This is the published version of a paper published in *International Journal for Mathematics Teaching and Learning*.

Citation for the original published paper (version of record):

Sollervall, H., Stadler, E. (2015) Validating affordances as an instrument for design and a priori analysis of didactical situations in mathematics
*International Journal for Mathematics Teaching and Learning*, : April 16th

Access to the published version may require subscription.

N.B. When citing this work, cite the original published paper.

Permanent link to this version:
http://urn.kb.se/resolve?urn=urn:nbn:se:lnu:diva-19263
Validating Affordances as an Instrument for Design and a Priori Analysis of Didactical Situations in Mathematics

Håkan Sollervall  
*Linnaeus University, Sweden and Malmö University, Sweden*  
hakan.sollervall@lnu.se

Erika Stadler  
*Stockholm University, Sweden*  
erika.stadler@mnd.su.se

Abstract  
The aim of the presented case study is to investigate how coherent analytical instruments may guide the a priori and a posteriori analyses of a didactical situation. In the a priori analysis we draw on the notion of affordances, as artefact-mediated opportunities for action, to construct hypothetical trajectories of goal-oriented actions that have not yet been initiated by an actor. These hypothetical action trajectories guide the design of a didactical situation, involving trigonometry in triangles and on the unit circle, for the specific purpose of illuminating how students in Swedish upper secondary school handle conceptually challenging tasks without making use of calculators. The a posteriori analysis puts corresponding focus on the actions that students have actually engaged in with respect to the available artefacts. We conclude that affordances, when embedded in the presented methodological framework, may be considered as a scientifically valid instrument for designing and evaluating didactical situations in mathematics.

Keywords  
Affordances, didactical situation, design-based research, validity

Rationale for the study  
On a daily basis, mathematics teachers deal with planning, implementing, evaluating, and improving their teaching practices. In comparison, research about mathematics teaching is dominated by studies that only evaluate observed classroom activities, albeit from different perspectives by using different theories and a variety of analytical instruments. Such instruments may also be used to guide teachers’ evaluation of their own classroom activities by adapting the corresponding research methodologies to the local school context. Regarding teachers’ planning of teaching activities, there are fewer analytical instruments available to guide the teachers’ decisions. Nevertheless, many experienced mathematics teachers are able to readily choose tasks and arrange teaching activities that provide opportunities for their students to achieve specific learning objectives by envisioning them working with the tasks and predicting what they will do. Since such tentative judgments cannot be validated, they are
not highly valued in research. Thus we are faced with the challenge, as researchers having the ambition to inform teachers about possible ways to improve their practices, to customize scientifically valid instruments that address not only the evaluation but also the design and planning of teaching activities. Since predictive instruments can only provide tentative conclusions, we have chosen to embed the predictive instruments in a methodological approach where issues related to both design and evaluation of classroom activities are addressed in different phases of the research process.

**Research question**

Our ambition as researchers is to identify and adapt analytical instruments that may be used by mathematics teachers to improve their teaching practices. In this study, we investigate if the notion of affordances may be suitable as an instrument for a priori analysis of a didactical situation, that is designed for a specific purpose. Our aim is to argue for the existence of a scientifically valid and reliable a priori analysis by addressing the following research question:

In what sense is it possible to embed affordances as a scientifically valid analytical instrument in a design methodological framework that specifically addresses the design and a priori analysis of a didactical situation?

The didactical situation in the presented case study is designed to illuminate a specific issue in mathematics education, namely how students in Swedish upper secondary school handle conceptually challenging tasks in trigonometry without making use of calculators. The internal validity concerns how well affordances guide us in designing a didactical situation that highlights the specific issue about trigonometry, while the external validity relates to how well they support our predictions of students’ goal-oriented actions.

**Methodological considerations**

Research addressing multiple issues, including the design, planning, implementation, and evaluation of a didactical situation, requires careful methodological considerations particularly if all these issues are going to be considered within a comprehensive methodological framework. Our methodological approach draws on the research traditions of didactical situations (Brousseau, 1997; Bessot, 2003) and design-based research (Brown, 1992; Collins, 1992; Cobb, Confrey, diSessa, Lehrer, and Schauble, 2003).

**A design methodological framework – the design cycle**

While many mathematical tasks in text books may be solved with pen and paper, other tasks require students to make use of material and immaterial artefacts that scaffold their
explorations and creative reasoning. Going beyond text books, the teacher may engage in designing new tasks for achieving specific learning objectives. Similarly, researchers may design tasks for achieving specific research objectives. In either case the task design process is guided by an a priori analysis, that serves the dual purpose of predicting outcomes and assessing the efficiency of the tasks with respect to the intended objectives. The tasks may be further adjusted to better fit the objectives, as an optional feature in a cyclic design process (indicated by the dotted line in Fig. 1).

Figure 1 Phases within a design cycle.

The design cycle (Fig. 1) illustrates the (possibly iterative) design process of planning, implementing, evaluating, and refining a teaching activity, guided by theories and specific learning objectives (Cobb et al., 2003).

