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Licentiate Thesis

Nelson-type Limits for α-Stable Lévy Processes

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To Rola, Adam
and
my parents

Abstract

Brownian motion has met growing interest in mathematics, physics and particularly in finance since it was introduced in the beginning of the twentieth century. Stochastic processes generalizing Brownian motion have influenced many research fields theoretically and practically. Moreover, along with more refined techniques in measure theory and functional analysis more stochastic processes were constructed and studied. Lévy processes, with Brownian motion as a special case, have been of major interest in the recent decades. In addition, Lévy processes include a number of other important processes as special cases like Poisson processes and subordinators. They are also related to stable processes.

In this thesis we generalize a result by S. Chandrasekhar [2] and Edward Nelson who gave a detailed proof of this result in his book in 1967 [12]. In Nelson's first result standard Ornstein-Uhlenbeck processes are studied. Physically this describes free particles performing a random and irregular movement in water caused by collisions with the water molecules. In a further step he introduces a nonlinear drift in the position variable, i.e. he studies the case when these particles are exposed to an external field of force in physical terms.

In this report, we aim to generalize the result of Edward Nelson to the case of α-stable Lévy processes. In other words we replace the driving noise of a standard Ornstein-Uhlenbeck process by an α-stable Lévy noise and introduce a scaling parameter β uniformly in front of all vector fields in the cotangent space, even in front of the noise. This corresponds to time being sent to infinity. With Chandrasekhar's and Nelson's choice of the diffusion constant the stationary state of the velocity process (which is approached as time tends to infinity) is the Boltzmann distribution of statistical mechanics. The scaling limits we obtain in the absence and presence of a nonlinear drift term by using the scaling property of the characteristic functions and time change, can be extended to other types of processes rather than α-stable Lévy processes.

In future, we will consider to generalize this one dimensional result to Euclidean space of arbitrary finite dimension. A challenging task is to consider the geodesic flow on the cotangent bundle of a Riemannian manifold with scaled drift and scaled Lévy noise. Geometrically the Ornstein-Uhlenbeck process is defined on the tangent bundle of the real line and the driving Lévy noise is defined on the cotangent space.

Keywords: Ornstein-Uhlenbeck position process, α-stable Lévy noise, scaling limits, time change, stochastic Newton equations

Sammanfattning

Brownsk rörelse har fått allt större intresse i matematik, fysik och särskilt i ekonomi sedan den introducerades i början av nittonhundratalet. Stokastiska processer som generaliserar Brownsk rörelse har påverkat många forskningsområden teoretiskt och praktiskt. Dessutom konstruerades och studerades mer stokastiska processer i samband med mer raffinerande metoder i måtteori och funktionalanalys. Lévy processer, med Brownsk rörelse som ett specialfall, har fått ett stort intresse under de senaste decennierna. Dessutom omfattar Lévy processer en rad andra viktiga processer som särskilda fall som Poisson processer och subordinatorer. De är också relaterade till stabila processer.

I denna avhandling generaliserar vi ett resultat av S. Chandrasekhar [2] och av Edward Nelson som gav ett detaljerat bevis av detta resultat i sin bok från 1967 [12]. I Nelsons första resultat studeras standard Ornstein-Uhlenbeck. Fysikalisk beskriver detta fria partiklar som utför en slumpmässig och en oregelbunden rörelse i vattnet som orsakas av kollisioner med vattenmolekylerna. I ett ytterligare steg introducerar han en olinjär drift av positionsvariabeln, dvs han studerar i fysiskaliska termer fallet när partiklarna utsätts för ett yttre kraftfält .

Vi kommer i denna rapport att generalisera resultatet av Edward Nelson till fallet med α-stabila Lévy processer. Med andra ord ersätter vi det drivande bruset för en standard Ornstein-Uhlenbeck process med ett α-stabilt Lévy brus och inför en skalningsparameter β likformigt framför alla vektorfält i cotangensrummet och framför bruset. Detta motsvarar att tiden går mot oändlighet. Med Chandrasekhars och Nelsons val av diffusionskonstanten har det stationära tillståndet av hastighetsprocessen (som fås då tiden går mot oändligheten) en Boltzmann fördelning av statistisk mekanik. Det skalningsgränsvärde vi uppnår i närvaro och frånvaro av en olinjär drift genom att använda skalningsegenskaper av karakteristiska funktioner och tidsförändring kan utvidgas till andra typer av processer snarare än α-stabila Lévy processer.

I framtiden tänker vi generalisera detta endimensionella resultat till euklidiska rummet för en godtycklig ändlig dimension. En utmanande uppgift är att betrakta det geodetiska flödet på cotangensknippet av en Riemann mångfald med skalad drift och skalad Lévy brus. Geometrisk definieras Ornstein-Uhlenbeck processen på tangentknippet av den reella linjen och det drivande Lévy bruset definieras på cotangensrummet.

Nyckelord: Ornstein-Uhlenbeck processer, α-stabila Lévy brus, skalning gränsvärde, tidsförändring, stokastisk Newton ekvationer

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List of papers

I H. Al-Talibi, A. Hilbert, and V. Kolokoltsov, Nelson-type Limit for a Particular Class of Lévy Processes, AIP Conference Proceedings, 1232(2009), pp. 189-193

II H. Al-Talibi, A Scaling Limit for Stochastic Newton Equations with α-Stable Lévy Noise, submitted to the journal Stochastics.

Contents

Chapter 1

Introduction

Brownian motion has been the most intensively studied Lévy process in both theory and applications. In fact, the studies of this process was initiated by a kinematic physical problem. In the nineteenth century biologists and physicists worked with phenomenas which finally lead to the Brownian motion we know today. The most well known scientist amongst them is the Scottish botanist Robert Brown who discovered it in 1827. In the beginning of the twentieth century Einstein and Smoluckowski introduced it as a model for the physical phenomenon of Brownian motion and Bachelier described with it the evolution of stock prices. The latter was the first one to give a mathematical theory of Brownian motion in 1900 in his PhD thesis ”The theory of speculation”.

In 1905 Einstein published his first paper on Brownian motion which became the keystone of a fully probabilistic formulation of statistical mechanics and an important subject in physics. Moreover Einstein's first paper contained the cornerstone for the modern theory of stochastic processes, see [5]. In his model a microscopic particle experiences a random number of collisions.

Later on, in 1906 Smoluchowski presented a similar equation to the one of Einstein. He worked on the molecular kinetic approach to Brownian motion independently of Einstein. This equation became of high importance in the theory of stochastic processes. This theory was placed a rigorous mathematical basis by Wiener in 1920.

Three years after Einstein i.e 1908, the French physicist Paul Langevin initiated a different but likewise successful description of Brownian motion. He showed that the time evolution of the position of the Brownian particle itself can be described approximately by an equation which involves taking into account a random force field rather than Einstein's prediction of the motion where the change in position is directly given by white noise. Both descriptions have since then been generalized into mathematically distinct but physically equivalent tools for studying an important class of continuous random processes, see [9].

In 1930 L. S. Ornstein and G. E. Uhlenbeck studied a free particle in Brownian motion moving in a gas and affected by a friction force proportional to the pressure [13].

Much more careful experiments supporting the kinetic theory were made by Gouy, see [12] and by S. Chandrasekhar [2].

The main objective of the present thesis is to generalize the result given in [12] which is based on Langevin equation and Ornstein-Uhlenbeck theory [2]. We would like to mention that there exist other works in this direction see e.g. the references given in [12]. The generalization we want to present is based on a wider class than Brownian motion, namely Lévy processes.

In general, stochastic processes are mathematical models of random phenomena evolving in time. Lévy processes are stochastic processes with independent increments where the increments are stationary in time. Their trajectories admit, however, jumps even though they are continuous in probability.

The thesis is organized as follows. In chapter 1 we give a few basic ideas about Lévy processes and stochastic integrals. While in chapter 2 we present our two papers where the first one is the generalization of the limit given in [12] without drift term and the second paper contains an additional nonlinear drift.

1.1 Infinite divisibility

Let us start to give some concepts which have a connection to Lévy processes The characteristic function, or inverse Fourier transform, is the basic tool in the analysis of the distributions of Lévy processes. Let X be a random variable, taking values in $\mathbb{R}^d$, defined on the probability space $(\Omega, \mathcal{F}, P)$ with probability law p_X. Then we define the characteristic function $\Phi_X : \mathbb{R}^d \to \mathbb{C}$ as

$$\Phi_X(u) = E\left(e^{i(u,X)}\right) = \int_{\mathbb{R}^d} e^{i(u,y)} p_X(dy),$$

where $u \in \mathbb{R}^d$.

