mathbf R^{2d})$. More precisely, assume that $0 \le \delta < \rho \le 1$ and $a \in S^{(\omega_0)}_{\rho,\delta}(\mathbf R^{2d})$. Then a and $\operatorname{Op}(a)$ are called (locally) *elliptic* with respect to $S^{(\omega_0)}_{\rho,\delta}(\mathbf R^{2d})$ or ω_0, if for each compact set $K \subseteq \mathbf R^d$, there are positive constants c and R such that

$$|a(x,\xi)| \ge c\omega_0(x,\xi), \quad x \in K,\ |\xi| \ge R.$$

Since $|a(x,\xi)| \le C\omega_0(x,\xi)$, it follows from the definitions that for each multi-index α, there are constants $C_{\alpha,\beta}$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta}|a(x,\xi)|\langle \xi\rangle^{-\rho|\beta|+\delta|\alpha|}, \quad x \in K,\ |\xi| > R,$$

when a is elliptic. (See e. g. [2,24].)

It immediately follows from the definitions that $\operatorname{Char}_{(\omega_0)}(a) = \emptyset$ when a is elliptic with respect to ω_0. The following result is now an immediate consequence of Theorem 4.1.

Theorem 4.5. *Let $\omega \in \mathscr P(\mathbf R^{2d})$, $\omega_0 \in \mathscr P_{\rho,0}(\mathbf R^{2d})$, $\rho > 0$, and let $a \in S^{(\omega_0)}_{\rho,0}(\mathbf R^{2d})$ be elliptic with respect to ω_0. Also let $\mathscr B$ be a translation invariant BF-space. If $f \in \mathscr S'(\mathbf R^d)$, then*

$$\mathrm{WF}_{\mathscr F\mathscr B(\omega/\omega_0)}(\operatorname{Op}(a)f) = \mathrm{WF}_{\mathscr F\mathscr B(\omega)}(f).$$

5. Wave-front sets of sup and inf types and pseudo-differential operators

In this section we put the micro-local analysis in a more general context compared to previous sections, and define wave-front sets with respect to sequences of Fourier BF-spaces.

Let $\omega_j \in \mathscr P(\mathbf R^{2d})$ and $\mathscr B_j$ be translation invariant BF-space on $\mathbf R^d$ when j belongs to some index set J, and consider the array of spaces, given by

$$(\mathcal B_j) \equiv (\mathcal B_j)_{j\in J}, \quad \text{where} \quad \mathcal B_j = \mathscr F\mathscr B_j(\omega_j), \quad j \in J. \tag{3.2$'$}$$

If $f \in \mathscr S'(\mathbf R^d)$, and $(\mathcal B_j)$ is given by (3.2)$'$, then we let $\Theta^{\sup}_{(\mathcal B_j)}(f)$ be the set of all $\xi \in \mathbf R^d \setminus 0$ such that for some $\Gamma = \Gamma_\xi$ and *each* $j \in J$ it holds $|f|_{\mathcal B_j(\Gamma)} < \infty$. We also let $\Theta^{\inf}_{(\mathcal B_j)}(f)$ be the set of all $\xi \in \mathbf R^d\setminus 0$ such that for some $\Gamma = \Gamma_\xi$ and *some* $j \in J$ it holds $|f|_{\mathcal B_j(\Gamma)} < \infty$. Finally we let $\Sigma^{\sup}_{(\mathcal B_j)}(f)$ and $\Sigma^{\inf}_{(\mathcal B_j)}(f)$ be the complements in $\mathbf R^d \setminus 0$ of $\Theta^{\sup}_{(\mathcal B_j)}(f)$ and $\Theta^{\inf}_{(\mathcal B_j)}(f)$ respectively.

Definition 5.1. Let J be an index set, $\mathscr B_j$ be translation invariant BF-space on $\mathbf R^d$, $\omega_j \in \mathscr P(\mathbf R^{2d})$ when $j \in J$, $(\mathcal B_j)$ be as in (3.2)$'$, and let X be an open subset of $\mathbf R^d$.