Teaching activities involve not only tasks but also physical and pedagogical arrangements that may further support achieving the learning objectives. In the next section, we address theories that allow us to consider the design of a mathematical teaching activity not only as a design of tasks but also as an embedding of mathematical tasks in a learning environment.

On the design of didactical situations

An important aspect of a mathematics teacher’s work is to provide mathematical tasks and activities that stimulate the students to discover and consolidate mathematical ideas. Following the terminology used in Theory of Didactical Situations (TDS; Brousseau, 1997) the teacher organizes a didactical situation where the students interact with tasks, artefacts, peers, and the teacher. When the students work autonomously, without guidance from the teacher, the situation is said to be adidactical. Inspired by Piaget’s theory of cognitive development, TDS regards learning as “adaptation (assimilation, accommodation) to an
environment that produces contradictions, difficulties, and imbalances” (Bessot, 2003, p. 2). Within TDS, adaption is interpreted beyond empirical adaptation to a “natural” environment. Instead, the adaptation processes take place in an educational environment that may be designed to provide opportunities for achieving specific learning objectives. The educational environment differs from a natural environment by being controlled by a teacher, representing institutional knowledge and being responsible for achieving institutional goals. In accordance with the definition of adidactical situations, TDS defines the milieu as the “natural” classroom environment consisting of resources such as tasks, artefacts, and peers, but not the teacher. An adidactical situation occurs, within the comprehensive didactical situation, when the students assume ownership and engage autonomously in the didactical project by interacting only with the milieu (Fig. 2; adapted from Bessot, 2003, p. 7).

**Figure 2** Illustration of adidactical situations within the didactical situation.

TDS puts focus on designing didactical situations that require the students to engage in processes of adaptation, that is, didactical situations with a strong element of adidacticity. The adidactical situations are designed to let the students encounter fundamental situations that serve to support achieving the learning objectives (Brousseau, 1997, p. 30).

Although the designer’s objective is to design for adidacticity and provide opportunities for students to engage in exploring fundamental situations, the entire didactical situation cannot be adidactical. Students’ own explorations of a situation has to be initiated by devolution, as transfer of ownership of the didactical situation from the teacher to the students, and followed by institutionalisation, where the teacher connects the students’ work to institutional knowledge. The processes of devolution and institutionalisation may be interpreted within the comprehensive didactical situation, but they are not adidactical (Fig. 3; adapted from Balacheff, 2013).
Figure 3 Devolution and institutionalisation within a didactical situation.

From a design perspective, the adidactical situations pose serious challenges since they cannot be controlled by the teacher during the implementation of the situation. Once the teacher intervenes, the adidacticity disappears. However, selected interventions may be needed to promote the students’ continued engagement in the adidactical situations. Such engagement may be interpreted as a trajectory of goal-oriented actions.

Artefact-mediated actions and objects of activity

The artefacts within a milieu affect what mathematical features the students will explore while engaging in the situation (Stacey, 2011). From a design perspective, the artefacts may be controlled and chosen by the teacher, or specifically designed, according to their supporting power for the unfolding of specific mathematical features.

Drawing on Vygotsky’s notion of explicit mediation (Wertsch, 2007) we may say that these artefacts mediate objects of activity, as actor-constructed objects that “singles out those properties that prove to be essential for developing social practice” (Engeström, 1999, p. 380). Within the framework of Activity Theory (ibid.) the objects of activity are identified based on actions that are actually observed, which makes them suitable as analytical instruments for a posteriori analysis of an implemented didactical situation.

However, regarding the a priori analysis of a situation that has not yet been implemented we need to consider actions that have not yet occurred.

Affordances and hypothetical action trajectories

Designing for adidacticity implies taking into account possible scenarios for students’ interaction with the milieu. From a constructivist perspective, this issue may be dealt with in terms of hypothetical learning trajectories (Simon, 1995; Drijvers, 2003; Empson, 2011).
Regarding the current study, we slightly deviate from the constructivist-oriented notion of hypothetical learning trajectories (HLT, compare Fig. 4) for the main reason that the HLT address learning processes that extend over a substantial period of time (Empson, 2011), while we just consider a singular didactical situation. Furthermore, our ambition is to propose and evaluate a methodology where the a priori and a posteriori analyses are well connected and thus support each other. For these reasons, we have adopted a socio-culturally oriented approach where we put focus on predicting students’ goal-oriented actions in a specific didactical situation. We will refer to predicted trajectories of such possible actions as hypothetical action trajectories, abbreviated HAT in our a priori analysis. The HAT are tentative constructs that may be updated and refined in an iterative design process, particularly in relation to the objects of activity that emerge in the a posteriori analysis.

A certain variety of HAT is needed in order to achieve adidacticity in the didactical situation. Such a variety of HAT may be achieved by considering specific artefacts and what actions they may stimulate the students to engage in.