Definition 1.1.1. *If X is a random variable in $\mathbb{R}^d$ then we say that X is infinitely divisible if there exist independent identically distributed random variables $Y_1, \ldots, Y_n$ such that*

$$X \stackrel{d}{=} Y_1 + \cdots + Y_n,$$

for all $n \in \mathbb{N}$.

Example 1.1.1 (Gaussian random variables). *Let $X = (X_1, \ldots, X_d)$ be a random vector. We say that the random vector is Gaussian if it has a probability density function (pdf) of the form*

$$f(x) = \frac{1}{(2\pi)^{n/2}\sqrt{\det(A)}} e^{-\frac{1}{2}(x-m, A^{-1}(x-m))},$$

for all $x \in I\!R^d$*, where* $m \in I\!R^d$ *is a vector and* A *is* $d \times d$ *matrix. For this we write that* X *has a Gaussian (normal) distribution with mean* m *and covariance matrix* A*, i.e.* $X \sim N(m, A)$*.*
Moreover, the characteristic function is given by

$$\Phi_X(u) = e^{i(m,u) - \frac{1}{2}(u,Au)},$$

respectively

$$[\Phi_X(u)]^{1/n} = e^{i(\frac{m}{n},u) - \frac{1}{2}(u,\frac{A}{n}u)}.$$

Thus, X *is infinitely divisible with* $Y_j \sim N(m/n, A/n)$ *for all* $1 \le j \le n$*, see e.g. [1, 19].*

1.2 Lévy-Khintchine formula

This formula was established by Paul Lévy and A. Ya. Khintchine in 1930. It was actually developed by de Finetti and Kolmogorov on $I\!R$ in some special cases, see [19]. This formula gives a representation of the characteristic functions of all infinitely divisible random variables. Before we present the Lévy-Khintchin theorem we need some preliminaries.
Let ν be a Borel measure defined on $I\!R^d/\{0\}$, we say that ν is a Lévy measure if

$$\int_{I\!R^d/\{0\}} (|y^2| \wedge 1)\nu(dy) < \infty,$$

where the symbol $\wedge$ stands for the minimum. There are other alternatives to characterize the Lévy measure, one of them is given by

$$\int_{I\!R^d/\{0\}} \frac{|y|^2}{1+|y|^2}\nu(dy) < \infty.$$

Of course one can define the Lévy measure on the whole $I\!R^d$ by letting $\nu(\{0\}) = 0$ as it is in [19]. It is worth to mention here that the Lévy measure we are dealing with later is of the form

$$\nu(dx) = \begin{cases} \frac{c_1}{x^{1+\alpha}} & \text{on } (0,\infty) \\ \frac{c_2}{|x|^{1+\alpha}} & \text{on } (-\infty,0) \end{cases}$$

where $0 < \alpha < 2$, $c_1 \ge 0$, $c_2 \ge 0$, and $c_1 + c_2 \ge 0$.

Theorem 1.2.1. *Let* $\mu \in \mathfrak{B}$*, where* $\mathfrak{B}$ *is a Borel set. If* μ *is infinitely divisible then for all* $u \in I\!R^d$

$$\begin{aligned} \Phi_\mu(u) &= \exp\Big[i(b,u) - \frac{1}{2}(u,Au) + \\ &\quad + \int_{I\!R^d/\{0\}} \left(e^{i(u,y)} - 1 - i(u,y)\mathbf{1}_D(y)\right)\nu(dy)\Big], \end{aligned} \tag{1.1}$$

where $b \in I\!\!R^d$ is a vector, A is a $d \times d$ matrix, ν is Lévy measure on $I\!\!R^d/\{0\}$, D is the closed unit ball and $\mathbf{1}_D$ is the indicator function of D.
The converse is also true i.e. every mapping of the form (1.1) is the characteristic function of an infinitely divisible probability measure on $I\!\!R^d$.

1.3 Lévy processes

Let us give a formal definition of Lévy processes. We mention here that our notation coincides with the one given in [1].

Definition 1.3.1. *A stochastic process $X = (X(t), t \geq 0)$ on a probability space $(\Omega, \mathcal{F}, P)$ is a Lévy process if the following conditions are satisfied*

1. *$X_0 = 0$ almost surely.*

2. *For any $n \in \mathbb{N}$ and $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables X_{t_0}, $X_{t_1} - X_{t_0}$, $X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.*

3. *X has stationary increments, i.e. $X_{s+t} - X_s \stackrel{d}{=} X_t$.*

4. *X is stochastically continuous, i.e. for every $s \geq 0$ and $a > 0$*

$$\lim_{t \to s} P\left(|X_t - X_s| > a\right) = 0.$$

5. *The sample path are right-continuous with left limits almost surely (càdlàg).*

Lemma 1.3.1. *If $X = (X(t), t \geq 0)$ is stochastically continuous, then the map $t \to \Phi_{X(t)}(u)$ is continuous for each $u \in I\!\!R^d$.*

Proof. Let $s, t \geq 0$ with $t \neq s$ and write $X(s,t) = X(t) - X(s)$. Fix $u \in I\!\!R^d$. Given any $\epsilon > 0$ we can find $\delta_1 > 0$ such that

$$\sup_{0 \leq |y| < \delta_1} \left| e^{i(u,y)} - 1 \right| < \frac{\epsilon}{2}, \tag{1.2}$$

where the map $y \to e^{i(u,y)}$ is continuous at the origin. And by stochastic continuity we can find $\delta_2 > 0$ such that whenever $0 < |t - s| < \delta_2$, we have

$P\left(|X(s,t)| > \delta_1\right) < \frac{\epsilon}{4}$. Thus for all $0 < |t-s| < \delta_2$ we have

$$\begin{aligned}
\left|\Phi_{X(t)}(u) - \Phi_{X(s)}(u)\right| &= \left|\int_\Omega e^{i(u,X(s)(\omega))}\left[e^{i(u,X(s,t)(\omega))} - 1\right] P(d\omega)\right| \\
&\leq \int_{\mathbb{R}^d} \left|e^{i(u,y)} - 1\right| p_{X(s,t)}(dy) \\
&= \int_{B_{\delta_1}(0)} \left|e^{i(u,y)} - 1\right| p_{X(s,t)}(dy) + \int_{B_{\delta_1}(0)^c} \left|e^{i(u,y)} - 1\right| p_{X(s,t)}(dy) \\
&\leq \sup_{0\leq|y|<\delta_1} |e^{i(u,y)} - 1| + 2P(|X(s,t)| > \delta_1) \\
&\leq \frac{\epsilon}{2} + 2\frac{\epsilon}{4} < \epsilon
\end{aligned}$$

where we used (1.2) and $P\left(|X(s,t)| > \delta_1\right) < \frac{\epsilon}{4}$ in the last step. Thus the result follows. □

Let us discuss the relationship between processes with stationary independent increments, which hold for Lévy process, and infinitely divisible distributions.

Lemma 1.3.2. *The characteristic function of a Lévy process X is given by*

$$\Phi_{X_t}(u) = e^{t\eta(u)},$$

where $u \in \mathbb{R}^d$, $t \geq 0$, and η is the Lévy symbol of $X(1)$.

Proof. Since by assumption X_t is a Lévy process which has stationary, independent increments we can write

$$\begin{aligned}
\Phi_{X(t+s)}(u) &= E\left(e^{i(u,X(t+s))}\right) = E\left(e^{i(u,X(t+s)-X(t))}e^{i(u,X(t))}\right) \\
&= E\left(e^{i(u,X(t+s)-X(t))}\right) E\left(e^{i(u,X(t))}\right) = E\left(e^{i(u,X(s))}\right) E\left(e^{i(u,X(t))}\right) \\
&= \Phi_{X(s)}(u)\Phi_{X(t)}(u). \qquad (1.3)
\end{aligned}$$

Because of the continuity in probability, Lemma 1.3.1, we conclude that $\Phi_{X(t)}(u)$ is continuous with respect to t. However, the unique solution of (1.3) and $\Phi_{X(0)}(u) = 1$ is $\Phi_{X(t)}(u) = e^{t\eta(u)}$, for some function $\eta : \mathbb{R}^d \to \mathbb{C}$. Furthermore $\Phi_{X(1)}(u) = e^{\eta(u)}$ which implies that $\Phi_{X(t)}(u) = (\Phi_{X(1)}(u))^t$. In addition we have that the Lévy-Khinchine formula for a Lévy process $X = (X(t), t \geq 0)$ is

$$\begin{aligned}
\Phi_\mu(u) \quad = \quad \exp\Bigg[t\Bigg(& i(b,u) - \frac{1}{2}(u,Au) + \\
&+ \int_{\mathbb{R}^d/\{0\}} \left(e^{i(u,y)} - 1 - i(u,y)\mathbf{1}_D(y)\right)\nu(dy)\Bigg)\Bigg],
\end{aligned}$$

for each $t \geq 0$, $u \in \mathbb{R}^d$, where (b, A, ν) are the characteristics of $X(1)$. $\square$

Theorem 1.3.3. *If $X = (X(t), t \geq 0)$ is a stochastic process and there exists a sequence of Lévy processes $(X_n, n \in \mathbb{N})$ such that each $X_n = (X_n(t), t \geq 0)$ converges in probability to $X(t)$ for each $t \geq 0$ and*

$$\lim_{n\to\infty} \limsup_{t\to 0} P(|X_n(t) - X(t)| > a) = 0,$$

for all $a > 0$, then X is a Lévy process [1].