(1) The wave-front set of $f \in \mathscr{D}'(X)$, $\mathrm{WF}^{\sup}_{(\mathcal{B}_j)}(f) = \mathrm{WF}^{\sup}_{(\mathscr{F}\!\mathscr{B}_j(\omega_j))}(f)$, of *sup-type* with respect to $(\mathcal{B}_j)$, consists of all pairs (x_0, ξ_0) in $X \times (\mathbf{R}^d \setminus 0)$ such that $\xi_0 \in \Sigma^{\sup}_{(\mathcal{B}_j)}(\varphi f)$ holds for each $\varphi \in C_0^\infty(X)$ such that $\varphi(x_0) \neq 0$;

(2) The wave-front set of $f \in \mathscr{D}'(X)$, $\mathrm{WF}^{\inf}_{(\mathcal{B}_j)}(f) = \mathrm{WF}^{\inf}_{(\mathscr{F}\!\mathscr{B}_j(\omega_j))}(f)$, of *inf-type* with respect to $(\mathcal{B}_j)$, consists of all pairs (x_0, ξ_0) in $X \times (\mathbf{R}^d \setminus 0)$ such that $\xi_0 \in \Sigma^{\inf}_{(\mathcal{B}_j)}(\varphi f)$ holds for each $\varphi \in C_0^\infty(X)$ such that $\varphi(x_0) \neq 0$.

Remark 5.2. Let $\omega_j(x, \xi) = \langle \xi \rangle^{-j}$ for $j \in J = \mathbf{N}_0$ and $\mathscr{B}_j = L^{q_j}$, where $q_j \in [1, \infty]$. Then it follows that $\mathrm{WF}^{\sup}_{(\mathcal{B}_j)}(f)$ in Definition 5.1 is equal to the standard wave-front set $\mathrm{WF}(f)$ in Chapter VIII in [24].

The following result follows immediately from Theorem 4.1 and its proof. We omit the details.

Theorem 4.1′. *Let $\rho > 0$, $\omega_j \in \mathscr{P}(\mathbf{R}^{2d})$ for $j \in J$, $\omega_0 \in \mathscr{P}_{\rho,0}(\mathbf{R}^{2d})$, $a \in S^{(\omega_0)}_{\rho,0}(\mathbf{R}^{2d})$ and $f \in \mathscr{S}'(\mathbf{R}^d)$. Also let $\mathscr{B}_j$ be a translation invariant BF-space on $\mathbf{R}^d$ for every j. Then*

$$\begin{aligned} \mathrm{WF}^{\sup}_{(\mathscr{F}\!\mathscr{B}_j(\omega_j/\omega_0))}(\operatorname{Op}(a)f) &\subseteq \mathrm{WF}^{\sup}_{(\mathscr{F}\!\mathscr{B}_j(\omega_j))}(f) \\ &\subseteq \mathrm{WF}^{\sup}_{(\mathscr{F}\!\mathscr{B}_j(\omega_j/\omega_0))}(\operatorname{Op}(a)f) \bigcup \operatorname{Char}_{(\omega_0)}(a), \quad (4.1)' \end{aligned}$$

and

$$\begin{aligned} \mathrm{WF}^{\inf}_{(\mathscr{F}\!\mathscr{B}_j(\omega_j/\omega_0))}(\operatorname{Op}(a)f) &\subseteq \mathrm{WF}^{\inf}_{(\mathscr{F}\!\mathscr{B}_j(\omega_j))}(f) \\ &\subseteq \mathrm{WF}^{\inf}_{(\mathscr{F}\!\mathscr{B}_j(\omega_j/\omega_0))}(\operatorname{Op}(a)f) \bigcup \operatorname{Char}_{(\omega_0)}(a). \quad (4.1)'' \end{aligned}$$

The following generalization of Theorem 4.5 is an immediate consequence of Theorem 4.1′, since $\operatorname{Char}_{(\omega_0)}(a) = \emptyset$, when a is elliptic with respect to ω_0.

