Local orthogonal mappings and operator formulation for varying cross-sectional ducts.
Abstract

A method is developed for solving the two dimensional Helmholtz equation in a duct with varying cross-section region bounded by a curved top and flat bottom, having one region inside. To compute the propagation of sound waves in a curved duct with a curved internal interface is difficult problem. One method is to transform the wave equation into a solvable form and making the curved interface plane. To this end a local orthogonal transformation is developed for the varying cross-sectional duct having one medium inside. This transformation is first used to make the curved top of the waveguide flat and to transform the Helmholtz equation into an initial value problem. Later on the local orthogonal transformation is developed for a waveguide having two media inside with flat top, a flat bottom and a curved interface. This local orthogonal transformation is used to flatten the interface and also to transform the Helmholtz equation into a simple, solvable ordinary differential equation. In this paper we present operator formulation for the part with flat bottom and curved top including a curved interface. In the ordinary differential equation with operators in coefficients, obtained after the transformation, all the operations related to the transverse variable are treated as operators while the derivative with respect to the range variable is kept.

Key-words: Helmholtz equation; Local orthogonal transform; operator equation
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1 Introduction

In many applications, two dimensional waveguides are involved. These two dimensional waveguides need to be studied and modelled with adequate mathematical techniques and solved for validating for more complicated models. In most of the cases, the model developed is based on Partial Differential Equations (PDE). These PDE are extremely important both from a mathematical as well as an application point of view. We are interested in application of acoustics in ducts with varying cross-section and the duct may be curved. The governing equation is a variable coefficient two-dimensional Helmholtz equation. The method that motivated this work is local orthogonal transformation and stable one-way methods for acoustic waveguides [9], [8].

The propagation of sound in a waveguide with constant cross-section and constant material properties can be solved easily by applying the separation of variable method. If the waveguide is not straight and has a varying cross-section, then it is not possible to solve the wave equation with the same method. When there is a waveguide with curved interface or curved boundary, the "staircase " approximation can be used. This approximation replaces curved boundary or interface by a constant boundary or interface. This piecewise constant formulation has discontinuities between different segments. According to [9], the use of such an approximation gives rise to a large error. So it is not appropriate to use this approximation.

We need to find an alternative method to avoid this approximation. A conformal mapping can be used to make the curved boundary or interface flat. This mapping is a global transformation that has many geometrical properties. It preserves the angle of intersection between two curves and it also keeps the same scaling of transformation in all directions. The conditions at the boundary or interface contain normal derivatives. So the sound pressure and a property proportional to the normal derivative should be continuous at the interface or boundary before and after the transformation. Since, the conformal mapping keep the same scaling, the points at the interface translate along the interface after the transformation if the top and bottom of the waveguide are forced to be plane. So the conformal mapping produces gliding at the interface. Gliding will be discussed in the coming chapters. Clearly, to transform the normal derivatives at the boundary or interface to a derivative that contains transverse variable (\(\hat{z}\)) we need an orthogonal transformation. However, the non-orthogonal transformation used in [5] changes the normal derivative at the interface to a derivative that is the combination of both range (\(\hat{x}\)) and transverse (\(\hat{z}\)) variable. For the numerical implementation point of view the range variable can produce difficulties. Therefore, we need something more general than the conformal mapping.

The local orthogonal transformation is more general and it has a property which is very important in the current problem that it does not keep the same scaling in all directions. So with the help of this property we can overcome the gliding problem.

One solution of this problem is to map a curved duct to a straight one which transforms the wave equation into a new equation with new coefficients. In the next step the local orthogonal transformation in which we are interested in, is introduced see [9] and [7].

Our thesis focuses on the local orthogonal transformation and the transformation of the wave equation into a new initial value problem with one variable, where the Helmholtz equation is solved in a waveguide with varying cross-section. One method to solve such a waveguide problem having two materials inside with varying cross-section is introduced in Ref. [7]. After the transformation, a simple ordinary differential equation is obtained. This differential equation contains the derivative with respect to the range variable and that equation can be solved [8].

To solve for the sound waves in a straight waveguide with constant cross-section, we
always split the acoustic pressure $p$ into the sum of $p_+$ and $p_-$. Here $p_+$ are waves moving to the right and $p_-$ are waves moving to the left. The waveguides we study are more general but their ends are straight with constant cross-section. In this case, the wave splitting is performed so that $p_+$ is right going and $p_-$ is left going in the ends of the waveguide.

The whole idea is that when we work with a long waveguide, we have problem with large error. The duct with a single layer is solved with a conformal mapping and there exists a stable method. In the stable method an operator formulation is used. This formulation for a waveguide having two materials is the main part of this thesis. The solution of this formulation is, however, outside the scope of this thesis. A similar method was developed by Nilsson [8] and also other techniques are used by Andersson with conformal mappings in [1] and [3]. Andersson has used stable methods also as in [4]. So we proceed with similar problems as Andersson has done.

1.1 Problem definition

Waveguides in two dimensions can, according to a theory of Nilsson [8], be treated in the following way. First the waveguide is mapped onto a straight waveguide with a conformal mapping. Then a stable solution is found by solving Riccati equations for either the transmission and reflection operators or the Dirichlet to Neumann operators. Recent research at the university has dealt with the construction of efficient numerical routines for conformal mappings as well as generalization of the boundary conditions from Dirichlet and Neumann to impedance conditions. The next step in the research is to treat two-layered waveguides where one layer may be air and the other an absorptive material. The current project is part of this work dealing with questions like local orthogonal mappings and operator formulation for the two-layered waveguide.

We consider acoustic waves in a two-dimensional waveguide with one layer. Later on we extend our work to a waveguide with two layers with an internal interface. Figure 3.1 shows the geometry.

1.2 Thesis outline

We start our thesis with the simple two-dimensional wave equation (Helmholtz equation) and discuss its solution with the help of separation of variables method in Sec. 2.1. A local orthogonal transformation for waveguide with one layer is discussed in Sec. 2.2 and after this the wave equation is transformed into a operator equation in Sec. 2.4.

Later we extend our work to two-layered varying cross-sectional duct in Sec 3.1. In Sec 3.2 the problem is solved and in Sec 3.3 the equations for the part below and above the interface are transformed into operator equation. Similarly in sec. 4.1, a local orthogonal transformation is discussed for the two-layered duct and in Sec. 4.2 wave equation is transformed into operator equation. AT the end we finish our this thesis with the conclusion in Sec. 6.

1.3 The aim

The main aim of the thesis is to develop a method which can be used to solve the wave equation in a two-dimensional waveguide. The aim is to introduce a local orthogonal transformation and to use this transformation on the acoustic waves in a varying cross-sectional waveguide and to transform our problem to a more simple one.
2 A duct with one medium and varying cross-section

In this chapter, we concentrate on the propagation of sound in a two- dimensional duct with flat lower and curved upper part. The duct consists of one medium only. The material which we are going to discuss is taken from [9] and [8].