Proof. We see that the first condition of the definition of Lévy processes is satisfied from the fact that $(X_n(0), n \in \mathbb{N})$ has a subsequence converging to 0 almost surely. For the third condition we obtain stationary increments by observing that for each $u \in \mathbb{R}^d$, $0 \leq s < t < \infty$,

$$\begin{aligned} E\left(e^{i(u,X(t)-X(s))}\right) &= \lim_{n\to\infty} E\left(e^{i(u,X_n(t)-X_n(s))}\right) \\ &= \lim_{n\to\infty} E\left(e^{i(u,X_n(t-s))}\right) \\ &= E\left(e^{i(u,X(t-s))}\right), \end{aligned}$$

where the convergence of the characteristic function follows by the argument used in Lemma 1.3.1 and the dominated convergence theorem is used in the last equality. The independence of the increments is proved similarly.
Since the limit process X was shown to be stationary it suffices to show continuity at $t = 0$. We have for each $a > 0$, $t \geq 0$, $n \in \mathbb{N}$ due to monotonicity of probability measures that

$$\begin{aligned} P\left(|X(t)| > a\right) &\leq P\left(|X(t) - X_n(t)| + |X_n(t)| > a\right) \\ &\leq P\left(|X(t) - X_n(t)| > \frac{a}{2}\right) + P\left(|X_n(t)| > \frac{a}{2}\right) \end{aligned}$$

and

$$\begin{aligned} &\limsup_{t\to 0} P\left(|X(t)| > a\right) \\ &\leq \limsup_{t\to 0} P\left(|X(t) - X_n(t)| > \frac{a}{2}\right) + \limsup_{t\to 0} P\left(|X_n(t)| > \frac{a}{2}\right). \end{aligned} \tag{1.4}$$

As each X_n is a Lévy process we find

$$\limsup_{t\to 0} P\left(|X_n(t)| > \frac{a}{2}\right) = \lim_{t\to 0} P\left(|X_n(t)| > \frac{a}{2}\right) = 0,$$

hence the result follows by taking $\lim_{n\to\infty}$ in (1.4). $\square$

1.3.1 Examples of Lévy processes

In the sequel we introduce the most prominent and frequently used examples of Lévy processes. For more details and examples see i.e. [1; 8]

1.3.1.1 Brownian motion From the definition of a Lévy process we see that Brownian motion in $\mathbb{R}^d$ is a Lévy process which possess even continuous sample paths almost surely, see [6; 7; 14; 15; 16]. The well known Gaussian distribution with mean 0 and variance t has the probability density function

$$f(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}.$$

Then the characteristic function of the standard Brownian motion $B = (B(t), t \geq 0)$ is given by

$$\Phi_{B(t)}(u) = e^{-\frac{1}{2}t|u|^2} = \left[e^{-\frac{1}{2}t\left|\frac{u}{\sqrt{n}}\right|^2} \right]^n,$$

which shows that it is an infinitely divisible distribution. Moreover $\eta = -\frac{|u|^2}{2}$ is called the characteristic exponent or Lévy symbol. For more details and deeper studies of Brownian motion we refer to i.e. Sato [19], Revuz and Yor [16], and Karatzas and Shreve [7].

1.3.1.2 Poisson processes For $\lambda > 0$ we consider the probability distribution of a Possion process with parameter λ:

$$P(n) = \frac{(\lambda)^n}{n!} e^{-\lambda}.$$

The characteristic function is obtained by calculating

$$\sum_{n \geq 0} e^{i\theta n} P(n) = e^{-\lambda(1-e^{i\theta})} = \left[e^{-\frac{\lambda}{n}(1-e^{i\theta})} \right]^n. \tag{1.5}$$

Thus, the characteristic function is in fact the sum of n independent Poisson processes with parameter λ/n as given by the right hand side of (1.5). Moreover, for the Poisson processes with parameter λt the characteristic function is given by

$$E(e^{i\theta N_t}) = e^{-\lambda t(1-e^{i\theta})},$$

and the characteristic exponent is $\eta = \lambda(1 - e^{i\theta})$ for any $\theta \in \mathbb{R}$. Poisson processes are called jump processes because they jump up to a higher state each time an event occurs. The applications of Poisson processes can be frequently seen in insurance mathematics.

1.3.1.3 Compound Poisson processes The compound Poisson processes is defined as

$$Y(t) = Z(1) + \cdots + Z(N(t)),$$

where $Z(n), n \in \mathbb{N}$, is a sequence of independent identically distributed random variables taking values in $\mathbb{R}^d$ with common law μ_Z and N is a Poisson process with parameter $\lambda > 0$. One can verify the properties specified in Definition 1.3.1 in the case of Compound Poisson processes i.e. $Y(0) = 0$ almost surely and $Y(t)$ has stationary independent increments. The continuity in probability can be achieved by considering

$$P(|Y(t)| > a) = \sum_{n=0}^{\infty} P\left[|Z(1) + \cdots + Z(n)| > a\right] P(N(t) = n),$$

where by the dominated convergence theorem we obtain the required result. The characteristic function of a Compound Poisson process can be determined as follows

$$\begin{aligned}
\Phi_X(u) &= \sum_0^{\infty} E\left(\exp\left[i(u, Z(1) + \cdots + Z(N))\right] | N = n\right) P(N = n) \\
&= \sum_0^{\infty} E\left(\exp\left[i(u, Z(1) + \cdots + Z(N))\right]\right) e^{-\lambda} \frac{\lambda^n}{n!} \\
&= e^{-\lambda} \sum_0^{\infty} \frac{\left(\lambda \Phi_Z(u)\right)^n}{n!} \\
&= \exp\left(\lambda\left(\Phi_Z(u) - 1\right)\right),
\end{aligned}$$

where we used independence and Taylor expansion. If we insert $\Phi_Z(u) = \int_{\mathbb{R}^d} e^{i(u,y)} \mu_Z(dy)$ we obtain the characteristic function of the Compound Poisson process, i.e. $\Phi_X(u) = \exp\left[\int_{\mathbb{R}^d}\left(e^{i(u,y)} - 1\right) \lambda \mu_Z(dy)\right]$.

1.3.2 Stable Lévy processes

A stable Lévy process X is a Lévy process where each $X(t)$ is a stable random variable. And a random variable $X(t)$ is said to have stable distribution if for all $n \geq 1$ the following equality holds in distribution

$$X_1 + \cdots + X_n \stackrel{d}{=} a_n X + b_n,$$

where $X_1, \ldots, X_n$ are independent copies of X, $a_n > 0$ and $b_n \in \mathbb{R}$. If $a_n = n^{1/\alpha}$ for $0 < \alpha \leq 2$ and $b_n = 0$ we obtain

$$X_1 + \cdots + X_n \stackrel{d}{=} n^{1/\alpha} X,$$

which classifies strictly stable distributions, see [18]. One may see that for the case $\alpha = 2$ we retrieve the case of Gaussian random variables with characteristic exponent of the form

$$\eta(u) = i\mu u - \frac{1}{2}\sigma^2 u^2.$$

On the other hand the characteristic exponents of stable Lévy process when $\alpha \in (0,1) \cup (1,2)$ is given by

$$\eta(u) = i\mu u - \sigma^\alpha |u|^\alpha \left[1 - i\beta \operatorname{sgn}(u) \tan\left(\frac{\pi\alpha}{2}\right)\right], \tag{1.6a}$$

and the characteristic exponent when $\alpha = 1$ is given by

$$\eta_1(u) = i\mu u - \sigma|u| \left[1 + i\beta\frac{2}{\pi} \operatorname{sgn}(u) \log(|u|)\right], \tag{1.6b}$$

where $\beta \in [-1,1]$, $\sigma > 0$ and $\mu \in \mathbb{R}$. In terms of Lévy measure the representation is given by

$$i(b,u) - \frac{1}{2}(u, Au) + \int_{\mathbb{R}^d/\{0\}} \left(e^{i(u,y)} - 1 - i(u,y)\mathbf{1}_D(y)\right) \nu(dy),$$

where $b \in \mathbb{R}^d$ is a vector, A is a $d \times d$ matrix, ν is Lévy measure on $\mathbb{R}^d/\{0\}$, D is the closed unit ball and $\mathbf{1}_D$ is the indicator function of D, see [1; 19]. It is worth mentioning that stable Lévy processes have many important applications because they exhibit self-similarity property, see [3].