Theorem 4.5′. *Let $\rho > 0$, $\omega_j \in \mathscr{P}(\mathbf{R}^{2d})$ for $j \in J$, $\omega_0 \in \mathscr{P}_{\rho,0}(\mathbf{R}^{2d})$ and let $a \in S^{(\omega_0)}_{\rho,0}(\mathbf{R}^{2d})$ be elliptic with respect to ω_0. Also let $\mathscr{B}_j$ be a translation invariant BF-space on $\mathbf{R}^d$ for every j. If $f \in \mathscr{S}'(\mathbf{R}^d)$, then*

$$\mathrm{WF}^{\inf}_{(\mathscr{F}\!\mathscr{B}_j(\omega_j/\omega_0))}(\operatorname{Op}(a)f) \, \mathrm{WF}^{\inf}_{(\mathscr{F}\!\mathscr{B}_j(\omega_j))}(f)$$

and

$$\mathrm{WF}^{\sup}_{(\mathscr{F}\!\mathscr{B}_j(\omega_j/\omega_0))}(\operatorname{Op}(a)f) \, \mathrm{WF}^{\sup}_{(\mathscr{F}\!\mathscr{B}_j(\omega_j))}(f).$$

Remark 5.3. We note that many properties valid for the wave-front sets of Fourier BF-type also hold for wave-front sets in the present section. For example, the conclusions in Remark 4.4 and Theorem 4.5 hold for wave-front sets of sup- and inf-types.

Finally we remark that there are some technical generalizations of Theorem 4.1 which involve pseudo-differential operators with symbols in $S^{(\omega_0)}_{\rho,\delta}(\mathbf{R}^{2d})$ with $0 \le \delta < \rho \le 1$. From these generalizations it follows that

$$\mathrm{WF}(\mathrm{Op}(a)f) \subseteq \mathrm{WF}(f) \subseteq \mathrm{WF}(\mathrm{Op}(a)f) \bigcup \mathrm{Char}_{(\omega_0)}(a),$$

when $0 \le \delta < \rho \le 1$, $\omega_0 \in \mathscr{P}_{\rho,\delta}(\mathbf{R}^{2d})$, $a \in S^{(\omega_0)}_{\rho,\delta}(\mathbf{R}^{2d})$ and $f \in \mathscr{S}'(\mathbf{R}^d)$. (Cf. Theorem 5.3′ and Theorem 5.5 in [31].)

6. Wave front sets with respect to modulation spaces

In this section we define wave-front sets with respect to modulation spaces, and show that they coincide with wave-front sets of Fourier BF-types. In particular, all micro-local properties for pseudo-differential operators in the previous sections carry over to wave-front sets of modulation space types.

We start with defining general types of modulation spaces. Let (the window) $\phi \in \mathscr{S}'(\mathbf{R}^d) \setminus 0$ be fixed, and let $f \in \mathscr{S}'(\mathbf{R}^d)$. Then the short-time Fourier transform $V_\phi f$ is the element in $\mathscr{S}'(\mathbf{R}^{2d})$, defined by the formula

$$(V_\phi f)(x,\xi) \equiv \mathscr{F}(f \cdot \overline{\phi(\cdot - x)})(\xi).$$

We usually assume that $\phi \in \mathscr{S}(\mathbf{R}^d)$, and in this case the short-time Fourier transform $(V_\phi f)$ takes the form

$$(V_\phi f)(x,\xi)(2\pi)^{-d/2} \int_{\mathbf{R}^d} f(y)\overline{\phi(y-x)}e^{-i\langle y,\xi\rangle}\, dy,$$

when $f \in \mathscr{S}(\mathbf{R}^d)$.

Now let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d}$, with respect to $v \in \mathscr{P}(\mathbf{R}^{2d})$. Also let $\phi \in \mathscr{S}(\mathbf{R}^d) \setminus 0$ and $\omega \in \mathscr{P}(\mathbf{R}^{2d})$ be such that ω is v-moderate. Then the modulation space $M(\omega) = M(\omega, \mathscr{B})$ consists of all $f \in \mathscr{S}'(\mathbf{R}^d)$ such that $V_\phi f \cdot \omega \in \mathscr{B}$. We note that $M(\omega, \mathscr{B})$ is a Banach space with the norm

$$\|f\|_{M(\omega,\mathscr{B})} \equiv \|V_\phi f \omega\|_{\mathscr{B}} \tag{6.1}$$

(cf. [9]).