2.1 Helmholtz Equation

We consider the two-dimensional reduced wave equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} + k^2 p = 0,$$

(2.1)

for $-\infty < x < \infty$, $0 < z < a$ and $k = \frac{\omega}{c}$ is the wave number, where $\omega$ is the angular frequency and $c$ is the speed of sound. Our study is motivated by applications in duct acoustics, where $p$ is the acoustics pressure, $z$ is the vertical axis and $x$ is the range variable in the horizontal direction. The top and the bottom of the duct is given by $z = a$ and $z = 0$, respectively. The normal component of the velocity vanishes at the boundaries. Therefore the top and bottom boundary conditions are

$$\frac{\partial p}{\partial z} \bigg|_{z=0} = 0, \quad \frac{\partial p}{\partial z} \bigg|_{z=a} = 0.$$  

(2.2)

We also assume that $k$ is independent of $x$ and $z$. By applying separation of variables $p(x, z) = \sum_{n=0}^{\infty} X_n(x)Z_n(z)$ on (2.1), we get the fundamental system ($n = 0, 1, 2, \cdots$)

$$Z_n(z) = \cos \mu_n z, \quad \mu_n = \frac{n\pi}{a},$$  

(2.3)

and

$$X_n(x) = A_ne^{i\alpha_n x} + B_ne^{-i\alpha_n x} = X_n^+ + X_n^-, \quad \alpha_n^2 = k^2 - \frac{n^2 \pi^2}{a^2}.$$  

(2.4)

Here we choose

$$\alpha_n = \begin{cases} \sqrt{k^2 - \frac{n^2 \pi^2}{a^2}}, & k \geq \frac{n\pi}{a} \\ i\sqrt{\frac{n^2 \pi^2}{a^2} - k^2}, & k < \frac{n\pi}{a} \end{cases}.$$  

If we choose time dependence $e^{-i\omega t}$, then

$$e^{i(\alpha_n x - i\omega t)} = \begin{cases} e^{i\sqrt{k^2 - \frac{n^2 \pi^2}{a^2}} x - i\omega t}, & k \geq \frac{n\pi}{a} \\ e^{-i\sqrt{\frac{n^2 \pi^2}{a^2} - k^2} x - i\omega t}, & k < \frac{n\pi}{a} \end{cases}.$$  

(2.5)

It is clear that $e^{i(\alpha_n x - i\omega t)}$ is propagating to the right when $k \geq \frac{n\pi}{a}$ and is attenuating for increasing $x$ when $k < \frac{n\pi}{a}$. We define this as a right going wave. A left going wave is defined by replacing $x$ with $-x$. If there is no wave coming from right, then

$$p = p^+ = \sum_{n=0}^{\infty} X_n(x) \cos \mu_n z,$$

where

$$X_n(x) = A_ne^{i\alpha_n x},$$
and

\[ X'_n(x) = i\alpha_n X_n(x). \]

If there is no wave from left, then

\[ p = p^− = \sum_{n=0}^\infty X_n(x) \cos \mu_n z, \]

with

\[ X_n(x) = B_n e^{-i\alpha_n x}, \]

and

\[ X'_n(x) = -i\alpha_n X_n(x). \]

The complete solution is

\[ p = p^+ + p^− = \sum_{n=0}^\infty A_n e^{i\alpha_n x} \cos \mu_n z + \sum_{n=0}^\infty B_n e^{-i\alpha_n x} \cos \mu_n z. \]

(2.6)

From the discussion above it follows that \( p^+ \) is a right going wave and \( p^− \) is a left going wave. Suppose that all the sources are enclosed in a finite region. In particular this means that there are no sources at infinity. In this case we have only \( p^+ \) to the right of the source region and \( p^− \) to the left. If we instead have only one source at \( x = −\infty \) but instead have a finite scattering region. Then we will have only \( p^+ \) to the right of the scattering region but both \( p^+ \) and \( p^− \) to the left.

In the next section a local orthogonal transformation will be developed. Here conformal mapping can be used to solve the problem of sound propagation in a waveguide having one media inside. The conformal mapping is a global transformation that has many geometrical properties. It keeps the angle of intersection between two curves and it has same scale of transformation in all directions. So for the current problem, the conformal mapping can be used even if we require same scaling in all directions. The problem comes when we solve the same problem in a duct having two materials inside. The conformal mapping can not be used for such kind of waveguides. In chapter 4 we will discuss some details.

2.2 Local orthogonal transform

In this section, we are dealing with a varying cross-sectional duct having one layer, \( 0 < z < h(x) \), where \( z = h(x) \) is the upper boundary of the duct and \( z = 0 \) is the lower boundary.

Here it is difficult to solve the Helmholtz equation in the given \((x, z)\) plane. For this we need a transformation that transforms our problem to the new plane i.e. \((\hat{x}, \hat{z})\) plane, that also keeps the boundary conditions. Under this transformation, for a given pair of new (or old) variables, the old (or new) variables can be calculated. In this thesis we are following the concept of local orthogonal transformation introduced in [9], because this transformation is natural for smooth boundaries. Consider

\[ \hat{x} = f(x, z), \quad \hat{z} = g(x, z), \]

(2.7)

that maps \( z = h(x) \) to \( \hat{z} = 1 \) and maps the horizontal line \( z = 0 \) to \( \hat{z} = 0 \). We consider also

\[ g(x, h(x)) = 1, \quad g(x, 0) = 0, x \in \mathbb{R}, \]