1.4 Stochastic integrals

In calculus, the Riemann-Steljes integral is defined by a limiting procedure arising from partitions getting finer. One defines the integral of a function in such a way that the integral represents the area under the graph. The next step is to extend the notion to a larger class of functions by approximation i.e. the integral of a function is defined as the limit of the sum of the function in subintervals in some sense.

One proceeds in the same way when defining the Itô integral i.e. by an approximation procedure. But here the step function is replaced by a process which is actually a random step function. The integral is then in several steps extended to larger classes of processes by taking the limit of the sum. By construction the integrator is not more deterministic in contrast to Riemann-Steljes integral rather than stochastic with respect to some process.

The most famous one is the one with respect to Brownian motion. One call this type Itô integrals after the discoverer Kiyoshi Itô. This kind of integral has been used widely in different field of mathematics and its applications [6; 7; 14; 15; 16]. The corresponding differential calculus, the Itô calculus extends the calculus of differential equations to one having stochastic processes as its driving process. Let us give a formal definition of the stochastic integrals

Definition 1.4.1. *Let B_t be a Brownian motion of dimension* 1 *on a probability space $(\Omega, \mathcal{F}, P)$. Then a stochastic integral is a stochastic process Y_t on $(\Omega, \mathcal{F}, P)$ of the form*

$$Y_t = Y_0 + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dB_s,$$

where u and v are functions in $I\!R$. One may sometimes write this integral equation in a shorter differential form

$$dY_t = udt + vdB_t.$$

For more details about stochastic integrals with respect to Brownian motion we refer to i.e. [6; 7; 14; 15; 16].
Let us now present the Itô formula where the same references as above are applied. For a generalization of this result see [4; 17].
Let x_t be an Itô process, i.e. a stochastic process such that

$$dx_t = udt + vdB_t.$$

Let $g(t,x) \in \mathbb{C}^2([0,\infty] \times I\!R)$ be twice continuously differentiable function then $Y_t = g(t, X_t)$ is also an Itô process and the Itô formula reads

$$\begin{aligned} dY_t &= \frac{\partial g}{\partial t}(t,x_t)dt + \frac{\partial g}{\partial x}(t,x_t)dx_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,x_t)\cdot(dx_t)^2, \\ &= \left(\frac{\partial g}{\partial t} + u\frac{\partial g}{\partial x} + \frac{v^2}{2}\frac{\partial^2 g}{\partial x^2}\right)dt + v\frac{\partial g}{\partial x}dB_t \end{aligned}$$

where one use the rules $dt\cdot dt = dt\cdot dB_t = dB_t\cdot dt = 0$ and $dB_t\cdot dB_t = dt$. Y_t is again an Itô process.

Let us take an example which is suitable for our calculations later on. Consider $g(t,B_t) = e^{tB_t}$. Here $g(t,x) = e^{tx}$ is twice continuous differentiable function. We have $\frac{\partial}{\partial t}g = xe^{tx}$, $\frac{\partial}{\partial x}g = te^{tx}$ and $\frac{\partial^2}{\partial x^2}g = t^2e^{tx}$. Thus we use Itô formula to obtain

$$\begin{aligned} d(e^{tB_t}) &= e^{tB_t}B_t dt + te^{tB_t}dB_t + \frac{1}{2}t^2e^{tx}dt \\ &= e^{tB_t}\left(B_t + \frac{1}{2}t^2e^{tx}\right)dt + te^{tB_t}dB_t. \end{aligned}$$

From the Itô formula we can derive the integration by parts formula i.e. suppose the function $f(s)$ is continuous and of bounded variation with respect to $s \in [0,t]$, then

$$\int_0^t f(s)dB_s = f(t)B_t - \int_0^t B_s df(s),$$

where the second integral is a Stieltjes integral i.e. an appropriate limit of the sum $\sum_j B(t_j)\left(f(t_{j+1}) - f(t_j)\right)$.
Of course the driving process for the stochastic integral above need not be Brownian motion. Recently Lévy processes have been of big interest in stochastic analysis and its applications. There are a lot of publications using the stochastic integral with respect to a Lévy process. Keeping in mind that a Lévy process has a Poisson and a Brownian part we say that the stochastic process $Y = (Y(t), t \geq 0)$ in $I\!R^d$ is a Lévy-type stochastic integral if it can be written in the following form

$$\begin{aligned} Y(t) = Y(0) &+ \int_0^t G(s)ds + \int_0^t F(s)dB(s) \\ &+ \int_0^t \int_{|x|<1} H(s,x)\tilde{N}(ds,dx) + \int_0^t \int_{|x|\geq 1} K(s,x)\tilde{N}(ds,dx), \end{aligned} \tag{1.7}$$

or it can be written as

$$\begin{aligned} dY(t) = G(t)dt + F(t)dB(t) &+ \int_{|x|<1} H(t,x)\tilde{N}(dt,dx) \\ &+ \int_{|x|\geq 1} K(t,x)N(dt,dx), \end{aligned} \tag{1.8}$$

where G, F, H are predictable mappings $F : [0,T] \times E \times \Omega \to I\!R$ for which $P\left(\int_0^T \int_E |F(t,x)|^2 \nu(dx)dt < \infty\right) = 1$, with ν as a Lévy measure and K is predictable. Moreover, B is a standard Brownian motion and N is an independent Possion process on $I\!R^+ \times I\!R^d/\{0\}$ with compensator $\tilde{N} = N(ds,dx) - ds\nu(dx)$, where ν is the intensity measure which is assumed to be a Lévy measure.
If Y is a Lévy-type stochastic integral of the form (1.8) then for each $f \in$

$\mathbb{C}^2(I\!R^d)$, $t \geq 0$ with probability 1 the Itô formula is given by

$$\begin{aligned}
df(Y(t)) &= f'(Y(t))G(t)dt + f'(Y(t))F(t)dB_t + \frac{1}{2}f''(Y(t))F(t)^2 dt \\
&+ \int_{|x|\geq 1} \left[f(Y(t-) + K(t,x)) - f(Y(t-))\right] N(dt,dx) \\
&+ \int_{0<|x|<1} \left[f(Y(t-) + H(t,x)) - f(Y(t-))\right] \tilde{N}(dt,dx) \\
&+ \int_{0<|x|<1} [f(Y(t-) + H(t,x)) - f(Y(t-)) \\
&-H(t,x)f'(Y(t-))]\, \nu(dx)dt.
\end{aligned}$$

For more details about Lévy-type stochastic integral, Itô formula and integration by parts we refer to the book by Applebaum [1] and references therein.

1.5 Ornstein-Uhlenbeck processes

Let us take a physical point of view, i.e. let us assume that $x(t)$ is the position of a Brownian particle at time t which exhibits a velocity $v(t) = \frac{d}{dt}x(t)$, $t \geq 0$, in distributional sense. Ornstein and Uhlenbeck studied this type of motion and argued that the total force on the particle is a sum of random bombardments between the particles in the fluid and a frictional force which damps the motion. Using Newton's law one can write

$$m\frac{dv}{dt} = -\beta m v + m\frac{dB}{dt},$$

where $\beta > 0$, m is the mass of the particle. In the form of a stochastic differential equation we write the latter equation as

$$dv(t) = -\beta v(t)dt + dB(t).$$

In order to find the solution of this stochastic differential equation one uses Itô formula to obtain

$$v(t) = e^{-\beta t}v_0 + \int_0^t e^{-\beta(t-s)}dB_s,$$

which we call in our work Ornstein-Uhlenbeck velocity process. For a deeper insight in the Ornstein-Uhlenbeck theory we refer to [2; 12] and for the existence and uniqueness we refer to e.g. [6; 15].

1.6 Time change

One can transform one stochastic process into another one by extending or shrinking the time scale. One possibility is to use the random time change , a pathwise change of time scale. Here we give a short introduction to random time change in the case of Brownian motion.