Remark 6.1. Assume that $p, q \in [1,\infty]$, $\omega \in \mathscr{P}(\mathbf{R}^{2d})$ and let $L^{p,q}_1(\mathbf{R}^{2d})$ and $L^{p,q}_2(\mathbf{R}^{2d})$ be the sets of all $F \in L^1_{loc}(\mathbf{R}^{2d})$ such that

$$\|F\|_{L^{p,q}_1} \equiv \Big(\int \Big(\int |F(x,\xi)|^p \, dx \Big)^{q/p} d\xi \Big)^{1/q} < \infty$$

and

$$\|F\|_{L^{p,q}_2} \equiv \Big(\int \Big(\int |F(x,\xi)|^p \, d\xi \Big)^{q/p} dx \Big)^{1/q} < \infty,$$

respectively (with obvious modifications when $p = \infty$ or $q = \infty$). Then $M(\omega, \mathscr{B})$ is equal to the usual modulation space $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ when $\mathscr{B} = L^{p,q}_1(\mathbf{R}^{2d})$. If instead $\mathscr{B} = L^{p,q}_2(\mathbf{R}^{2d})$, then $M(\omega, \mathscr{B})$ is equal to the space $W^{p,q}_{(\omega)}(\mathbf{R}^d)$, related to Wiener-amalgam spaces.

In the following proposition we list some important properties for modulation spaces. We refer to [16] for the proof.

Proposition 6.2. *Assume that $\mathscr{B}$ is a translation invariant BF-space on $\mathbf{R}^{2d}$ with respect to $v \in \mathscr{P}(\mathbf{R}^{2d})$, and that $\omega_0, v_0 \in \mathscr{P}(\mathbf{R}^{2d})$ are such that ω is v-moderate. Then the following is true:*

1. *if $\phi \in M^1_{(v_0 v)}(\mathbf{R}^d) \setminus 0$, then $f \in M(\omega, \mathscr{B})$ if and only if $V_\phi f \omega \in \mathscr{B}$. Furthermore, (6.1) defines a norm on $M(\omega, \mathscr{B})$, and different choices of ϕ gives rise to equivalent norms;*
2. *$M^1_{(v_0 v)} \subseteq M(\omega, \mathscr{B}) \subseteq M^\infty_{(1/(v_0 v))}$.*

The following generalization of Theorem 2.1 in [33] shows that modulation spaces are locally the same as translation invariant Fourier BF-spaces. We recall that if $\varphi \in \mathscr{S}(\mathbf{R}^d) \setminus 0$ and $\mathscr{B}$ is a translation invariant BF-space on $\mathbf{R}^{2d}$, then it follows from Proposition 1.4 that

$$\mathscr{B}_0 \equiv \{\, f \in \mathscr{S}'(\mathbf{R}^d)\, ;\, \varphi \otimes f \in \mathscr{B} \,\} \tag{6.2}$$

is a translation invariant BF-space on $\mathbf{R}^d$ which is independent of the choice of φ.

Proposition 6.3. *Let $\varphi \in C_0^\infty(\mathbf{R}^d) \setminus 0$, $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d}$, and let $\mathscr{B}_0$ be as in (6.2). Also let $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, and $\omega_0(\xi) = \omega(x_0, \xi)$, for some fixed $x_0 \in \mathbf{R}^d$. Then*

$$M(\omega, \mathscr{B}) \cap \mathscr{E}'(\mathbf{R}^d) = \mathscr{F}\mathscr{B}_0(\omega_0) \cap \mathscr{E}'(\mathbf{R}^d).$$

Furthermore, if $K \subseteq \mathbf{R}^d$ is compact, then

$$C^{-1}\|f\|_{\mathscr{F}\mathscr{B}_0(\omega_0)} \le \|f\|_{M(\omega,\mathscr{B})} \le C\|f\|_{\mathscr{F}\mathscr{B}_0(\omega_0)}, \quad f \in \mathscr{E}'(K), \tag{6.3}$$

for some constant C.

We need the following lemma for the proof.

Lemma 6.4. *Assume that $f \in \mathscr{E}'(\mathbf{R}^d)$. Then the following is true:*

1. *if $\phi \in C_0^\infty(\mathbf{R}^d)$, then there exists $0 \le \varphi \in C_0^\infty(\mathbf{R}^d)$ such that*
$$(V_\phi f)(x,\xi) = \varphi(x)(\widehat{f} * (\mathscr{F}(\overline{\phi(\cdot - x)})))(\xi)\, ; \tag{6.4}$$
2. *if $\varphi \in C_0^\infty(\mathbf{R}^d)$, then there exists $\phi \in C_0^\infty(\mathbf{R}^d)$ such that*
$$(\varphi \otimes \widehat{f})(x,\xi) = \varphi(x) V_\phi f(x,\xi). \tag{6.5}$$

Proof. (1) From the support properties it follows that there is a compact set $K \subseteq \mathbf{R}^d$ such that $\operatorname{supp} V_\phi f \subseteq K \times \mathbf{R}^d$. The assertion now follows from Fourier's inversion formula, by choosing $\varphi \in C_0^\infty$ such that $\varphi(x) = (2\pi)^{d/2}$ when $x \in K$.