(2.8)
and we assume that \( \hat{z} \) is a linear function of \( z \) for all \( x \). Assuming
\[
g(x, z) = a(x)z + b(x),
\] (2.9)
and using (2.8), we get
\[
\left\{ \begin{align*}
g(x, h(x)) &= 1 = a(x)h(x) + b(x) \\
g(x, 0) &= 0 = a(x).0 + b(x)
\end{align*} \right., \quad x \in \mathbb{R}. \tag{2.10}
\]
Using the simultaneous solution of the above two equations for \( a \) and \( b \), (2.7) becomes
\[
\hat{z} = g(x, z) = \frac{z}{h(x)}. \tag{2.11}
\]
The transformation (2.7) should be orthogonal. If \( h'(x) \neq 0 \), then the orthogonality condition \( f_x g_x + f_z g_z = 0 \) leads to the equation
\[
f_z - \frac{zh'(x)}{h(x)} f_x = 0,
\] or
\[
h(x)f_z = zh'(x)f_x. \tag{2.12}
\]
We consider an integration on the curve \( f(x, z) = c \) (constant), to find the function \( f \), that is \( z = z(x) \) and also let \( \hat{x} = f(x, 0) = x \). Since \( f(x, z) = c \),
\[
\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \left( \frac{dz}{dx} \right) = 0. \tag{2.13}
\]
From (2.12) and (2.13) we get
\[
-z \frac{\partial f}{\partial z} \frac{dz}{dx} = \frac{h(x)}{h'(x)} \frac{\partial f}{\partial z}.
\]
Assuming \( \frac{\partial f}{\partial z} \neq 0 \), the above equation reduces to
\[
\frac{h(x)}{h'(x)} dx + z dz = 0.
\]
By integrating both sides from 0 to \( z \) about \( z \), we get
\[
\int_{\hat{x}}^{x} \frac{h(t)}{h'(t)} dt + \frac{1}{2} z^2 + c = 0, \tag{2.14}
\]
where \( c \) is the constant of integration. If \( x \) is equal to \( \hat{x} \) and \( z = 0 \), then the above equation yields \( c = 0 \). If \( h'(x) = 0 \), we set \( \hat{x} = x \). Given \( (x, z) \), we can, if \( h'(x) = 0 \) solve \( \hat{x} \) from the above equation, say, by Newton’s method. If \( (\hat{x}, \hat{z}) \) is given, we first solve \( x \) from
\[
\int_{\hat{x}}^{x} \frac{h(t)}{h'(t)} dt + \frac{z^2}{2}h^2(x) = 0, \tag{2.15}
\]
then obtain \( z \) by \( z = \hat{z}h(x) \). Now we check for any given pair of variables \( (x, z) \), does there exist a unique solution \( (\hat{x}, \hat{z}) \)? This has already been discussed in [9] and we follow this presentation. Since \( \hat{z} \) is given by \( \hat{z} = \frac{z}{h(x)} \), we only consider \( \hat{x} = f(x, z) \). We assume that \( h' \) is continuous. It is clear that \( \hat{x} = x \) if \( h'(x) = 0 \), and only consider the case when \( h'(x) \neq 0 \).
and we assume \( x_m < x < x_n \), where \( x_m \) and \( x_n \) are the two closest zeros of \( h' \). Now let us define a function by

\[
F(\tau) = \int_{\tau}^{x} \frac{h(t)}{h'(t)} dt + \frac{1}{2} \tau^2
\]

and assume that \( h' \) is positive in \((x_m, x_n)\). Under the assumption that \( h \) has a continuous and bounded second order derivative in \((x_m, x_n)\), we have

\[
\lim_{\tau \to x_m^+} F(\tau) = +\infty, \quad \lim_{\tau \to x_n^-} F(\tau) = -\infty.
\]

Since

\[
F'(\tau) = -\frac{h(\tau)}{h'(\tau)} < 0.
\]

\( F \) is strictly decreasing in \((x_m, x_n)\). Therefore, there must be a unique solution \( \hat{x} \) satisfying \( F(\hat{x}) = 0 \).

Here we only developed the local orthogonal transformation. In Appendix we will discuss in details the comparison of the conformal mapping and the local orthogonal mapping. We will show how the local orthogonal transformation transforms the straight duct to a varying cross-sectional duct.

### 2.3 Transformation of the equation

In this section we transform Helmholtz equation to a new equation by using local orthogonal transformation. Now using (2.7) and (2.11), instead of (2.1) we get the transformed equation

\[
\frac{\partial^2 p}{\partial \hat{x}^2} + \alpha(\hat{x}, \hat{z}) \frac{\partial^2 p}{\partial \hat{x} \partial \hat{z}} + \beta(\hat{x}, \hat{z}) \frac{\partial p}{\partial \hat{z}} + \gamma(\hat{x}, \hat{z}) \frac{\partial^2 p}{\partial \hat{z}^2} + k^2 p = 0 \tag{2.16}
\]

where \( \alpha, \beta \) and \( \gamma \) are given by

\[
\begin{align*}
\alpha(\hat{x}, \hat{z}) &= -\frac{2h'(x)}{h(x)}, \\
\beta(\hat{x}, \hat{z}) &= -\frac{2h'(x) - h(x)h''(x)}{h^2(x)}, \\
\gamma(\hat{x}, \hat{z}) &= \frac{2h^2(x)}{h^2(x)} + 1,
\end{align*}
\]

\( \alpha, \beta \) and \( \gamma \) given above are the same as those introduced in [9] but expressed in a different way, because we did not follow the method which is discussed in [9]. Here to transform the Helmholtz equation to an operator equation we follow the technique introduced in [8]. The boundary conditions are

\[
\frac{\partial p}{\partial \hat{z}}|_{\hat{z}=0} = 0, \quad \frac{\partial p}{\partial \hat{z}}|_{\hat{z}=a} = 0. \tag{2.18}
\]

### 2.4 Operator equation

Now we reformulate problem (2.1) for the duct having constant cross-section with refractive index \( \mu(\hat{x}, \hat{z}) = 1 \). Here we choose refractive index \( \mu(\hat{x}, \hat{z}) \) equal to one for simplicity. We can write (2.16) in the following form

\[
\frac{d^2 p}{d\hat{x}^2} + A(\hat{x}) \frac{dp}{d\hat{x}} - B^2(\hat{x}) p = 0. \tag{2.19}
\]

Instead of the PDE (2.16) we have (2.19), where all operations related to \( \hat{z} \) are treated as operators while the derivative with respect to \( \hat{x} \) is kept. Equation (2.19) is an ordinary
differential equation with operator valued coefficients. In case of (2.1), the operator $A = 0$ and the operator

$$B^2 = -\frac{\partial^2}{\partial z^2} - k^2(x, z).$$

For (2.16) the operator

$$A(\hat{x}) = \alpha(\hat{x}, \hat{z}) \frac{\partial}{\partial \hat{z}},$$

and the operator

$$B^2(\hat{x}) = -\beta(\hat{x}, \hat{z}) \frac{\partial}{\partial \hat{z}} - \gamma(\hat{x}, \hat{z}) \frac{\partial^2}{\partial \hat{z}^2} - k^2(\hat{x}, \hat{z}).$$

We have developed a transformation (2.11) that maps the varying cross-sectional duct to a duct with constant cross-section i.e $\hat{z} = 1$, and finally we have transformed our problem into a new operator equation (2.19). Since the duct is described by co-ordinate surfaces $\hat{x} = 0$ and $\hat{x} = a$ and the boundary conditions (2.18) are the same as (2.2), it is possible to expand

$$p(\hat{x}, \hat{z}) = \sum_{n=0}^{\infty} p_n(\hat{x}) Z_n(\hat{z}),$$

where $Z_n$ is given by (2.3). In this way (2.19) is transformed to a matrix valued ordinary differential equation for the vector $(p_0(\hat{x}), p_1(\hat{x}), p_2(\hat{x}), \cdots)$, which requires a stable method for its solution. This is stated in detail in [8].
3 A duct with two media and plane intersection

In this chapter, we concentrate on the propagation of sound in a two-dimensional duct with flat top, flat bottom and plane internal interface. Here the duct consists of two media. Figure 3.1 shows the geometry. We obtain the solutions for propagation of sound in a duct having two media with constant speed \(c_1\) in the bottom layer and also the constant speed \(c_2\) in the top layer. The solution are either trigonometric or exponential, since both the speeds are constant. In this chapter the material is taken from [6].