The construction of HAT require us to consider inherent qualities of artefacts. For example, a knife naturally affords cutting, but it also affords peeling and several other actions. Artefact-mediated actions that are anticipated but have not yet happened are sometimes referred to as affordances of the artefact (Kirschner, Strijbos, Kreijns, and Beers, 2004; Bower, 2008). The affordances are qualities that the milieu offers to an actor, whether the actor makes use of it or not, and “must be perceivable and meaningful so that they can be used

**Figure 4** A mathematics teacher’s design cycle (Simon, 1995, p. 136).
and must furthermore support or anticipate an action” (Kirschner et al., 2004, p. 49). The unfolding of a certain affordance is related to specific actions during a situation and may be stimulated through the provision of appropriate artefacts in the milieu. However, what is actually afforded in the implemented situation depends strongly on socio-cultural factors that influence the actions undertaken by the actors. Nevertheless, affordances may be used both as a design instrument, underpinning decisions of what artefacts should be made available in a specific didactical situation, and to support implementation of the situation by reducing the number of in-the-moment decisions needed to regulate students’ actions (Schoenfeld, 2010).

We will construct the HAT in terms of affordances and interpret the HAT as predicted objects of activity, thus achieving coherence between the a priori and a posteriori analyses.

**Affordances for representation and communication**

Regarding design features in mathematical learning environments, Hegedus and Moreno-Armella (2009) argue that the emergence of students’ representational expressivity can be supported by infrastructures for communication and representation. All mathematical tasks and activities involve representations. A mathematical object may be represented in a didactical situation by written or spoken natural language, with a picture, a diagram, arithmetic symbols or algebraic symbols, or physical manipulatives. For example, a unit circle may be described synthetically as “the locus of all points in a plane located one length unit from a given point”, or analytically as the set of all points \((x, y)\) that satisfy the equation \(x^2 + y^2 = 1\), or by a drawn picture of a circle with radius 1. Each representation adds a dimension of meaning to the object making it more flexible and adaptable for the actor particularly when the situation calls for solving a mathematical task, requiring the actor to choose efficient representations of the objects in the task (Winsløw, 2003; Sollervall, 2011).

In the process of designing a mathematical task and embedding it in a didactical situation, we particularly need to consider affordances for representation and communication. In this initial part of the design process, we may switch between the development of a preliminary situation and an a priori analysis of its predicted outcomes in terms of affordances. After the situation has been implemented, we may analyze its outcomes in terms of objects of activity.

We will now proceed to account for a case study of a didactical situation involving trigonometry and the unit circle. In this case study, we particularly put focus on the students’ capabilities and preferences for making use of various artefacts. The study serves to illustrate how the a priori analysis, where HAT are constructed in terms of affordances, can be validated and strengthened by an a posteriori analysis based on objects of activity.
The purpose and set-up of the current study

The aim of the current study is to illustrate how affordances and objects of activity may be used to account, respectively, for the prediction and evaluation of upper secondary students’ problem solving strategies regarding trigonometry in triangles and on the unit circle.

A specific didactical situation has been designed to afford several different strategies for calculating values of sine and cosine for acute, right, and obtuse angles in triangles. The study was conducted with nine students, age 18-19, in their last year of study at the Natural Science program in upper secondary school. All nine students had elected to take the optional last mathematics course in the program that covered complex numbers and differential equations. The didactical situation was implemented in three groups, with three students in each group. They were informed that their performance during the situation would not affect their grades.

Trigonometry as topic for the study

The unit circle was chosen as topic for the study due to its transitional character, as an object that is rarely used in Swedish secondary school but has a central role in trigonometry at university level. According to three experienced Swedish university teachers who were interviewed as part of the preparation for the study, the unit circle mediates an abstract and general understanding of mathematics needed in mathematics studies at university. While the unit circle is not needed for developing procedural understanding, which according to the university teachers dominates secondary school mathematics, it plays an essential role in developing conceptual understanding. The unit circle also mediates a “deeper understanding” of mathematics, characterized as a holistic understanding involving connections and relations between concepts. The teachers argued that excessive use of calculators may hamper students’ development of conceptual understanding, that students also need to engage in algorithmic processes and experience mathematical structures by working only with pen and paper.

From an epistemological perspective, trigonometry involves angles as a concept hard to define. Angles are commonly used in two qualitatively different ways, as a geometrical object (a region bounded by rays) as well as a measure (of “open-ness”) in degrees or radians (Thompson, 2008). In secondary school trigonometry, the problematic notion of angle may be replaced by a name (such as \(u\) or \(v\)) placed as a label inside the triangle together with a tiny arc indicating the angle’s “opening” (Fig. 5, left pane) or the name of a corner of a right-angled triangle (such as \(A\) or \(B\)) placed outside the triangle (Fig. 5, right pane).
Calculating $\sin \, v$ and $\cos \, v$ (or $\sin \, A$ and $\cos \, A$) may then be reduced to interpreting the label $v$ (or $A$) as suggesting a specific perspective on the triangle rather than an angle. As a consequence, sine and cosine may be interpreted as measures in (right-angled) triangles rather than measures of angles, which in turn may cause trigonometry of triangles and trigonometry of periodic functions to be regarded as two unrelated trigonometries (ibid.). The Cartesian coordinate plane may serve as an additional useful register for trigonometry (Presmeg, 2007). Working with trigonometry in the coordinate plane may afford interpreting the hypotenuse of a right-angled triangle both as the radius of a circle and as a unit of measurement (Moore, 2010), serving to connect the trigonometries of triangles and periodic functions.