The assertion (2) follows in a similar way by choosing $\phi \in C_0^\infty$ such that $\phi = 1$ on $\operatorname{supp} f - \operatorname{supp} \varphi$. □

Proof of Proposition 6.3. We may assume that $\omega = \omega_0 = 1$ in view of Remark 1.2. Assume that $f \in \mathscr{E}'$ and $\varphi \in C_0^\infty(\mathbf{R}^d) \setminus 0$. From (2) of Lemma 6.4 it follows that there exists $\phi \in C_0^\infty$ such that

$$\|f\|_{M(\mathscr{B})} = \|V_\phi f\|_{\mathscr{B}} \|\varphi \otimes \widehat{f}\|_{\mathscr{B}} = \|\widehat{f}\|_{\mathscr{B}_0},$$

and (6.3) follows. The proof is complete. $\square$

Let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d}$, $\phi \in \mathscr{S}(\mathbf{R}^d) \setminus 0$ be fixed, $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $\Gamma \subseteq \mathbf{R}^d \setminus 0$ be an open cone, and let χ_Γ be the characteristic function of Γ. For any $f \in \mathscr{S}'(\mathbf{R}^d)$ we set

$$|f|_{\mathcal{B}(\Gamma)} |f|_{M(\omega,\Gamma,\mathscr{B})} = |f|_{M^\phi(\omega,\Gamma,\mathscr{B})} \equiv \|(V_\phi f)\, \omega\, (1 \otimes \chi_\Gamma)\|_{\mathscr{B}} \qquad \text{when} \quad \mathcal{B} = M(\omega, \mathscr{B}). \tag{6.6}$$

We note that $|\cdot|_{\mathcal{B}(\Gamma)}$ defines a semi-norm on $\mathscr{S}'$ which might attain the value $+\infty$. If $\Gamma = \mathbf{R}^d \setminus 0$, then $|f|_{\mathcal{B}(\Gamma)} = \|f\|_{M(\omega,\mathscr{B})}$.

The sets $\Theta_{\mathcal{B}}(f)$ and $\Sigma_{\mathcal{B}}(f)$, and the wave-front set $\mathrm{WF}_{\mathcal{B}}(f)$ of f with respect to $\mathcal{B} = M(\omega, \mathscr{B})$ are now defined in the same way as in Section 3, after replacing the semi-norms of Fourier BF-types in (3.3) with the semi-norms in (6.6).

In Theorem 6.9 below we prove that wave-front sets of Fourier BF-spaces and modulation spaces agree with each others. As a first step we prove that $\mathrm{WF}_{M(\omega,\mathscr{B})}(f)$ is independent of ϕ in (6.6).

Proposition 6.5. *Let $X \subseteq \mathbf{R}^d$ be open, $f \in \mathscr{D}'(X)$, $\omega \in \mathscr{P}(\mathbf{R}^{2d})$ and let $\mathcal{B} = M(\omega, \mathscr{B})$. Then $\Theta_{\mathcal{B}}(f)$, $\Sigma_{\mathcal{B}}(f)$ and $\mathrm{WF}_{\mathcal{B}}(f)$ are independent of the window function $\phi \in \mathscr{S}(\mathbf{R}^d) \setminus 0$ in* (6.6).

We need some preparation for the proof, and start with the following lemma. We omit the proof, since the result can be found in [3].