3.1 Helmholtz Equation for the duct with two media.

As we are dealing with a two-layered duct, we need to develop Helmholtz equation for region \(0 \leq z \leq a\) and for the region \(a \leq z \leq b\) separately, one with \(k = k_1\) and another with \(k = k_2\).

![Figure 3.1: A straight duct with two media and plane intersection with sound speed \(c_1\) and density \(\rho_1\) in the first part and sound speed \(c_2\), density \(\rho_2\) in the second part.](image)

The sound speed is constant within each layer but changes across the interface. Therefore we separate the two dimensional wave equation

\[
\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} + k^2 p = 0, \tag{3.1}
\]

into two equations one for each region,

\[
\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} + k_1^2 p = 0, \quad 0 \leq z < a, \tag{3.2}
\]

\[
\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} + k_2^2 p = 0, \quad a < z \leq b. \tag{3.3}
\]

for \(-\infty < x < +\infty\), and \(k_1 = \frac{\omega}{c_1}\) is the wave number in first medium and \(k_2 = \frac{\omega}{c_2}\) is the wave number in the second medium, where \(\omega\) is angular velocity, \(c_1\) and \(c_2\) is the speed in the bottom and top part respectively. The boundary conditions are assumed to be

\[
\frac{\partial p}{\partial z} = 0, \quad z = 0 \tag{3.4}
\]

\[
\frac{\partial p}{\partial z} = 0, \quad z = b \tag{3.5}
\]
The coupling condition at the interface \( z = a \) is

\[
p_1|_{z=a} = p_2|_{z=a}, \quad \frac{1}{\rho_1} \left( \frac{\partial p_1}{\partial z} \right)_{z=a} = \frac{1}{\rho_2} \left( \frac{\partial p_2}{\partial z} \right)_{z=a}, \tag{3.6}
\]

where \( \rho_1 \) and \( \rho_2 \) are constant densities in the bottom and top part respectively, we have \( \rho = \rho_1 \) for the layer \( 0 < z < a(x) \) and \( \rho = \rho_2 \) for the layer \( a(x) < z < b(x) \). The coupling condition reveals that the pressure and the normal velocities at the interface are equal.

### 3.2 Solution of the problem

In this section, we solve our problem at the given boundary conditions. We assume that \( k_1 \) and \( k_2 \) are independent of \( x \) and \( z \). First we solve equation (3.2), by applying separation of variables assuming that \( p \) is the product of two functions \( X(x) \) and \( Z(z) \),

\[
p(x,z) = \sum_{n=0}^{\infty} X_n(x) Z_n(z),
\]

substituting and dividing by \( X(x)Z(z) \) gives

\[
\frac{X''(x)}{X(x)} + \frac{Z''(z)}{Z(z)} + k_1^2 = 0.
\]

Since the first term is a function of \( x \) and the second term is only a function of \( z \), both the functions must be constant, otherwise the sum of three terms can not be zero for all \( x \) and \( z \). We get the following pair of equations

\[
\frac{X''(x)}{X(x)} + \alpha_n^2 = 0, \quad \frac{Z''(z)}{Z(z)} + \mu_n^2 = 0. \tag{3.7}
\]

Where the relation between constants \( \alpha_n^2 \) and \( \mu_n^2 \) is given by

\[
k_1^2 = \alpha_n^2 + \mu_n^2,
\]

from (3.7), we get

\[
Z_n(z) = A \cos \mu_n z + B \sin \mu_n z. \tag{3.8}
\]

By applying condition (3.4) on (3.8), gives \( B = 0 \) and the above equation becomes

\[
Z_n(z) = \cos \mu_n z, \tag{3.9}
\]

Now again from equation (3.7),

\[
X''(x) + \alpha_n^2 X(x) = 0, \quad \alpha_n^2 = k_1^2 - \mu_n^2,
\]

which implies that

\[
X_n(x) = Ce^{i\alpha_n x} + De^{-i\alpha_n x},
\]

\[
= X_n^+ + X_n^-.
\]

If there is no wave coming from right, then the fundamental system \( (n = 0, 1, 2, \cdots) \) is

\[
p(x,z) = e^{i\alpha_n x} \cos \mu_n z, \quad 0 \leq z \leq a \tag{3.11}
\]
Now we solve equation (3.3) for \(a \leq z \leq b\), by applying separation of variable again, and doing the same process as above, we get

\[
Z_n(z) = E \cos \phi_n z + F \sin \phi_n z, \tag{3.12}
\]

and

\[
X_n(x) = Ce^{i\beta_n x} + De^{-i\beta_n x}, \tag{3.13}
\]

\[
\beta_n = \sqrt{k_n^2 - \phi_n^2}. \quad \text{If there is no wave coming from right, then the required solution is}
\]

\[
p = p^+(x, z) = \sum_{n=0}^{\infty} e^{i\beta_n x}(E \cos \phi_n z + F \sin \phi_n z). \tag{3.14}
\]

Now by applying (3.5) on (3.14), we get

\[
0 = -E \sin \phi_n b + F \cos \phi_n b, \tag{3.15}
\]

now using (3.6) and by equating (3.11) and (3.14), we get

\[
e^{i\alpha_n x} \cos \mu_n a = e^{i\beta_n x}(E \cos \phi_n a + F \sin \phi_n a), \tag{3.16}
\]

by solving equation (3.15) and (3.16) and substituting the values of constants \(E\) and \(F\) in (3.14), we finally get

\[
p^+(x, z) = \frac{e^{i\alpha_n x} \cos \mu_n a \cos \phi_n (b - z)}{\cos \phi_n (b - a)}, \quad a \leq z \leq b. \tag{3.17}
\]

Hence the required result of the two layered duct, which satisfy the boundary conditions is given by

\[
p^+(x, z) = \begin{cases} 
  e^{i\alpha_n x} \cos \mu_n z, & 0 \leq z \leq a \\
  \frac{e^{i\alpha_n x} \cos \mu_n a \cos \phi_n (b - z)}{\cos \phi_n (b - a)}, & a \leq z \leq b
\end{cases}
\]

Similarly if there is no wave from the left, then the left going wave is defined by replacing \(x\) with \(-x\). Hence for left going wave we get the following expression.

\[
p^{-}(x, z) = \begin{cases} 
  e^{-i\alpha_n x} \cos \mu_n z, & 0 \leq z \leq a \\
  \frac{e^{-i\alpha_n x} \cos \mu_n a \cos \phi_n (b - z)}{\cos \phi_n (b - a)}, & a \leq z \leq b
\end{cases}
\]