Theorem 1.6.1. *Let* $dY_t = \sum_{i=1}^{n} v_i(t,\omega)dB_i(t,\omega)$, $Y_0 = 0$, *where* $B = (B_1,\ldots,B_n)$ *is a Brownian motion in* $\mathbb{R}^d$. *Then*

$$\widehat{B}_t = Y_{a_t}, \qquad \textit{is a } 1\textit{-dimensional Brownian motion}$$

where $a_t = \inf\{s; b_s > t\}$ *is the right inverse of*

$$b_s = \int_0^s \left\{ \sum_{i=1}^{n} v_i^2(r,\omega) \right\} dr.$$

This means that a_t is a random time change as defined in [14]. For more details about random time change we refer to [14; 15]. And for time change with respect to càdlàg processes and Lévy processes we refer to [10; 11; 19]

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Chapter 2

Papers

I Nelson-type Limit for a Particular Class of Lévy Processes.

II A Scaling Limit for Stochastic Newton Equations with α-Stable Lévy Noise.

Paper I

2.1 Nelson-type Limit for a Particular Class of Lévy Processes

Haidar Al-Talibi, Astrid Hilbert and Vassili Kolokoltsov

Nelson-type Limit for a Particular Class of Lévy Processes

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Abstract. Brownian motion has been constructed in different ways. Einstein was the most outstanding physicists involved in its construction. From a physical point of view a dynamical theory of Brownian motion was favorable. The Ornstein-Uhlenbeck process models such a dynamical theory and E. Nelson amongst others derived Brownian motion from Ornstein-Uhlenbeck theory via a scaling limit. In this paper we extend the scaling result to α-stable Lévy processes.

Keywords: Ornstein-Uhlenbeck process, α-stable Lévy noise, scaling limits
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1. INTRODUCTION

Anyone looking at water through a microscope is apt to see little things moving around. Robert Brown conducted a systematic investigation of this motion showing in particular that it was not vital in origin but the chaotic perpetual motion of small particles which is the result of collisions with the molecules of the surrounding fluid. The molecular collisions with the particle occur in very rapid succession. Hence the mean free path of the molecules is small compared with the particle's size respectively the relaxation time β^{-1} between two successive collisions is small.
The Einstein-Smoluchowski theory is different from Newtonian mechanics of particles although numerically, i.e. experimentally, indistinguishable from the Ornstein-Uhlenbeck theory which describes a dynamical model. Examples where the Einstein-Smoluchswski theory breaks down but the Ornstein-Uhlenbeck theory is successful may be found in the book by E. Nelson [3].
In the physical model $x(t)$ describes the position of the Browninan particle at time $t > 0$. It is assumed that the velocity $\frac{dx}{dt} = v$ exists and satisfies the Langevin equation. Mathematically the two ordinary differential equations combine to the initial value problem:

$$\begin{aligned} dx_t &= v_t\,dt \\ dv_t &= -\beta v_t dt + dB_t, \end{aligned} \tag{1}$$

with initial value $(x_0, v_0) = (x(0), v(0))$, where $B_t,\ t \geq 0$, is mathematical Brownian motion on the real line and $\beta > 0$ is a constant which physically represents the inverse relaxation time between two successive collisions.

The solution of system (1) is

$$v_t = e^{-\beta t} v_0 + \int_0^t e^{-\beta(t-u)} dB_u,$$

which is called Ornstein-Uhlenbeck velocity process, and

$$x_t = x_0 + \int_0^t e^{-\beta s} v_0 ds + \int_0^t \int_0^s e^{-\beta s} e^{\beta u} dB_u ds, \tag{2}$$

which is called Ornstein-Uhlenbeck position process.
For β tending to infinity the Ornstein-Uhlenbeck position process converges to Browninan motion. A mathematically rigorous exposition of the limiting procedure is given in [3, chap. 9] as well as further references. We stress that Nelson is not using standard Brownian motion but introduces the diffusion constant $\sqrt{2\frac{\beta kT}{m}}$ where k, m, T are physical constants.

2. α-STABLE LÉVY NOISE CASE

In this paper we introduce a modified Ornstein-Uhlenbeck position process driven by βX_t, where $\{X_t\}_{t\geq 0}$ is an α-stable Lévy process, $0 < \alpha < 2$ and $\beta > 0$ is a scaling parameter as above

$$x_t = x_0 + \int_0^t e^{-\beta s} v_0 ds + \int_0^t \int_0^s e^{-\beta s} e^{\beta u} \beta dX_u ds. \tag{3}$$

The second term of (3), a double integral, includes a stochastic integral with respect to a Lévy process the existence of which is guaranteed e.g. by the results in [1, section 4.2].

Our notation coincides with the one in [1] from where we also recall that for arbitrary Lévy processes Y the characteristic function is of the form $\phi_{Y_t}(u) = e^{t\eta(u)}$ for each $u \in I\!R$, $t \geq 0$, where η is called the Lévy-symbol of $Y(1)$. For a centered α-stable Lévy processes the Lévy-symbol for $\alpha \neq 1$ is given by:

$$\eta(u) = -\sigma^\alpha |u|^\alpha \left[1 - i\beta \operatorname{sgn}(u) \tan\left(\frac{\pi\alpha}{2}\right)\right] \tag{4a}$$

and for $\alpha = 1$ is given by:

$$\eta_1(u) = -\sigma|u| \left[1 + i\beta \frac{2}{\pi} \operatorname{sgn}(u) \log(|u|)\right]. \tag{4b}$$

Proposition 2.1. *Assume that Y is an α-stable Lévy process, $0 < \alpha < 2$, and g is a continuous function on the interval $[s,t] \subset T \subsetneq I\!R$. Let η be the Lévy symbol of Y_1 and ξ be the Lévy symbol of $\psi(t) = \int_s^t g(r)\, dY_r$. Then we have*

$$\xi(u) = \int_s^t \eta(ug(r))\, dr\,.$$

The proof is a direct consequence of Theorem 1 in [2].

For $g(\ell) = e^{\beta(\ell-t)}, \ell \geq 0$ and the α-stable process X in (3) the symbol of $Z_t = \int_s^t e^{\beta(r-t)}\,dX_r$ is:

$$\xi(u) = \begin{cases} \int_s^t e^{\alpha\beta(r-t)}\,dr \cdot \eta(u) & \text{for } 0 < \alpha < 2, \alpha \neq 1 \\ \int_s^t e^{\alpha\beta(r-t)}\,dr \cdot \eta_1(u) & \text{for } \alpha = 1 \end{cases}$$

with η, η_1 as in (4a) and (4b), respectively, and $0 \leq s \leq t$. We are thus lead to introduce the random time change $\tau^{-1}(t)$ where

$$\tau(t) = \int_0^t e^{-\alpha\beta t} e^{\alpha\beta u} du = \frac{1}{\alpha\beta}\left(1 - e^{-\alpha\beta t}\right)$$

which is actually deterministic. This means that X and $Z_{\tau^{-1}(t)}$ have the same distribution. Let us now formulate the main result of this paper.

Theorem 2.1. *Let $t_1 < t_2$, $t_1, t_2 \in T$ and T a compact subset of $[0,\infty)$. Then for every $\delta > 0$ there exists $\varepsilon > 0$ depending on N_1 and N_2 satisfying:*

$$(i)\ t_2 - t_1 \geq \frac{N_1}{\beta} \quad \text{and} \quad (ii)\ \beta^\alpha \geq N_2 v_0^\alpha, \tag{5}$$

with $0 < \alpha < 2$ such that

$$I\!P[|x_t - X_t| > \varepsilon] < \delta$$

for any $t_1 \leq t \leq t_2$ where $\{x_t\}_{t \geq 0}$ is the Ornstein-Uhlenbeck position process (3) and $\{X_t\}_{t \geq 0}$ is its driving α-stable Lévy Noise.

Proof. The statement of the theorem means that the Ornstein-Uhlenbeck-type position process x_t in (3) converges uniformly to X_t on any compact subset of the time axis $[0,\infty)$ almost surely as N_1 and N_2 tend to infinity. The increment of the Ornstein-Uhlenbeck process (3) is given by

$$\tilde{x}_t = \int_{t_1}^{t_2} e^{-\beta s} v_0 ds + \int_{t_1}^{t_2} \int_0^s e^{-\beta(s-u)} \beta dX_u ds, \tag{6}$$

where the first integral of (6) is $\int_{t_1}^{t_2} e^{-\beta s} v_0 ds = \frac{v_0}{\beta}\left(e^{-\beta t_1} - e^{-\beta t_2}\right)$. From now on let us denote $\Delta t = t_2 - t_1$.
Taking the latter expression to the power α, where $0 < \alpha < 2$, and taking into account that $e^{-\beta t_1} - e^{-\beta t_2} \leq 1$ we obtain that

$$\frac{v_0^\alpha}{\beta^\alpha}\left|e^{-\beta t_1} - e^{-\beta t_2}\right|^\alpha = \frac{v_0^\alpha}{\beta^\alpha} e^{-\alpha\beta t_1}\left|-(1 - e^{-\beta\Delta t})\right|^\alpha \leq \frac{1}{N_2} e^{-\alpha N_1}\left|-(1 - e^{-N_1})\right|^\alpha,$$

where we used $((5)(i),(ii))$ and the fact that $e^{-\alpha\beta t_1} \leq e^{-\alpha' N_1}$, where $\alpha' = \alpha \min\{s \in T\}$.