Lemma 6.6. *Let $f \in \mathscr{E}'(\mathbf{R}^d)$ and $\phi \in \mathscr{S}(\mathbf{R}^d)$. Then for some constant N_0 and every $N \ge 0$, there are constants C_N such that*

$$|V_\phi f(x,\xi)| \le C_N \langle x \rangle^{-N} \langle \xi \rangle^{N_0}.$$

The following result can be found in [16]. Here $\widehat{*}$ is the twisted convolution, given by the formula

$$(F \mathbin{\widehat{*}} G)(x,\xi) = (2\pi)^{-d/2} \iint F(x-y, \xi-\eta) G(y,\eta) e^{-i\langle x-y,\eta \rangle}\, dy d\eta,$$

when $F, G \in \mathscr{S}(\mathbf{R}^{2d})$. The definition of $\widehat{*}$ extends in such way that one may permit one of F and G to belong to $\mathscr{S}'(\mathbf{R}^{2d})$, and in this case it follows that $F \mathbin{\widehat{*}} G$ belongs to $\mathscr{S}' \cap C^\infty$.

Lemma 6.7. *Let $f \in \mathscr{S}'(\mathbf{R}^d)$ and $\phi_j \in \mathscr{S}(\mathbf{R}^d)$ for $j = 1, 2, 3$. Then*

$$(V_{\phi_1} f) \mathbin{\widehat{*}} (V_{\phi_2} \phi_3) = (\phi_3, \phi_1)_{L^2} \cdot V_{\phi_2} f.$$

Proof of Proposition 6.5. We may assume that $f \in \mathscr{E}'(\mathbf{R}^d)$ and that $\omega(x,\xi) = \omega(\xi)$, since the statements only involve local assertions. Assume that $\phi, \phi_1 \in \mathscr{S}(\mathbf{R}^d) \setminus 0$ and let Γ_1 and Γ_2 be open cones in $\mathbf{R}^d$ such that $\overline{\Gamma_2} \subseteq \Gamma_1$. The assertion follows if we prove that

$$|f|_{M^{\phi}(\omega,\Gamma_2,\mathscr{B})} \le C(|f|_{M^{\phi_1}(\omega,\Gamma_1,\mathscr{B})} + 1) \tag{6.7}$$

for some constant C.

When proving (6.7) we shall mainly follow the proof of (3.5). Let $v \in \mathscr{P}$ be chosen such that ω is v-moderate, and let

$$\Omega_1 = \mathbf{R}^d \times \Gamma_1 \subseteq \mathbf{R}^d \times (\mathbf{R}^d \setminus 0) \qquad \text{and} \qquad \Omega_2 = \complement \Omega_1,$$

with characteristic functions χ_1 and χ_2 respectively. Here the complement is taken with respect to $\mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$. Also set

$$F_k(x,\xi) = |V_{\phi_1} f(x,\xi)\omega(\xi)\chi_k(x,\xi)| \qquad \text{and} \qquad G = |V_\phi \phi_1(x,\xi)v(\xi)|.$$

By Lemma 6.7, and the fact that ω is v-moderate we get

$$|V_\phi f(x,\xi)\omega(x,\xi)| \le C((F_1 + F_2) * G)(x,\xi),$$

for some constant C, which implies that

$$|f|_{M^{\phi}(\omega,\Gamma_2,\mathscr{B})} \le C(J_1 + J_2), \tag{6.8}$$

where

$$J_k = \|(F_k * G)(1 \otimes \chi_{\Gamma_2})\|_{\mathscr{B}}$$

and χ_{Γ_2} is the characteristic function of Γ_2. Since G turns rapidly to zero at infinity, (1.4) gives

$$J_1 \le \|F_1 * G\|_{\mathscr{B}} \le \|G\|_{L^1_{(v)}} \|F_1\|_{\mathscr{B}} = C|f|_{M^{\phi_1}(\omega,\Gamma_1,\mathscr{B})}, \tag{6.9}$$

where $C = \|G\|_{L^1_{(v)}}$.

Next we consider J_2. Since, for each $N \ge 0$, there are constants C_N such that

$$F_2(x,\xi) = 0, \qquad \text{and} \qquad \langle \xi - \eta \rangle^{-2N} \le C_N \langle \xi \rangle^{-N} \langle \eta \rangle^{-N}$$

when $\xi \in \Gamma_2$ and $\eta \in \complement \Gamma_1$, Lemma 6.6 and the computations in (3.9) give

$$(F_2 * G)(x,\xi) \le C_N \langle x \rangle^{-N} \langle \xi \rangle^{-N}, \qquad \xi \in \Gamma_2.$$

Hence $(F_2 * G) \in \mathscr{B}$ in view of Remark 1.3, which implies $J_2 < \infty$. The estimate (6.7) is now a consequence of (6.8) and (6.9). This completes the proof. $\square$

We are now able to prove the following.