**Designing the didactical situation**

We will proceed to account for the tasks that we were inspired to construct based on the literature review and the interviews with the university teachers. But first, we discuss the pedagogical arrangements and the milieu for the didactical situation.

With the ambition to make the student groups engage in adidactical situations, working autonomously without having to ask the teacher for help, we provided written instructions in the form of prompts with detailed descriptions both about the tasks themselves and how the group was expected to work with them. During the devolution phase, the students were given an oral introduction inviting them to engage in collaborative efforts of problem solving by discussing, writing, making use of the available materials, and interacting with their peers. They were informed that they should attempt to come up with solutions that each of them could understand and agree on. They were also informed that they may ask the teacher, but only if they cannot figure out how to proceed based on their own initiatives. The tasks were presented in a group room, with a wide variety of materials present on the table: pens, eraser, ruler, graph paper, overhead transparencies, overhead pens, protractor, compass, and scissors, with the purpose of affording a diversity of solution strategies. However, we decided to follow up on the university teachers’ and avoid using calculators. The tasks were presented hand-written on graph paper (5 mm square pattern, compare Fig. 6) intended to afford constructing a coordinate system, constructing circles, and measuring horizontal and vertical lengths by visual inspection, as the corresponding number of squares.
For convenience of notation, we refer to the three tasks in Figure 6 as Task A, Task B, and Task C, respectively. We proceed with an a priori analysis with focus on affordances, followed by an a posteriori analysis of objects of activity, with primary focus on representation.

A priori analysis of the triangle task
The a priori analysis not only predicts the outcomes of a preliminary didactical situation, but also informs us how it may be improved in order to better support the hypothetical action trajectories. In this section we briefly account for these HAT, with focus on affordances.

The available artefacts in the milieu were: pens, eraser, ruler, graph paper, overhead transparencies, overhead pens, protractor, compass, and scissors, but no calculators. Providing a multitude of artefacts was motivated by our desire to understand students’ preferences and capabilities to engage in mathematical problem solving, specifically when calculators were not available.

The graph paper was intended to afford copying and enlargements of angles and triangles. The overhead transparencies and overhead pens afford geometric representation of (unit) circles to be placed on top of angles and triangles. The compass and protractor both afford the construction of circles. The scissors afford cutting out circles and triangles.

Hypothetical action trajectories for Task A
The standardized shape of the triangle and position of the angle in Task A invite the students to make use of the definition of cosine for an acute angle in a right-angled triangle, as the length of the adjacent side divided by the length of the hypotenuse. However, as the students had also worked with the unit circle and the general definition of cosine for any angle as the x-coordinate for the corresponding point on the unit circle, they could also approach Task A in this manner.
**HAT 1.** The students apply the rule of dividing the length of the adjacent side with the length of the hypotenuse. The length 4 cm of the adjacent side may be measured by inspection (as the length of 8 adjacent squares) or by using the ruler. Measuring of the hypotenuse is afforded by the ruler and the Pythagorean Theorem or by recall as the third length in the well-known 3-4-5 Pythagorean Triple, or by using the compass.

**HAT 2a.** The students measure the angle with the protractor. The angle is copied onto a new graph paper, preferably in a larger scale to afford a smaller error of measurement. The angle could alternatively be copied without the protractor, by counting the appropriate number of squares in the original picture and possibly enlarging the angle by proportional reasoning. Thereafter, the compass affords the drawing of a circle with any radius (within reasonable limits) which may be chosen as 10 cm (as in Fig. 7) to afford simple calculations, or just by inspection to obtain the cosine value as the \(x\)-coordinate (reasonably interpreted on the graph paper) divided by the radius, or by interpreting 10 cm = 1 dm as the length unit.

**HAT 2b.** The compass, overhead transparencies, and the overhead pens, afford the drawing of a circle that can be applied for measuring the angles in the tasks, either as is or in enlarged versions. The remaining parts of HAT 2b agree with HAT 2a.

**HAT 2c.** The compass affords the drawing of a circle on a separate graph paper. The compass, overhead transparencies, and the overhead pens, afford the triangles to be copied and possibly enlarged on the transparencies. These transparencies may be placed on top of the circle. The remaining parts of HAT 2c agree with HAT 2a.

**Hypothetical action trajectories for Task B**

Since the angle in Task B is obtuse, the students cannot directly apply the definition of cosine for an acute angle in a right-angled triangle. However, they may make use of an auxiliary
right-angled triangle (Fig. 8) which involves the supplementary angle (HAT 1) or make use of the unit circle (HAT 2). A third strategy involves the Cosine Theorem (HAT 3).

**HAT 1.** The given triangle is extended as in Figure 8. The right-angled triangle with the acute angle \( u \) is treated as in Task A, to obtain \( \cos u \). As \( u \) is the supplementary angle to \( v \), \( \cos v \) is obtained as \( -\cos u \).