If we choose N_1 and N_2 large enough then $\frac{1}{N_2}e^{-\alpha N_1}\left|-(1-e^{-N_1})\right|^{\alpha}$ tends to zero as N_1, N_2 tend to infinity.

The second part of (6) is estimated by first splitting the double integral into two integrals. We have

$$\beta\left[\int_{t_1}^{t_2}\int_{t_1}^{s} e^{-\beta s}e^{\beta u}dX_u ds + \int_{t_1}^{t_2}\int_{0}^{t_1} e^{-\beta s}e^{\beta u}dX_u ds\right] \tag{7}$$

The double integral of the second part of (7) can be written as

$$\begin{aligned}\beta\int_{t_1}^{t_2}\int_{0}^{t_1} e^{-\beta s}e^{\beta u}\,dX_u ds &= \beta Z_{\tau(t_1)}\int_{t_1}^{t_2} e^{-\beta s}e^{\beta t_1}ds = -Z_{\tau(t_1)}\left(e^{-\beta t_2}-e^{-\beta t_1}\right)e^{\beta t_1}\\ &= \left(1-e^{-\beta\Delta t}\right)Z_{\frac{1}{\alpha\beta}\left(1-e^{-\alpha\beta t_1}\right)} = \frac{1}{\sqrt[\alpha]{\beta}}\left(1-e^{-\beta\Delta t}\right)Z_{\frac{1}{\alpha}\left(1-e^{-\alpha\beta t_1}\right)}\end{aligned}$$

where we used that Z is an α-stable Lévy process. Moreover, the scaling property of Lévy processes we used in the last step, i.e. $Z_{\gamma\tau} = \gamma^{\alpha}Z_{\tau}$, where $\gamma > 0$, is actually a special case of Proposition 2.1. Using the assumption $(5(i))$ we obtain

$$e^{-\beta\Delta t} \leq e^{-N_1}.$$

Thus, for N_1 and N_2 tending to infinity, the latter expression converges to zero and $Z_{\frac{1}{\alpha}\left(1-e^{-\alpha\beta t_1}\right)}$ converges to $Z_{\frac{1}{\alpha}}$ a.e. which is almost surely finite. Hence the product converges almost surely to zero.

Let us turn to the first part of (7), we use partial integration to have

$$\begin{aligned}\beta\int_{t_1}^{t_2}\int_{t_1}^{s} e^{-\beta s}e^{\beta u}dX_u ds &= -\left[e^{-\beta s}\int_{t_1}^{s} e^{\beta u}dX_u\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} e^{-\beta s}e^{\beta s}dX_s\\ &= -e^{-\beta t_2}\int_{t_1}^{t_2} e^{\beta u}dX_u + (X_{t_2}-X_{t_1})\end{aligned}$$

By introducing a random time change similar to the one before, for the first term on the right hand side of (8) we obtain

$$-e^{-\beta t_2}\int_{t_1}^{t_2} e^{\beta u}dX_u = Z_{\frac{1}{\alpha\beta}\left(1-e^{-\alpha\beta\Delta t}\right)} = \frac{1}{\sqrt[\alpha]{\beta}}Z_{\frac{1}{\alpha}\left(1-e^{-\alpha\beta\Delta t}\right)}$$

where we used again the scaling property of Lévy processes $Z_{\gamma\tau} = \gamma^{\alpha}Z_{\tau}$ with $\gamma > 0$. By assumption $(5(i))$ we see that $e^{-\alpha\beta\Delta t} \leq e^{-\alpha N_1}$ which tends to zero for large N_1 and $Z_{\frac{1}{\alpha}\left(1-e^{-\alpha\beta\Delta t}\right)}$ converges to $Z_{\frac{1}{\alpha}}$. In analogy to the argument above the product $\frac{1}{\sqrt[\alpha]{\beta}}Z_{\frac{1}{\alpha}\left(1-e^{-N_1}\right)}$ tends to zero almost surely for N_1 and N_2 tending to infinity.

This means that the increments of the Ornstein-Uhlenbeck position process are the sum of the increments of the originally driving α-stable Lévy process

$$X_{t_2} - X_{t_1},$$

and three terms which are uniformly bounded by e^{-N_1} and e^{-N_2} for all $t_1, t_2 \in T$, T a compact subset of $[0,\infty)$, and which converge to zero as N_1 and N_2 tend to infinity.
Since we have uniform convergence to zero we need not consider that we have been using versions of the error terms in the course of estimation.

$\square$

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Paper II

2.2 A Scaling Limit for Stochastic Newton Equations with α-Stable Lévy Noise

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A Scaling Limit for Stochastic Newton Equations with α-Stable Lévy Noise

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Abstract

Edward Nelson derived Brownian motion from Ornstein-Uhlenbeck theory by a scaling limit. Previously we extended the scaling limit to an Ornstein-Uhlenbeck process driven by an α-stable Lévy process. In this paper we extend the scaling result to α-stable Lévy processes in the presence of a nonlinear drift, an external field of force in physical terms.
keywords Ornstein-Uhlenbeck process, α-stable Lévy noise, scaling limits
AMS Subject classification: 60G52; 60G15; 60G51; 60H05

1 Introduction

In [10] E. Nelson constructed Brownian motion as a scaling limit of a one parameter family of Ornstein-Uhlenbeck position processes. See also the references in [10] for previous results. In a further step he extended the scaling limit by adding a nonlinear drift to the evolution equation in the cotangent space. Processes of this type are solutions of stochastic Newton equations which where studied e.g. in [1; 2; 9]. Geometrically the Ornstein-Uhlenbeck process is defined on the tangent bundle of the real line. The driving Brownian motion of the system is defined in the tangent space. The scaling procedure recovers the driving process in the limit and a drift term which physically represents the external field of force, see [10].
In our previous work [3] we have extended the result in [10] to α-stable Lévy processes. In this paper we introduce Ornstein-Uhlenbeck processes driven by an α-stable Lévy process as in [3] with an additional nonlinear drift term (βK), $\beta > 0$. For the new model we derive the limit, but first let us give a description of the case where the Ornstein-Uhlenbeck position process is driven by a Brownian motion.

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In physical models $x(t)$ describes the position of a particle at time $t > 0$. It is assumed that the velocity $\frac{dx}{dt} = v$ exists and satisfies the Langevin equation with an additional nonlinear drift. Mathematically the two ordinary differential equations combine to the initial value problem:

$$\begin{aligned} dx_t &= v_t\, dt \\ dv_t &= -\beta v_t dt + \beta K(x_t)dt + dB_t, \end{aligned} \tag{1}$$

with initial value $(x_0, v_0) = (x(0), v(0))$, where $B_t,\ t \geq 0$, is mathematical Brownian motion on the real line, $\beta > 0$ is a constant which physically represents the inverse relaxation time between two successive collisions, and $K(x_t)$ is a nonlinear drift. As mentioned before we assume that a global solution exists. Sufficient conditions for the existence of a unique solution of (1) can be found in e.g. [2; 9] and references therein.
The solution of system (1) is

$$v_t = e^{-\beta t}v_0 + \beta \int_0^t e^{-\beta(t-u)}K(x_u)du + \int_0^t e^{-\beta(t-u)}dB_u,$$

which is called velocity process, and

$$x_t = x_0 + \int_0^t e^{-\beta s}v_0 ds + \beta \int_0^t \int_0^s e^{-\beta(s-u)}K(x_u)duds + \int_0^t \int_0^s e^{-\beta s}e^{\beta u}dB_u ds, \tag{2}$$

which is called position process. We introduce this physical notation for the solution to the stochastic Newton equation since it is more adequate for our studies than the mathematical one. For β tending to infinity the position process converges almost surely to Brownian motion with drift. A rigorous description of the limiting procedure is given in [10, chap. 10]. We emphasize that Nelson is not using standard Brownian motion but introduces the diffusion constant $\sqrt{2\frac{\beta kT}{m}}$ where k, m, T are physical constants.