Let us decompose \(p\) by \(p = p^+ + p^-\), so finally the solutions that satisfy the boundary conditions at the bottom, at the top and at the interface is

\[
p(x, z) = \begin{cases} 
  \sum_{n=0}^{\infty} (e^{i\alpha_n x} + e^{-i\alpha_n x}) \cos \mu_n z, & 0 \leq z \leq a \\
  \sum_{n=0}^{\infty} \frac{(e^{i\alpha_n x} + e^{-i\alpha_n x}) \cos \mu_n a \cos \phi_n (b - z)}{\cos \phi_n (b - a)}, & a \leq z \leq b
\end{cases} \tag{3.18}
\]

From the above discussion it follows that \(p^+\) is a right going wave and \(p^-\) is a left going wave.
3.3 Operator equation for the duct with plane intersection

In this section we formulate the equations for the two-layered duct as one ordinary differential equation in the $x-$ variable. The partial derivatives in the $z-$ variable are included in the operators so that the ordinary differential equations has operator-valued coefficients. In order to have only one differential equation for the combined two layers, the coupling conditions are introduced in the requirements for the space $Z$ to which the function belong.

For the space $Z$ we consider functions depending on $z$, where $x$ is treated as parameter. We define the space

$$Z = \begin{cases} f(z) = f_1(z), & 0 \leq z < a \\ f(z) = f_2(z), & a < z \leq b \end{cases}$$

with $f_1$ and $\frac{df_1}{dz} \in L_2(0, a)$, $f_2$ and $\frac{df_2}{dz} \in L_2(a, b)$,

$$\begin{cases} \frac{df_1}{dz}(0) = 0, \frac{df_2}{dz}(b) = 0, f_1(a) = f_2(b) \\ \frac{1}{\rho_1} \frac{df_1}{dz}(a) = \frac{1}{\rho_2} \frac{df_2}{dz}(a) \end{cases}$$

and the operator

$$B^2 = \begin{cases} -\frac{\partial^2}{\partial z^2} - k_1^2, & 0 \leq z < a \\ -\frac{\partial^2}{\partial z^2} - k_2^2, & a < z \leq b \end{cases}$$

Then we can combine partial differential equations (3.2) and (3.3) with the boundary conditions (3.4) and (3.5) and the coupling conditions (3.6) to get the ordinary differential equation

$$\frac{d^2 p}{dx^2} - B^2 p = 0, \quad p \in Z,$$

with the operator coefficient $B^2$ in (3.21), allowing the use of ordinary differentiation in $x$.

4 A duct with two media and curved intersection

In this section we are dealing with a duct with two media but with curved intersection. The figure 4.1 shows the geometry. In order to map the curved duct to straight one and to transform the Helmholtz equation to an operator equation, we need to develop an orthogonal transformation. In this chapter the material and the techniques are mainly taken from [7] and [8].

4.1 Local orthogonal transform for duct with two media

In this section, the problem that we are dealing with is a varying cross-sectional duct with two materials inside i.e. $0 < z < h(x)$ and $h(x) < z < D$, where $z = h(x)$ is the smooth internal interface. Figure 4.1 shows the geometry. We need to develop a transformation for both parts.

A conformal mapping that has many geometrical properties can be used to solve the problem of sound propagation in a two layered duct. It takes much time to calculate and it has same scaling in all directions. The property that it has same scaling in all directions
creates problem when the conformal mapping is operated. Because of this property we get gliding at the interface.

So, to avoid the gliding problem we need a transformation that does not keep same scaling of transformation in all directions. It is only possible with the local orthogonal transformation. The local orthogonal transformation is easy to calculate and it does not keep same scaling in all directions. One should keep in mind that the transformation should be orthogonal. Because, the orthogonal transformation transforms the interface conditions that contains the normal derivative to a derivative that contains only the transverse $\hat{z}$ variable.

For the sound propagation in a two layered duct, the local orthogonal transformation is more suitable than the conformal mapping. It is because of the scaling property mentioned above. Let $a$ and $b$ be two points at the interface as shown in figure 4.2. The pressure and the normal derivatives at these points are equal i.e $p(a) = p(b)$ and $\frac{\partial p}{\partial \hat{z}}(a) = \frac{\partial p}{\partial \hat{z}}(b)$.

As the conformal mapping keeps the same scaling in all direction, the points at the interface have moved apart after the transformation. We need a transformation that can help us to avoid gliding at the interface. It is only possible if we do not require the same scaling in all directions and this can be done with the local orthogonal transformation. It is mentioned above that the local orthogonal transformation does not keep the same scaling in all directions. So by using this transformation we can overcome the gliding problem. Figure 4.3 shows the gliding at the interface when the conformal mapping is used.
4.1.1 Local orthogonal transformation for the part below the interface

For the first part $0 < z < h(x)$ we have already developed the transformation in the previous section (A duct with one medium and varying cross-section).

Let $(\hat{x}, \hat{z})$ be the new variables. For the layer, $0 < z < h(x)$, consider

$$\hat{x} = f(x, z), \quad \hat{z} = g(x, z). \tag{4.1}$$

The equation 4.1 maps $z = \xi h(x)$ to $\hat{z} = \xi$ and map the horizontal line $z = 0$ to $\hat{z} = 0$.

We consider

$$g(x, \xi h(x)) = \xi, \quad g(x, 0) = 0, x \in \mathbb{R}, \tag{4.2}$$

we assume that $\hat{z}$ is a linear function of $z$ for all $x$. We have

$$g(x, z) = a(x) z + b(x) \tag{4.3}$$

by using (2.8), we get

$$\left\{ \begin{array}{l} g(x, \xi h(x)) = \xi = a(x) \xi h(x) + b(x) \\ g(x, 0) = 0 = a(x) 0 + b(x), \end{array} \quad x \in \mathbb{R}. \right. \tag{4.4}$$

using the simultaneous solution of the above two equations for $a(x)$ and $b(x)$, (4.3) becomes

$$\hat{z} = g(x, z) = \frac{z}{h(x)}. \tag{4.5}$$

The transformation should be orthogonal, if $h'(x) \neq 0$ then the orthogonality condition $f_x g_z + f_z g_x = 0$ leads to the equation

$$f_z - \frac{zh'(x)}{h(x)} f_x = 0,$$

or

$$h(x) f_z = zh'(x) f_x. \tag{4.6}$$

We consider an integration on the curve $f(x, z) = c$ (constant), to find the function $f$, that is $z = z(x)$ and also let $\hat{x} = f(x, 0) = x$. Since $f(x, z) = c$,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \frac{dz}{dx} = 0$$

From (4.6), we get

$$-\frac{z}{\partial z} \frac{dz}{dx} = h(x) \frac{\partial f}{\partial z}$$
Assuming $\frac{\partial f}{\partial z} \neq 0$, the above equation reduces to
\[
\frac{h(x)}{h'(x)} dx + z dz = 0. 
\]
By integrating both sides, we get
\[
\int_{\tilde{x}}^{x} \frac{h(t)}{h'(t)} dt + \frac{1}{2} z^2 = 0. 
\] (4.7)
If $h'(x) = 0$, we set $\hat{x} = x$. For given $(x, z)$, we can solve if $h'(x) = 0$, $\hat{x}$ from the above equation, say, by Newton’s method. If $(\hat{x}, \hat{z})$ is given, we first solve $x$ from
\[
\int_{\tilde{x}}^{x} \frac{h(t)}{h'(t)} dt + \frac{1}{2} h^2(x) = 0, 
\] (4.8)
then obtain $z$ by $z = \hat{z} h(x)$.