**HAT 2.** Similar to HAT 2abc for Task A. The obtuse angle is copied onto a graph paper and a drawn circle affords the determination of the cosine value (Fig. 9).

**HAT 3.** The Cosine Theorem

\[ c^2 = a^2 + b^2 - 2ab \cos v \]

affords solving for the value \( \cos v \). The length \( a \) of the horizontal leg is readily measured by inspection (Fig. 6b), while the length \( b \) of the slanted leg is determined by applying the Pythagorean Theorem to an auxiliary triangle (with catheti lengths 8 and 6 squares, or 4 cm and 3 cm, compare Fig. 6b and Fig. 8). The length \( c \) of the side opposite to the angle \( v \) is also afforded by the same theorem, applied to a larger auxiliary triangle (with catheti lengths 18 and 6, compare Fig. 6b and Fig. 8).

**Hypothetical action trajectories for Task C**

The students are familiar with the 90 degree angle, primarily as a right angle in triangles but also as corresponding to the point \((0,1)\) on top of the unit circle, which invites the students to solve Task C simply by recall of known facts. This recall may be based on various known representations, as indicated in HAT 1a, HAT 1b, and HAT 1c.

**HAT 1a.** The angle is identified as “90 degrees”. The cosine value is afforded by verbal recall “cosine for 90 degrees equals zero”. This phrase may, or may not, evoke mental pictures of the point \((0,1)\) located on top of a unit circle.

**HAT 1b.** The angle is identified as “90 degrees”. Recalling and writing the known formula \( \cos 90^\circ = 0 \) affords the cosine value 0.

**HAT 1c.** The angle is identified as “90 degrees”. The student verbally relates 90 degrees to the point located on top of the unit circle. This point has \( x \)-coordinate 0.

**HAT 2a.** Similar to HAT 2 for Task A and Task B. The right angle is copied onto a graph paper and a circle is drawn with the compass. The circle affords the determination of the cosine value as the \( x \)-coordinate 0 for the point on the top of the circle.

**HAT 2b.** A unit circle is drawn schematically to support the conclusion that the right angle corresponds to the point located on top of the unit circle. The drawn circle affords the conclusion that this point has \( x \)-coordinate 0, which is also the cosine value.
Results and a posteriori analysis of the triangle task

While engaging in the implemented activity and attempting to solve the three subtasks A, B, and C, the students make use of material artefacts such as the graph paper, the protractor, and the compass. They also make use of immaterial artefacts such as known facts about cosine values of angles in triangles, the Pythagorean Theorem and the Cosine Theorem, as well as familiar strategies such as engaging in the identification of the $x$-coordinate for a point on the unit circle.

Group 1 solves the tasks readily, but the students in Group 2 entangle themselves in symbolic expressions that they cannot understand or even manage at a procedural level. In our analysis we pay more attention to the work of Group 3, where there is apparent tension between the group members because of differences both in mathematical capacity, preferences for strategies, as well as regarding available and perceived affordances.

The transcripts from each of the three groups reveal that their work is hampered by the absence of calculators, as mediators for the calculation of cosine values.

**Group 1. Three girls: Evelina, Karin, and Sara.**

The students start by measuring the sides of the triangles. When they see the protractor on the table they measure the size of the angles.

Evelina: Okay we can measure the angle but that does not give the cosine value. But… with cosine value, do they mean what cos like 30 is?

Karin: Or what it becomes if you enter it.

Evelina: Yes, oh yes. That is what is meant by the cosine value.

Karin: Yes, I think so. Or I do not know.

Evelina: Yes, maybe you should make a unit circle.

Karin: Should we construct such a one maybe? Then we can just read it off afterwards.

Regarding Task A, they make use of strategy HAT 2a. That is, they copy the acute angle onto an empty graph paper with the protractor, use the compass to draw a circle with radius 10 cm, and read off the cosine value on the $x$-axis. The same strategy is used for Task B. Regarding the 90 degree angle in Task C, they recall verbally that cosine 90 degrees is 0 (HAT 1a) and also confirm their answer by making use of a schematically drawn unit circle (HAT 2b).

The phrase “what it becomes if you enter it” refers to intended actions of evaluation that would be mediated by a calculator. Since calculators were not available during the activity,
the students have to shift their attention towards the available artefacts. The unit circle is not available as a material artefact, but the students’ mathematical knowledge serves as an immaterial artefact that initiates an action directed at constructing a unit circle. This action of construction is mediated by the compass.

**Group 2. Three boys: Tobias, Daniel, and Niklas.**

Niklas looks at the first triangle and says quickly “there you have 45 degrees”.

Tobias: 45 degrees. How do you get it to that?
Daniel: But it was like we guessed, or?
Niklas: It is not 45 degrees.
Tobias: No, it is not.
Daniel: But I thought it looked like that.
Niklas: Yes yes…

Their reasoning is hard to follow. They do not use many words and communicate mainly by pointing at the pictures in the tasks.

Niklas: Then what does it become?
Daniel: I do not know. Sine and cosine are kind of adjacent… It must be such and such… then it becomes like that.