2 Driving Lévy Noise with an External Force

Let us modify the stochastic Newton equation (1) as in [3]. We introduce a stochastic Newton equation driven by βX_t, where $\{X_t\}_{t\geq 0}$ is an α-stable Lévy process, with $0 < \alpha < 2$ and β is a scaling parameter. Sufficient conditions for the existence of a unique solution may be found in [4; 6; 7]. In this case the solution of this stochastic differential equation can be represented as given in the proposition below.

Proposition 2.1. *Let A be a linear map from $\mathbb{R}$ to $\mathbb{R}$. Furthermore, let X be a Lévy process on $\mathbb{R}$. Let $f : [0, \infty] \to \mathbb{R}$ be a continuous function. Then the solution of the stochastic differential equation*

$$dx_t = Ax_t dt + f(t)dt + dX_t, \qquad t \geq 0,$$

with initial value $x(0) = x_0$, is

$$x_t = e^{At}x_0 + \int_0^t e^{A(t-s)}f(s)ds + \int_0^t e^{A(t-s)}dX_s.$$

Proof. We derive the representation of the solution using integration by parts or Itô formula, respectively, i.e.

$$e^{-At}x_t = x_0 + \int_0^t x_s\left(-Ae^{-As}\right)ds + \int_0^t e^{-As}dx_s,$$

and inserting for $dx_t = Ax_tdt + f(t)dt + dX_t$ we obtain

$$e^{-At}x_t = x_0 + \int_0^t e^{-As}f(s)ds + \int_0^t e^{-As}dX_s,$$

which finishes the proof of the proposition. □

For simplicity reason we treat the case where K in (1) is independent of time. Then the stochastic Newton equation is given by

$$\begin{aligned} dx_t &= v_t\,dt \\ dv_t &= -\beta v_tdt + \beta K(x_t)dt + \beta dX_t, \end{aligned} \tag{3}$$

where $\beta > 0$ and K satisfies sufficient conditions to guarantee existence and uniqueness of solutions see e.g. [4; 7]. Let us focus on the position process $\{x_t\}_{t\geq 0}$. Due to Proposition 2.1 it has the form

$$x_t = x_0 + \int_0^t e^{-\beta s}v_0ds + \beta\int_0^t\int_0^s e^{-\beta(s-u)}K(x_u)duds + \int_0^t\int_0^s \beta e^{-\beta s}e^{\beta u}dX_uds. \tag{4}$$

There is a natural extension of these results to $I\!R^d$, $d > 1$. We observe that the third term in (4), a double integral, includes a stochastic integral with respect to a Lévy process.

Our notation coincides with the one in [4] from where we also recall that for arbitrary Lévy processes Y the characteristic function is of the form $\phi_{Y_t}(u) = e^{t\eta(u)}$ for each $u \in I\!R,\ t \geq 0$, where η is the Lévy-symbol of $Y(1)$. For a centered α-stable Lévy processes the Lévy-symbol at $t = 1$ for $\alpha \neq 1$ is given by:

$$\eta(u) = -\sigma^\alpha|u|^\alpha\left[1 - i\beta\,\mathrm{sgn}\,(u)\tan\left(\frac{\pi\alpha}{2}\right)\right], \tag{5a}$$

and for $\alpha = 1$ is given by:

$$\eta_1(u) = -\sigma|u|\left[1 + i\beta\frac{2}{\pi}\,\mathrm{sgn}\,(u)\log\left(|u|\right)\right]. \tag{5b}$$

Proposition 2.2. *Assume that Y is an α-stable Lévy process, $0 < \alpha < 2$, and g is a continuous function on the interval $[s, t] \subset T \subsetneq I\!R$. Let η be the Lévy symbol of Y_1 and ξ_t be the Lévy symbol of $\psi(t) = \int_s^t g(r)\, dY_r$. Then we have*

$$\xi_t(u) = \int_s^t \eta(ug(r))\, dr \ .$$

The proof is a direct consequence of Theorem 1 in [8].
For $g(\ell) = e^{\beta(\ell - t)}, \ell \geq 0$, and the α-stable process X in (4) the symbol of $Z_t = \int_s^t e^{\beta(r-t)}\, dX_r$ is:

$$\xi(u) = \begin{cases} \int_s^t e^{\alpha\beta(r-t)}\, dr \cdot \eta(u), & \text{for } 0 < \alpha < 2, \alpha \neq 1 \\ \int_s^t e^{\alpha\beta(r-t)}\, dr \cdot \eta_1(u), & \text{for } \alpha = 1 \end{cases}$$

with η, η_1 as in (5a) and (5b), respectively, and $0 \leq s \leq t$. We are thus lead to introduce the time change $\tau^{-1}(t)$ where

$$\tau(t) = \int_0^t e^{-\alpha\beta t} e^{\alpha\beta u} du = \frac{1}{\alpha\beta}\left(1 - e^{-\alpha\beta t}\right), \tag{6}$$

which is actually deterministic. This means that X_t and $Z_{\tau^{-1}(t)}$ have the same distribution.

3 Scaling limit for the stochastic Newton equation

Let us now formulate the main result of this paper.

Theorem 3.1. *Let $t_1 < t_2$, $t_1, t_2 \in T$, T a compact subset of $[0, \infty)$, and $\beta > 0$. Assume that $N_1 > 0$ and $N_2 > 0$ satisfy*

$$(i)\ t_2 - t_1 \geq \frac{N_1}{\beta} \quad and \quad (ii)\ \beta^\alpha \geq N_2 v_0^\alpha, \tag{7}$$

with $0 < \alpha < 2$. Furthermore, let

$$dy_t = K(y_t)dt + dX_t, \tag{8}$$

with $y(0) = x_0$ and $K : I\!R \to I\!R$ satisfy a global Lipschitz condition, then for N_1 and N_2 tending to infinity we have

$$\lim_{\beta \to \infty} x_t = y_t, \tag{9}$$

in probability for any $t \in T$, where $\{x_t\}_{t \geq 0}$ is the position process (4) and $\{y_t\}_{t \geq 0}$ is the solution of (8) with $\{X_t\}_{t \geq 0}$ as its driving α-stable Lévy noise.

Proof. The statement of the theorem means that the position process x_t in (4) converges uniformly in probability to y_t on any compact subset of the time axis $[0, \infty)$, as N_1 and N_2 tend to infinity. The increment of the process (4), according to Proposition 2.1, is given by

$$\begin{aligned} x_{t_2} - x_{t_1} &= \int_{t_1}^{t_2} e^{-\beta s} v_0 ds + \beta \int_{t_1}^{t_2} \int_0^s e^{-\beta(s-u)} K(x_u) du ds + \\ & \int_{t_1}^{t_2} \int_0^s e^{-\beta(s-u)} \beta dX_u ds. \end{aligned} \tag{10}$$

From now on let us denote $\Delta t = t_2 - t_1$. The first integral of (10) tends to zero as β tends to infinity, see [3].

The third part of (10) is estimated by first splitting the double integral into two integrals. We have

$$\beta \left[\int_{t_1}^{t_2} \int_{t_1}^s e^{-\beta s} e^{\beta u} dX_u ds + \int_{t_1}^{t_2} \int_0^{t_1} e^{-\beta s} e^{\beta u} dX_u ds \right]. \tag{11}$$

The double integral of the second part of (11) tends to zero as β and N_1 tend to infinity. For more details we refer to [3].