4.1.2 Local orthogonal transformation for the part above the interface

In this section the local orthogonal transformation for the part above the interface i.e $h(x) < z < D$, where $D$ is larger than $h(x)$ for all $x$ will be developed. We consider
\[
\hat{x} = \hat{f}(x, z), \quad \hat{z} = \hat{g}(x, z) 
\]
that maps the interface $z = \xi h(x)$ to $\hat{z} = \bar{\xi}$ (constant) and keep the horizontal line $z = D$ unchanged. We get
\[
\hat{g}(x, \xi h(x)) = \bar{\xi} , \quad \hat{g}(x, D) = D. 
\] (4.9)
we assume that $\hat{g}$ is a linear function of $z$ for all $x$. We get
\[
\hat{g}(x, z) = a(x) z + b(x) 
\] (4.10)
by using equation (4.9) in above equation, we have
\[
\begin{cases}
\hat{g}(x, \xi h(x)) = \xi = a(x) \xi h(x) + b(x) \\
\hat{g}(x, D) = D = a(x) D + b(x)
\end{cases}, \quad x \in \mathbb{R}. 
\] (4.11)
using the simultaneous solution of the above two equations for $a(x)$ and $b(x)$, (4.10) becomes,
\[
\hat{z} = \hat{g}(x, z) = \frac{\bar{\xi} - D}{\xi h(x) - D} z + \frac{D[\xi h(x) - \bar{\xi}]}{\xi h(x) - D}, 
\] (4.12)
From the orthogonality condition
\[
\hat{f}_{x} \hat{g}_{x} + \hat{f}_{z} \hat{g}_{z} = 0, 
\] (4.13)
we obtain
\[
\int_{x_{s}}^{x} \frac{D - h(t)}{h'(t)} dt - \frac{1}{2} [z - D]^2 - (\xi h(x) - D)^2 = 0. 
\] (4.14)
where $(x_{s}, h(x_{s}))$ corresponds to $(\hat{x}, \hat{\xi})$ and $x_{s}$ can be determined as in the first layer. If $h'(x) \neq 0$ and for given $(\hat{x}, \hat{z})$, we can solve $x$ and $z$ from Eq.(4.14) and Eq.(4.12). And for given $(x, z)$, we can obtain $\hat{z}$ from Eq.(4.12) and $\hat{x}$ from
\[
\int_{\tilde{x}}^{x} \frac{h(t)}{h'(t)} dt + \frac{1}{2} [\xi h(x)]^2 = 0, 
\] (4.15)
Finally, if $h'(x) = 0$, we have $\hat{x} = x$ and the linear relationship between $z$ and $\hat{z}$ remains same.
4.2 Transformation of the equation

In this section we transform (3.2) and (3.3) into new simple equations. So by applying the transformation (4.1) for the lower part $0 < z < h(x)$, we get the transformed equation instead of (3.2)

\[
\frac{\partial^2 p}{\partial \xi^2} + \alpha(\xi, \hat{z}) \frac{\partial^2 p}{\partial \hat{\xi} \partial \hat{z}} + \beta(\xi, \hat{z}) \frac{\partial p}{\partial \hat{z}} + \gamma(\xi, \hat{z}) \frac{\partial^2 p}{\partial \hat{z}^2} + k_1^2(\xi, \hat{z})p = 0, \quad (4.16)
\]

\[\alpha, \beta \text{ and } \gamma \text{ are given by:}
\]

\[
\begin{cases}
\alpha(\xi, \hat{z}) = -\frac{2\xi h'(x)}{h(x)}, \\
\beta(\xi, \hat{z}) = -\frac{2(2h^2(x)-h(x)h''(x))}{h'(x)}, \\
\gamma(\xi, \hat{z}) = \frac{\xi h^2(x)}{h'^2(x)} + 1.
\end{cases}
\]

while for the upper part $h(x) < z < D$, we get the following transformed equation

\[
\frac{\partial^2 p}{\partial \xi^2} + \alpha(\xi, \hat{z}) \frac{\partial^2 p}{\partial \hat{\xi} \partial \hat{z}} + \beta(\xi, \hat{z}) \frac{\partial p}{\partial \hat{z}} + \gamma(\xi, \hat{z}) \frac{\partial^2 p}{\partial \hat{z}^2} + k_2^2(\xi, \hat{z})p = 0, \quad (4.17)
\]

where \(\alpha, \beta \text{ and } \gamma\) are given by

\[
\begin{cases}
\alpha(\xi, \hat{z}) = -\frac{2\xi (\xi h(x)-D)(\xi+D)h''(x)+4\xi^2 Dh'(x)}{(\xi h(x)-D)^2}, \\
\beta(\xi, \hat{z}) = -\frac{\xi (\xi h(x)-D)(\xi+D+2D\xi)h''(x)-2\xi^2 h'(x))}{(\xi h(x)-D)^2}, \\
\gamma(\xi, \hat{z}) = \frac{(\xi-D)^2-\xi^2 (\xi h(x)-D)(\xi+D)+2D\xi h'(x)}{(\xi-D)^4} + 1.
\end{cases}
\]

\(\alpha, \beta \text{ and } \gamma\) given above for the parts $0 < z < h(x)$ and $h(x) < z < D$ are same as those introduced in [7] but expressed in a different way, because we did not follow the same technique which is discussed in Ref. [7]. Here to transform the Helmholtz equation to an operator equation we follow the technique introduced in Ref. [8], consequently these values are identical to those discussed in Ref. [7].