Daniel determines the lengths of the triangle’s legs by counting squares on the graph paper. He writes \( v = \sin^{-1} \left( \frac{8}{6} \right) \) and concludes that \( \cos v = \cos \left( \sin^{-1} \left( \frac{8}{6} \right) \right) \). Niklas does not like that answer.

Niklas: Let us see here. That was the cosine value. Positive. But you do not need that. Sine raised to minus one you do not need.
Daniel: Why is that?
Niklas: Cosine, that is adjacent divided by the hypotenuse.
Daniel: Yes.
Niklas: And that is 6. So take away sine.
Daniel: Then we write so. Like that.
Daniel writes $\cos \frac{8}{6}$, which is an incorrect answer for Task A. The group proceeds and eventually agree on the (incorrect) answer $\cos^{-1} \left(180^\circ - \cos^{-1} \left(\frac{8}{6}\right)\right)$ for the obtuse angle in Task B. When proceeding with Task C, they somewhat surprisingly involve a unit circle. Tobias initiates the discussion by saying “cos 90 becomes… cos 90 is either one or zero. One of them.” Daniel follows up on Tobias’ initiative, draws the unit circle and marks the angle 90 degrees but reads off the cosine value on the $y$-axis. This attempt resembles HAT 2a, but they implement the strategy incorrectly as they should have read off the cosine value on the $x$-axis.

**Group 3. Two girls and one boy: Jenny, Anna, and Bjorn.**

Regarding Task A, Jenny quickly states that “cosine, then you take the hypotenuse and the adjacent”. They measure the triangle’s legs, calculate the length of the hypotenuse by using the Pythagorean Theorem, and conclude that “$\cos v$ equals $5$ divided by the root of $41$”. This strategy agrees with HAT 1, but is implemented with incorrect values as the students estimate the lengths of the legs incorrectly as $5$ cm and $4$ cm, instead of $4$ cm and $3$ cm as in the picture. However, their strategy may be regarded as conceptually correct. They apply the Pythagorean Theorem in a correct manner, but receive an incorrect length for the hypotenuse due to the incorrect values $5$ and $4$. They cannot see how they should proceed since they do not have a calculator, and they are not sure that $\cos v = \frac{5}{\sqrt{41}}$ is an acceptable answer.

Jenny: But it says the cosine value for the angle. It does not say…

Bjorn: It does not say that you should answer in degrees.

Their doubts concern their strategy and the format of their answer, which both are appropriate for the task, and not the incorrect values that are due to measurement. Hence, it is unlikely that they would have discovered these errors by pursuing their discussion. Although not convinced that their answer to Task A is acceptable, they agree to move on to the next task.

Regarding Task B, they quickly agree that the strategy used in the previous task only works for acute angles. Jenny suggests several times that they should use the Cosine Theorem but Anna and Bjorn do not follow up on her suggestions. Instead, the three students attempt to consider the supplementary angle $u$ and propose that they may calculate $\cos(180^\circ - u)$ but they do not pursue working in this direction. Although the instructions explicitly mention that the protractor may be used, they are very reluctant to make use of it.
Jenny: We also have a protractor here, but maybe we are not allowed to use it.
Teacher: Yes, you may use it.
Anna: But it feels like we are supposed to solve everything without using such things that are...
Bjorn: But what is the problem?!
Anna: But then we do not need. We can get it directly.
Bjorn: Then we do not need cosine or anything.
Jenny: But… the question is about the cosine value… the cosine value…

Jenny: If we can find the angle then we are done. If you just take cosine angle you get the cosine value. Shall we do that instead?
Anna: Yes… the cosine value. Okay, that is like cos…
Jenny: Yes, we cannot continue like this.
Anna: What is the meaning of these tasks?
Jenny: There must be a reason that we have a protractor and such. Let’s use it.
Anna: But the answer cannot be just like this.
Jenny: Why couldn’t it be?
Anna: Should that take an hour? Okay, we just have to measure.
Bjorn: It is just because nobody else has understood that.
Jenny: Why should the protractors lie there if nobody is supposed to use them?
Anna: But it can’t… cos 143…
Jenny: Yes, that is the cosine value.

When they conclude that the cosine value for the obtuse angle is \( \cos 143^\circ \), they do not know how to proceed without making use of a calculator.

Jenny: Well if we had had a calculator we could have calculated this so that it becomes kind of…
Bjorn: This is actually really easy. It is just because we do not have a calculator.
Jenny: Yes, exactly. I hate not having a calculator. I have not worked without a calculator for…
Anna: But at the university… there they don’t even have calculators?
Jenny: Yes, I know.
Certainly, the expression $\cos 143^\circ$ may be considered as an acceptable answer to Task B. However, the students feel that they are expected to simplify the answer and become frustrated when they cannot take action to perform this simplification.