Proposition 6.8. *Let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d}$, $\mathscr{B}_0$ be given by* (6.2) *and let $\omega \in \mathscr{P}(\mathbf{R}^{2d})$. Also let $f \in \mathscr{E}'(\mathbf{R}^d)$. Then*

$$\Theta_{M(\omega,\mathscr{B})}(f) = \Theta_{\mathscr{F}\mathscr{B}_0(\omega)}(f) \quad \textit{and} \quad \Sigma_{M(\omega,\mathscr{B})}(f) = \Sigma_{\mathscr{F}\mathscr{B}_0(\omega)}(f). \tag{6.10}$$

Proof. We may assume that $\omega = 1$ in view of Lemma 1.2. Let Γ_1, Γ_2 be open cones in $\mathbf{R}^d \setminus 0$ such that $\overline{\Gamma_2} \subseteq \Gamma_1$, let χ_{Γ_2} be the characteristic function of Γ_2, and let φ and ϕ be chosen such that (1) in Lemma 6.4 is fulfilled.

By (6.4) it follows that

$$|V_\phi f(x,\xi)| \le \varphi(x)(|\widehat{f}| * |\mathscr{F}\check{\phi}|)(\xi),$$

where $\check{\phi}(x) = \phi(-x)$. This gives

$$\begin{aligned} |f|_{M^\phi(\Gamma_2,\mathscr{B})} = \|V_\phi f\,(1\otimes \chi_{\Gamma_2})\|_{\mathscr{B}} &\le C\|\big(\varphi\otimes(|\widehat{f}| * |\mathscr{F}\check{\phi}|)\big)\,(1\otimes\chi_{\Gamma_2})\|_{\mathscr{B}} \\ &= C\|(|\widehat{f}| * |\mathscr{F}\check{\phi}|)\chi_{\Gamma_2}\|_{\mathscr{B}_0} \le C(J_1+J_2), \end{aligned}$$

for some constant C, where J_1 and J_2 are the same as in (3.6) and (3.7) with $\mathscr{B}_2 = \mathscr{B}_0$, $\psi = |\mathscr{F}\check{\phi}|$ and $F = |\widehat{f}|$.

A combination of the latter estimate, (3.8) and (3.9) now gives that for each $N \ge 0$, there is a constant C_N such that

$$|f|_{M^\phi(\Gamma_2,\mathscr{B})} \le C_N\Big(|f|_{\mathscr{F}\mathscr{B}_0} + \sup_\xi |\widehat{f}(\xi)\langle\xi\rangle^{-N}|\Big).$$

Hence, by choosing N large enough it follows that $|f|_{M^\phi(\Gamma_2,\mathscr{B})}$ is finite when $|f|_{\mathscr{F}\mathscr{B}_0} < \infty$. Consequently,

$$\Theta_{\mathscr{F}\mathscr{B}_0}(f) \subseteq \Theta_{M(\mathscr{B})}(f). \tag{6.11}$$

In order to get a reversed inclusion we choose φ and ϕ such that Lemma 6.4 (2) is fulfilled. Then (6.5) gives

$$\begin{aligned} |f|_{\mathscr{F}\mathscr{B}_0(\Gamma)} &= \|\varphi\otimes(\widehat{f}\chi_\Gamma)\|_{\mathscr{B}} \\ &= \|(\varphi\otimes 1)(V_\phi f\,(1\otimes\chi_\Gamma))\|_{\mathscr{B}} \le C_1\|\varphi\|_{L^\infty}\|V_\phi f(1\otimes\chi_\Gamma)\|_{\mathscr{B}} \\ &= C_2|f|_{M^\phi(\Gamma,\mathscr{B})}, \end{aligned}$$

for some constants $C_1, C_2 > 0$. This proves that (6.11) holds with reversed inclusion. The proof is complete. □

The following result is now an immediate consequence of Proposition 6.8.

Theorem 6.9. *Let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d}$, $\mathscr{B}_0$ be given by (6.2), $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $X \subseteq \mathbf{R}^d$ be open and let $f \in \mathscr{D}'(X)$. Then*

$$\mathrm{WF}_{\mathscr{F}\mathscr{B}_0(\omega)}(f) = \mathrm{WF}_{M(\omega,\mathscr{B})}(f).$$

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