4.3 The coupling condition

In this section the coupling condition for the duct with two media and curved intersection will be developed. Obviously the conditions (3.4) and (3.5) becomes

\[
\begin{align*}
&\frac{\partial p}{\partial \hat{z}}(0) = 0, \\
&\frac{\partial p}{\partial \hat{z}}(b) = 0. \quad (4.18)
\end{align*}
\]

The coupling condition (3.6) can be written as

\[
\frac{1}{\rho_1} \left( \tau_1 \frac{\partial p_1}{\partial \hat{z}} \right) = \frac{1}{\rho_2} \left( \tau_2 \frac{\partial p_2}{\partial \hat{z}} \right), \quad (4.19)
\]

where $\frac{\partial p_1}{\partial \hat{z}} = \tau_1 \frac{\partial p}{\partial \hat{z}}$ and $\frac{\partial p_2}{\partial \hat{z}} = \tau_2 \frac{\partial p}{\partial \hat{z}}; \tau_1$ is the scale factor for the part below the interface and $\tau_2$ is scale factor for the part above the interface. The procedure for calculating the
scale factor is explained in [2, section 9.6]. By using the procedure for the scale factor, \( \tau_1 \) and \( \tau_2 \) can be calculated and are given as

\[
\tau_1 = h(x) \sqrt{1 + \hat{z}^2 h(x)^2},
\]

\[
\tau_2 = \frac{h(x) - D}{(1 - D)^2} \sqrt{(1 - D)^2 + (h(x) - D)^2 (\hat{z} - D)^2}.
\]

By using these values in (4.19) we get the required coupling condition

\[
\frac{1}{\rho_1} \left( h(x) \sqrt{1 + \hat{z}^2 h(x)^2} \right) \left( \frac{\partial p_1}{\partial \hat{z}} \right) = \frac{1}{\rho_2} \left( \frac{h(x) - D}{(1 - D)^2} \sqrt{(1 - D)^2 + (h(x) - D)^2 (\hat{z} - D)^2} \right) \left( \frac{\partial p_2}{\partial \hat{z}} \right).
\]

### 4.4 Scattering of waves in the duct with curved intersection

The prime objective for the transformation of the curved intersection to a plane one, is to generalize the stable scattering theory of Nilsson [8] to this type of duct. This scattering theory, which is a combination of wave splitting and invariant embedding, is based on the following ingredients.

(I) The duct is first transformed to a plane duct so that Fourier methods can be employed, which enforces that also the intersection is plane.

(II) Both ends of the duct must have constant properties in the axial direction so that the wave can be uniquely splitted into left going and right going parts; the duct treated in section 3 meets this requirement.

(III) An operator formulation is required and the eigen function in the straight ends are used to transform the operator equations to matrix equations that are treated with numerical software.

The construction of the eigen functions is found in section 3.2 and the corresponding operator formulation in section 3.3. Whereas it is outside the scope of this thesis to solve the scattering problem, we present briefly in this section an operator formulation for the part with curved interface.

The sound pressure \( p \) solves the ordinary differential equation

\[
\frac{d^2 p}{d\hat{z}^2} + A(\hat{z}) \frac{dp}{d\hat{z}} + B^2(\hat{z}) p = 0, \quad p \in \hat{z},
\]

with operator coefficients \( A(\hat{z}) \) and \( B^2(\hat{z}) \).

Like before we write

\[
\hat{Z} = \begin{cases} 
  f(\hat{z}) = f_1(\hat{z}), & 0 \leq \hat{z} < a \\
  f(\hat{z}) = f_2(\hat{z}), & a < \hat{z} \leq b 
\end{cases}
\]

\( f_1 \) and \( \frac{df_1}{d\hat{z}} \in L_2(0, a) \), \( f_2 \) and \( \frac{df_2}{d\hat{z}} \in L_2(a, b) \),

\[
\frac{df_1}{d\hat{z}}(0) = \frac{df_2}{d\hat{z}}(b) = 0,
\]

and include also in the definition of \( \hat{z} \) the coupling conditions (4.18) and (4.20).
For the operators we have

\[
\begin{align*}
A_1(\hat{x}) &= \alpha(\hat{x}, \hat{z}) \frac{\partial}{\partial \hat{z}}, \\
A_2(\hat{x}) &= \alpha(\hat{x}, \hat{z}) \frac{\partial}{\partial \hat{z}},
\end{align*}
\]

and

\[
\begin{align*}
B_1^2(\hat{x}) &= -\beta(\hat{x}, \hat{z}) \frac{\partial}{\partial \hat{z}} - \gamma(\hat{x}, \hat{z}) \frac{\partial^2}{\partial \hat{z}^2} - k_1^2(\hat{x}, \hat{z}), \\
B_2^2(\hat{x}) &= -\beta(\hat{x}, \hat{z}) \frac{\partial}{\partial \hat{z}} - \gamma(\hat{x}, \hat{z}) \frac{\partial^2}{\partial \hat{z}^2} - k_2^2(\hat{x}, \hat{z}).
\end{align*}
\]

Where sub index 1 and 2 refers to values in the lower and upper layers respectively.
5 Appendix

Consider a straight horizontal channel in the \((\hat{x}, \hat{z})\) plane as shown in figure 5.1. The bottom of the waveguide is at \(\hat{z} = 0\) and the top boundary is at \(\hat{z} = 1\). The coordinates in the \((\hat{x}, \hat{z})\) plane are orthogonal because for every point \(p_0\) the two coordinate lines \(\hat{z} = \hat{z}_0\) and \(\hat{x} = \hat{x}_0\) intersect orthogonally at \(p_0\). As the boundaries are horizontal, the normal vector at each point of the boundary is parallel to the \(\hat{z}\)-axis.

A conformal mapping can be used to transform a straight horizontal channel to a varying cross-sectional channel, like the one in figure 5.2 or vice versa. It is mentioned earlier that the conformal mapping has many geometrical properties. It preserves the angle of intersection between two curves. The conformal mapping has the same scale of transformation in all directions, but this is not the case in general for the local orthogonal transformation. A disadvantage of the conformal mapping is that, when the waveguide is very long and the boundaries or the interface are complicated, it requires much effort for calculations.

On the other hand the local orthogonal mapping (2.7) has almost the same properties, but it has two advantages when compared with conformal mapping.

1) The local orthogonal mapping is usually easier to calculate numerically when the boundaries or the interface are complicated.

2) A property that is of importance in the current problem is that the scale is not the same in all directions.

The original \((x, z)\) and the new coordinates \((\hat{x}, \hat{z})\) are related by nonlinear equations (2.7). These nonlinear equations can be solved easily by Newton’s method and the coordinates \((\hat{x}, \hat{z})\) and the coordinates \((x, z)\) can be calculated from each other. Therefore the local orthogonal transformation is more suitable for such kind of problems. So we have found a transformation that is more general than the conformal mapping.

The conditions at the interface contains the normal derivative, so it is needed to use an orthogonal transform that changes the normal derivative to a derivative that contains only the transverse \((\hat{z})\) variable. If the transformation is non orthogonal at the interface, it changes the normal derivative at the interface to a derivative that is the combination of both the range \((\hat{x})\) and the transverse \((\hat{z})\) variable. This would produce a problem that is
difficult to solve. Therefore, the transformation should be orthogonal. A nonorthogonal transformation is used in [5].