The students briefly consider two possible strategies for calculating the cosine value of the angle, namely the Cosine Theorem (HAT 3) and using the supplementary angle (HAT 1). The first strategy is not pursued because Jenny is not able to convince Anna and Bjorn about engaging in this strategy, possibly because they do not know the Cosine Theorem well enough to make use of it in solving the task. While they agree to consider the supplementary angle, they suddenly abandon this object of activity. Instead, they engage in discussing affordances of the protractor, where they agree that it can be used to determine the angle but not to solve the task completely.

Besides the task-oriented objects of activity, we may interpret Bjorn’s comments as attempts to make him appear more important in front of the girls. He is very assertive about his claims, which do not contribute much to the progression of the group’s work. This behavior becomes even more apparent when the group continues working with the next task.

They move on to Task C, where they recall the cosine value for $90^\circ$.

Jenny: Cosine 90 degrees is 0, isn’t it?
Bjorn: Yes, $\cos 0$ is 90.
Jenny: Cos 90 is zero.
Bjorn: No, the other way around. $\cos 0$ is 90 degrees.
Anna: Yees… Yes, that’s the way it is.

Bjorn claims that he can prove that his statement is true by drawing “the f-ing unit circle”, thus showing awareness of HAT 2b. This strategy is implemented by Jenny who justifies her claim by drawing the angle in the unit circle. She explains for the others how you can read off the cosine value for an angle in the unit circle.

Jenny: But it is like that. Cos, that is. If you have the unit circle it is cosine on that one [points on the $x$-axis]. And on this, here is cosine. Cos 90 is 0.
Jenny: Imagine… imagine this like a pointer that turns. Then it turns up to here and then you can read off cosine on the $x$-axis. Then you get up to there [points towards the point $(0,1)$ on the $y$-axis].
Anna: But then it becomes 0. You can get 0 for many… Cos 90 is 0.
Bjorn: I told you that cos 0 is 90!!
Jenny: But cos, that is, 90 degrees becomes this.

When they are done with the three tasks, Jenny wants to return to Task B with the obtuse angle because she “wants to know the Cosine Theorem”. Anna argues that it is not needed and Bjorn is very reluctant to participate.

Bjorn: But the Cosine Theorem becomes too messy? Can you use it without a calculator?
Anna: I don’t think you can use it in this case. We don’t have cosine.
Jenny: But this is cosine… Then we get the cosine value. Let’s test this.

Jenny proceeds to apply the Cosine Theorem. At one point she asks Bjorn to calculate 9,5 times 9,5 but else he has stopped listening. Anna tries to follow Jenny’s work. Jenny remembers the Cosine Theorem by heart, but the teacher allows her to use a book with formulas for confirmation. She arrives at the result -39,75/50 and concludes that “this should be correct because it becomes negative when it is larger than 90”.

After discussing a while with the teacher, Anna says that “there must be a reason for there being so much of the unit circle”. Jenny expresses he frustration: “It feels that this didn’t turn out very well. It feels as if we are rather hopeless without calculators. No, this isn’t good”. After Anna has drawn a circle around the obtuse triangle, Jenny questions if this is allowed since the radius is 5 and not 1. They continue discussing about possibly changing the radius but they run out of time and excuse themselves.

The three students do not communicate effectively during the session. Their individual preferences for different strategies cause them to spend considerable efforts on negotiating and agreeing on a common strategy. The affordances that each of them perceive simply do not agree very well, so that they may engage in a common effort directed at solving the tasks. They all know how to use a calculator to find cosine values, but they are not sure how to proceed when calculators are not available.

**Assessing the validity of affordances**

By making use of corresponding theoretical instruments, hypothetical action trajectories for the a priori analysis and objects of activity for the a posteriori analysis, we can account for the comprehensive effort of designing, implementing and evaluating a didactical situation. The a
posteriori analysis is efficiently related to the hypothetical action trajectories that were constructed in the a priori analysis. In the research process, affordances were used as instruments for designing and a priori analyzing the didactical situation.

**Internal validity of affordances as an instrument for design**

Affordances may be questioned due to their tentative nature. A proposed affordance could very well be considered as nothing more than a pure guess. The predictive value of a set of affordances can only be assessed a posteriori, in our case when the didactical situation has been implemented. The purpose of our didactical situation was to illuminate how students in upper secondary school handle conceptually challenging tasks without making use of calculators. When assessing the internal validity of affordances, we ask ourselves if we achieved that purpose and to what extent the identified affordances guided us towards that end.

Obviously, the students were not allowed to use calculators in our didactical situation. Instead, several other artefacts were introduced that afforded conceptually oriented reasoning. For example, the three tasks (Fig. 6) were chosen to successively challenge the students’ understanding of trigonometric values, starting with a task about the cosine value for an acute angle in a right-angled triangle directly followed by a task involving an obtuse angle, and finally a task about the cosine value for a right angle. The tasks were presented hand-written on graph paper with the intention to afford constructing a coordinate system, constructing circles, and measuring horizontal and vertical lengths by visual inspection, thus stimulating conceptually oriented strategies.

The absence of calculators explicitly affects the work in Group 1, as they measure the angle but are initially not able to figure out its cosine value since they cannot “enter it”. However, they readily handle this denied affordance of calculating cosine values by letting the compass afford construction of a unit circle with radius 10 cm, thus implementing the hypothetical learning trajectory HAT 2a.