Let us turn to the first part of (11) which reveals the increment of the driving Lévy process. We use partial integration to have

$$\beta \int_{t_1}^{t_2} \int_{t_1}^s e^{-\beta s} e^{\beta u} dX_u ds = -e^{-\beta t_2} \int_{t_1}^{t_2} e^{\beta u} dX_u + (X_{t_2} - X_{t_1}). \tag{12}$$

By introducing a time change in analogy to (6) on the right hand side of (12) we obtain

$$-e^{-\beta t_2} \int_{t_1}^{t_2} e^{\beta u} dX_u = Z_{\frac{1}{\alpha\beta}\left(1-e^{-\alpha\beta\Delta t}\right)} = \frac{1}{\sqrt[\alpha]{\beta}} Z_{\frac{1}{\alpha}\left(1-e^{-\alpha\beta\Delta t}\right)},$$

where we used the scaling property of α-stable Lévy processes, i.e. $Z_{\gamma\tau} \overset{\Delta}{=} \gamma^\alpha Z_\tau$ with $\gamma > 0$.
By assumption $(7(i))$ we see that $e^{-\alpha\beta\Delta t} \leq e^{-\alpha N_1}$ which tends to zero when N_1 tends to infinity and $Z_{\frac{1}{\alpha}\left(1-e^{-\alpha\beta\Delta t}\right)}$ converges to $Z_{\frac{1}{\alpha}}$. In analogy to the argument above the product $\frac{1}{\sqrt[\alpha]{\beta}} Z_{\frac{1}{\alpha}\left(1-e^{-N_1}\right)}$ tends to zero almost surely for N_1 and N_2 tending to infinity.
This means that the increments related to the position process, i.e. the terms independent of the drift K, are the sum of the increments of the originally driving α-stable Lévy process

$$X_{t_2} - X_{t_1},$$

and three terms which are uniformly bounded by e^{-N_1} and e^{-N_2} for all $t_1, t_2 \in T$, T a compact subset of $[0, \infty)$, and which converge to zero as N_1 and N_2 tend to infinity.
The second term in (10) can be rewritten as

$$\int_{t_1}^{t_2} \beta e^{-\beta s} \int_0^s e^{\beta u} K(x_u) du ds.$$

Let $t_1 = 0$ we obtain

$$\int_0^{t_2} \beta e^{-\beta s} \int_0^s e^{\beta u} K(x_u) du ds.$$

Using integration by parts, we obtain

$$\begin{aligned} \left[-e^{-\beta s} \int_0^s e^{\beta u} K(x_u) du \right]_0^{t_2} + \int_0^{t_2} K(x_s) ds &= -e^{-\beta t_2} \int_0^{t_2} e^{\beta u} K(x_u) du + \\ &+ \int_0^{t_2} K(x_s) ds. \end{aligned} \tag{13}$$

The first integral of (13) can be estimated by

$$\begin{aligned} \left| \int_0^{t_2} e^{-\beta(t_2-u)} K(x_u) du \right| &\leq \int_0^{t_2} e^{-\beta(t_2-u)} \left| K(x_u) - K(x_0) \right| du + \\ &+ K(x_0) \int_0^{t_2} e^{-\beta(t_2-u)} du. \end{aligned} \tag{14}$$

The last integral of (14) is $K(x_0) \left(-\frac{1}{\beta} + \frac{1}{\beta} e^{-\beta t_2} \right)$ which tends to zero as β tends to infinity. Let κ be the Lipschitz constant of K i.e. $|K(x_1) - K(x_2)| \leq \kappa |x_1 - x_2|$ for $x_1, x_2 \in I\!R$. Looking at the first integral in (14) we see that it is bounded by

$$\int_0^{t_2} e^{-\beta(t_2-u)} \left| K(x_u) - K(x_0) \right| du \leq \kappa \sup_{0 \leq u \leq t_2} |x_u - x_0| \int_0^{t_2} e^{-\beta(t_2-u)} du. \tag{15}$$

Now reconsider (4), observing that $\int_0^s e^{-\beta(s-u)} du \leq 1$ and letting $t_2 \kappa \leq \frac{1}{2}$, we have

$$\begin{aligned} x_t - x_0 &= \int_0^{t_2} e^{-\beta s} v_0 ds + \beta \int_0^{t_2} \int_0^s e^{-\beta(s-u)} K(x_u) du ds + \\ &+ \int_0^{t_2} \int_0^s \beta e^{-\beta s} e^{\beta u} dX_u ds. \end{aligned}$$

The absolute value of this difference may be estimated by using the triangle inequality, monotonicity of Lebesgue integrals and by neglecting negative terms as follows

$$
\begin{aligned}
|x_t - x_0| \le & \int_0^{t_2} e^{-\beta s} |v_0| ds + \beta \int_0^{t_2} \int_0^{s} e^{-\beta (s-u)} |K(x_u)| du ds + \\
& + \beta | \int_0^{t_2} \int_0^{s} e^{-\beta s} e^{\beta u} dX_u ds | \\
\le & \int_0^{t_2} e^{-\beta s} |v_0| ds - e^{-\beta t_2} \int_0^{t_2} e^{\beta u} |K(x_u)| du + \int_0^{t_2} |K(x_s)| ds + \\
& + | - e^{-\beta t_2} \int_0^{t_2} e^{\beta u} dX_u + (X_{t_2} - X_0) | \\
\le & \int_0^{t_2} e^{-\beta s} |v_0| ds + \int_0^{t_2} |K(x_s)| ds + |e^{-\beta t_2} \int_0^{t_2} e^{\beta u} dX_u | + \\
& + | (X_{t_2} - X_0) | .
\end{aligned}
$$

Due to the Lipschitz continuity of K with constant κ, taking suprema on both sides of the inequality reveals

$$
\begin{aligned}
\sup_{0 \le t \le t_2} |x_t - x_0| \le & |v_0| + t_2 \kappa \sup_{0 \le s \le t_2} |x_s - x_0| + t_2 |K(x_0)| + \\
& + |e^{-\beta t_2} \int_0^{t_2} e^{\beta u} dX_u | + \sup_{0 \le u \le t_2} | (X_u - X_0) | .
\end{aligned}
$$

Algebraic calculation yields

$$
\begin{aligned}
\sup_{0 \le t \le t_2} |x_t - x_0| \le & |v_0| + \frac{1}{2} \sup_{0 \le s \le t_2} |x_s - x_0| + t_2 |K(x_0)| + \\
& + |e^{-\beta t_2} \int_0^{t_2} e^{\beta u} dX_u | + \sup_{0 \le u \le t_2} | (X_u - X_0) | \\
\frac{1}{2} \sup_{0 \le t \le t_2} |x_t - x_0| \le & |v_0| + t_2 |K(x_0)| + |e^{-\beta t_2} \int_0^{t_2} e^{\beta u} dX_u | + \sup_{0 \le u \le t_2} |X_u| .
\end{aligned}
$$

For β tending to infinity $|e^{-\beta t_2} \int_0^{t_2} e^{\beta u} dX_u|$ vanishes. Hence we neglect this term in the sequel and find

$$
\sup_{0 \le t \le t_2} |x_t - x_0| \le 2|v_0| + c|K(x_0)| + 2 \sup_{0 \le u \le t_2} |X_u|, \tag{16}
$$

where $c = \frac{1}{\kappa} > 0$. We see that the right hand side of this inequality is bounded in probability. In an analogous way we see that for each interval $[t_1, t_2] \subset T$ such that $(t_2 - t_1)\,\kappa \le \frac{1}{2}$ we have that

$$
\zeta_2 = \sup_{t_1 \le t \le t_2} |x_t - x_{t_1}| ,
$$

is bounded. If $(t_2 - t_1)\,\kappa > \frac{1}{2}$ we slice the time interval $[t_1, t_2]$ and use the induction. Thus, for all $t_1 \le t \le \tau_n \le t_2$, $n = 1, 2, \ldots$, and any $t_1, \tau_n \in [0, T]$ we have

$$
\zeta_n = \sup_{t_1 \le t \le \tau_n} |x_t - x_{t_1}| .
$$

For $n = 2$ we have seen that ζ_n is bounded. We assume that ζ_n is bounded for $n = p$ and we use the supremum property to show that it is bounded for $n = p + 1$, i.e. for $\tau_{p+1} \leq t_2$

$$\zeta_{p+1} = \sup_{t_1 \leq t \leq \tau_{p+1}} |x_t - x_{t_1}| \leq \sup_{t_1 \leq t \leq \tau_p} |x_t - x_{t_1}| + \sup_{\tau_p \leq t \leq \tau_{p+1}} \left| x_t - x_{\tau_p} \right|,$$

where the first term of the right hand side is bounded by assumption and the second term is bounded by an analogous argument to the one given in the first step of the induction. Inserting (16) into (15) we obtain

$$\int_0^{t_2} e^{-\beta(t_2-u)} \left| K(x_u) - K(x_0) \right| du \leq$$
$$\leq \kappa \left[2|v_0| + c|K(x_0)| + 2 \sup_{0 \leq u \leq t_2} |X_u| \right] \left[\frac{1}{\beta} \left(1 - e^{-\beta t_2} \right) \right].$$

Then, the integral $\int_0^{t_2} e^{-\beta(t_2-u)} \left| K(x_u) - K(x_0) \right| du$ vanishes when β tends to infinity. Finally, the remaining, non vanishing part of (13) is the integral $\int_{t_1}^{t_2} K(x_s) ds$ as proposed in (9). □

Interesting applications of the Nelson-type scaling limit for α-stable Lévy processes are to study Lévy processes on manifolds. A generalization of Nelson's result on Brownian motion to Banach spaces and Riemannian manifolds is proven in [5].

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