When the local orthogonal maps (2.7) transforms the straight horizontal channel $-\infty < \hat{x} < +\infty$, $0 < \hat{z} < 1$ in the $(\hat{x}, \hat{z})$ plane into a similar channel $-\infty < x < +\infty$, $0 < z < h(x)$ with varying cross-section, it maps the upper boundary $\hat{z} = 1$ to the upper boundary in the $xz$ plane and $\hat{z} = 0$ to the lower boundary. For each point generated by the intersection of coordinate lines in the $(\hat{x}, \hat{z})$ plane, there is a corresponding point in the $(x, z)$-plane. So the lines in the $(x, z)$-plane form a grid. Figure 5.2 is the figure obtained after the transformation. The normal vector at each point of the boundary in the $(\hat{x}, \hat{z})$ plane which is parallel to the $z$-axis, is mapped to the normal at each point of the boundary in the $(x, z)$ plane. The horizontal lines in figure 5.1 are coordinate lines with $\hat{z} = \text{constant}$ and $\hat{x} = \text{constant}$. Similarly, the local orthogonal transformation (2.7) can be used to transform the problem in the varying cross-sectional duct into a similar problem in a straight duct as shown in figure 5.1. After the transformation the result will be the set of orthogonal and constant lines in the $(\hat{x}, \hat{z})$-plane. Here, we found a new coordinate system to transform the problem because it is not convenient to analyze the problem in the original coordinate system in which the physical problem is defined.

![Figure 5.2: Sketch map of the waveguide having one material inside.](image)

If a varying cross-sectional waveguide consists of more than one part having different internal properties and has complicated geometry, then it is more interesting to examine the problem in such a waveguide. So far we have been studying the problem in a duct having one material inside. Now, we move to a waveguide that contains two materials inside. A waveguide with such a geometry is assumed to have two materials inside separated by an internal interface. Figure 4.1 shows the geometry.

In our case, we only consider a varying cross-sectional duct having $z = h(x)$ as an internal interface. The interface divides the waveguide into two layered media where the first layer is located at $0 < z < h(x)$, the second layer is located at $h(x) < z < D$. It is more complicated to analyze the problem in such a waveguide. For such kind of problems it is convenient to use a coordinate transformation to flatten the curved interface. According to [7], a conformal mapping can be performed to solve the problem in a waveguide having two media. Although, it has many geometrical properties but it requires much effort for
its computations when the waveguide is large or the interface is complicated. The local orthogonal transformation is more suitable for long and complicated boundaries or interface because it is usually easier to generate numerically. As the conditions at the interface contains the normal derivative, it is needed to use an orthogonal transform that change the normal derivative to a derivative that contains only the transverse ($\hat{z}$) variable. If the transformation is non orthogonal, then it changes the normal derivative at the interface to a derivative that is the combination of both range ($\hat{x}$) and transverse ($\hat{z}$) variable. For the numerical implementation point of view the range variable can produce difficulties. Therefore the transformation should be orthogonal. A nonorthogonal transformation is used in [5].

As the conformal mapping has a property that it keeps the same scale in all directions. So, because of this property we get "gliding" at the interface when we use the conformal mapping for a duct with an internal interface. And the problem can not be solved if the same scaling in all directions is required. We have seen that the local orthogonal transformation is more general than the conformal mapping. The local orthogonal transformation does not keep the same scale in all direction. Therefore, with the help of this transformation we can avoid gliding at the interface.

The original ($x, z$) and the new coordinates ($\hat{x}, \hat{z}$) in the parts below and above the interface $h(x)$ are related by the nonlinear equations (4.1) and (4.9). These nonlinear equations can be solved easily by Newton’s method and the coordinates ($\hat{x}, \hat{z}$) and the coordinates ($x, z$) can be calculated from each other. Therefore the local orthogonal transformation is more suitable for such kind of problems. So we have found a transformation that is more general than the conformal mapping.

Consider a straight horizontal channel in the ($\hat{x}, \hat{z}$) plane as shown in figure 5.3. The bottom of the waveguide is at $\hat{z} = 0$, the top boundary is at $\hat{z} = 3$ and $\hat{z} = 1$ is the internal interface. The constant lines of rectangular grid in the ($\hat{x}, \hat{z}$) plane are orthogonal because for every point $p_0$ any two lines intersect orthogonally at $p_0$. As the boundaries are horizontal, normal vector at each point of the boundary is parallel to $\hat{z}$- axis.

Figure 5.3: Sketch map of the waveguide having two materials inside.

When the local orthogonal maps (4.1) and (4.9) transforms the straight horizontal channel $-\infty < \hat{x} < +\infty, 0 < \hat{z} < 1$ and $1 < \hat{z} < D = 3$ in the ($\hat{x}, \hat{z}$) plane into a similar channel
\(-\infty < x < +\infty, 0 < z < h(x) \) and \( h(x) < z < D = 3 \) with varying cross-section where \( h(x) \) is an internal interface, it maps \( \xi = D \) to the upper boundary in the \( xz \) plane, the internal interface \( \xi \) to the internal interface \( z = \xi h(x) \) (we choose \( \xi = 1 \)) and \( \xi = 0 \) to the lower boundary. Finally after the transformation figure 5.4 is obtained.

Figure 5.4: Sketch map of the waveguide having two materials inside with curved internal interface.
6 Conclusion

This work provides a basis to develop a numerical technique for the propagation of sound in waveguides with curved top and flat bottom (but no internal interface). It also provides a basis for waveguides with flat top, flat bottom, and smooth internal interface. For the varying cross-sectional duct, one way is to use the staircase approximation. The staircase approximation approximates and replaces the curved boundary or interface by a constant function. Such an approximation produces large errors. We found an alternative method to avoid the staircase approximation. The local orthogonal transformation is introduced to flatten the boundary or the interface. The transformation is orthogonal, it transforms the conditions at the boundary or the interface that contains normal derivative to a derivative that does not contain range variable \( \hat{x} \). The non orthogonal transformation would change the normal derivative to a derivative that contains both the range variable \( \hat{x} \) and the transverse variable \( \hat{z} \). In the numerical solutions, the non orthogonal transformation produces large error.

The new variables \((\hat{x}, \hat{z})\) and the old variables \((x, z)\) are related by non linear equations. Newton’s method can be used to solve these non linear equations. The new and old variables can be calculated easily from each other since this transformation is local. The operator formulation for the part having flat bottom and curved top, and the part with internal interface is presented. Helmholtz equation is transformed into an ordinary differential equation that contains operators in its coefficients. The transformed equation only contains derivative with respect to new range variable \( \hat{x} \), whereas the derivatives with respect to \( \hat{z} \) are treated as operators.
References