In comparison, Group 2 attempts a symbolic representation of the solution, which would have been correct if they had used, for example, $v = \tan^{-1} \left( \frac{6}{8} \right)$ instead of their own incorrect representation $v = \sin^{-1} \left( \frac{\theta}{6} \right)$. In their continued work, they are not able to simplify the (incorrect) symbolic expression $\cos \left( \sin^{-1} \left( \frac{\theta}{6} \right) \right)$, possibly because they cannot access a calculator. They make use of a unit circle in the third task but incorrectly interpret the cosine value on the $y$-axis, thus attempting but not succeeding to implement the conceptually oriented strategy HAT 2a. One possible reason may be that they are used to calculating
trigonometric values on their calculators, instead of applying conceptually oriented strategies for confirming values.

Group 3 readily implement HAT 1 to solve the first task. They recall the definition of cosine in a right-angled triangle, let the square pattern afford measurement of the catheti, and let the Pythagorean theorem afford calculating the length of the hypotenuse. In the second task, with the obtuse angle, they initially consider HAT 3 and HAT 1 as conceptually oriented strategies afforded by the Cosine Theorem and the supplementary angle, respectively. Instead of implementing these strategies, they choose to let the protractor afford measuring the angle but get stuck when they cannot simplify the answer $\cos 143^\circ$. They explicitly express that they “could have calculated this” with a calculator. At the end of the session, they return to the Cosine Theorem but one of the students questions if the theorem an be used without a calculator. Our conclusion is that the students are aware of several conceptually oriented strategies, but are reluctant to implement them when calculators are not available.

We conclude that the design process is supported by the construction of HAT in terms of affordances. The HAT have afforded the design of a didactical situation highlighting students’ conceptually oriented strategies for handling conceptually challenging tasks in trigonometry without making use of calculators. Furthermore, the implemented didactical situation has also provided insights into the students’ preferences and dependence on calculators for doing mathematics. These insights may also be attributed the HAT and their affordances.

**External validity of affordances as an instrument for analysis**

We measure external validity in terms of agreement between a priori and a posteriori analyses, thereby claiming to address notions such as confirmability, credibility, transferability, and dependability (Lincoln & Guba, 1985). In our case, most of the observed strategies identified in the a posteriori analysis can be confirmed in relation to the HAT in the a priori analysis. Only the (incorrect) symbolic representation $\cos \left( \sin^{-1} \left( \frac{9}{16} \right) \right)$, used by Group 2, is not found among the HAT. However, since the students did not succeed in simplifying this symbolic expression, it may be considered as not affording a solution and thus not qualifying as an HAT, at least not in the current didactical situation. Furthermore, all the HAT in the a priori analysis were observed, thus implying high credibility. However, all the HAT were not observed in all forms. For example, the students did not use the compass as in HAT 2a for solving the third task. Instead they favored the schematic drawing in HAT 2b, thus implementing the same conceptual strategy but in a different manner.
Regarding transferability of our findings, we can only make a preliminary assessment based on subjective judgments based on our knowledge of mathematics education in the Swedish context. The transferability may certainly be enhanced by making accumulated studies in several design cycles. Furthermore, credibility as well as transferability is addressed already in our methodological approach, as our design methodological framework involves cross-validation between HAT and affordances and objects of activity, as constructs within the well established framework of Activity Theory. We need this interconnectedness between constructs particularly to validate the a priori analysis. We claim that the coherence between analytical constructs, particularly regarding predicted actions and observed actions as well as the possibility to accumulate empirical data, implies high external validity in our research process. We regard affordances as a necessary instrument to achieve this necessary coherence.

**Discussion**

Although affordances and objects of activity have individual merits, as established scientific concepts, we achieve additional insights by connecting them in our analyses. Not only do the hypothetical action trajectories help to focus and simplify the a posteriori analysis, but they also guide the designer to promote or suppress specific objects of activity already in the design phase. For example, not allowing the students to make use of calculators in the didactical situation obviously suppressed (denied) certain objects of activity. This particular design decision has provided insights regarding students’ capabilities and readiness to engage in conceptually oriented strategies when calculators are not available.

From a research perspective, our design-based approach allows us to design didactical situations that are optimized to inform us about specific research issues. Instead of randomly looking around for evidence in mathematics classrooms, we can design and nurture situations that provide a good crop of evidence for answering research questions and addressing research hypotheses. In order to do this efficiently, the a priori and a posteriori analyses have to be closely related. In this study, we have illustrated one possible way to achieve such a close connection through affordances and objects of activity.

An inherent limitation in our approach concerns the quality of our analyses. Since we address multiple aspects within each design cycle, we have to distribute our research efforts accordingly and do not analyse the outcomes at a very fine-grained level. Our ambition has been to provide a balanced and “lean” exposition of the comprehensive design process including the underpinning theories, the design phase, the a priori analysis, and the a
posteriori analysis, without emphasizing either aspect more than the other. This style of presentation may also appeal to teachers who are used to working with limited resources.

References


