Accurate techniques for 2D electromagnetic scattering

Imad Kassar Akeab

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Department of Physics and Electrical Engineering
Faculty of Technology
Linnaeus University
Sweden
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Abstract

This thesis consists of three parts. The first part is an introduction and references some recent work on 2D electromagnetic scattering problems at high frequencies. It also presents the basic integral equation types for impenetrable objects. A brief discussion of the standard elements of the method of moments is followed by summaries of the papers.

Paper I presents an accurate implementation of the method of moments for a perfectly conducting cylinder. A scaling for the rapid variation of the solution improves accuracy. At high frequencies, the method of moments leads to a large dense system of equations. Sparsity in this system is obtained by modifying the integration path in the integral equation. The modified path reduces the accuracy in the deep shadow.

In paper II, a hybrid method is used to handle the standing waves that are prominent in the shadow for the TE case. The shadow region is treated separately, in a hybrid scheme based on a priori knowledge about the solution. An accurate method to combine solutions in this hybrid scheme is presented.
Keywords: Integral equations, method of moments, sparsity, scaling, shadow boundary, B-Spline, hybrid method, complex contour, Fock functions, standing wave pattern
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Introduction

High frequency scattering problems formulated with integral equations lead to linear systems with a large number of unknowns. The solution of these problems, by means of efficient and accurate numerical techniques, is the topic of many recent papers [1],[2],[3],[4], [5], [6].

The Fast multipole method (FMM) is a general technique that has been developed to handle a large number of unknowns ($N$). The method requires $O(N\log N)$, or even $O(N)$, operations. Obtaining high accuracy with the FMM requires a large $N$, also for 2D problems, and even with the $O(N\log N)$ complexity [7], the computational time will be considerable. The balance between accuracy and execution time is a point of interest for the FMM method. The standard FMM methods are derived from approximations of the kernel, in the form of Taylor expansions, or as expansions in terms of eigenfunctions. In a recent study [7], one handles the kernel with Cauchy’s integral formula and the Laplace transform. This method reformulates the kernel so that it leads to diagonal multipole-to-local operators, which reduce the computational time without losing accuracy.

Methods based on wavelets have been developed to solve the 2D Helmholtz problem for large wave numbers. The idea is to obtain a sparse matrix by means of a wavelet basis. The number of operations needed to solve this sparse matrix is $O(N\log^2 N)$. An investigation showed that the sparsity that can be achieved with this method is lost for high frequencies [8].
The investigations in this thesis seek a fast and accurate method to solve two dimensional Helmholtz problems for both medium and high frequency electromagnetic scattering. A formulation in terms of an integral equation has the advantage that it reduces to a 1D problem. Dirichlet and Neumann boundary conditions at the surface of the scatterer are considered.

For large wave numbers, the oscillatory solution requires a large number of basis functions to be represented correctly. Since the phase of the solution resembles that of the incident field [1], according to physical optics (PO), one could factor the total solution into a smooth part and a phase factor. Extracting the phase from the solution substantially reduces the size of the linear system. One way of handling this problem was investigated by Bruno, et al. in 2004 [1]. They used the principle of stationary phase for local integration at high frequencies. In 2007 Huybrechs and Vandewalle [9] obtained a sparse linear system by modifying the integral equation by means of a suitable complex integration contour. The decay of the kernel makes it possible to decouple the basis functions. Leaving the real axis and doing integration in the complex plane leads to a sparse matrix. This approach is accurate in the lit zone only [9] and the deep shadow is excluded with the argument that the field is approximately zero there at high frequencies.

Some background and notation for the integral equations for transverse electric and magnetic waves follows. The combined field integral equation (CFIE) is used to eliminate internal resonances. Section (1.4) briefly introduces the uniform and nonuniform B-splines. Integration paths for numerical quadrature are discussed as a preparation for the implementation of sparsity. A few words on numerical quadrature and solvers for linear systems conclude this introduction to the papers. Short summaries of paper I and paper II are presented.

1.1 The Electric Field Integral Equation (EFIE)

In scattering problems, the total field is the sum of the incident field and the scattered field,
\[ u(\rho) = u_i(\rho) + u_s(\rho). \]  

(1.1)

From Green’s theorem, the scattered field at a point \( \rho \),

\[ u_s(\rho) = \int_l \left( G_0(\rho, \rho') \frac{\partial u(\rho')}{\partial n'} - u(\rho') \frac{\partial G_0(\rho, \rho')}{\partial n'} \right) dl', \]

(1.2)

leads to the relation,

\[ u(\rho) = u_i(\rho) + \int_l \left( G_0(\rho, \rho') \frac{\partial u(\rho')}{\partial n'} - u(\rho') \frac{\partial G_0(\rho, \rho')}{\partial n'} \right) dl', \]

(1.3)

Here \( \rho \) and \( \rho' \) represent the observation and source points. \( \hat{n}' \) is a unit normal to the scatterer surface and \( G_0(\rho, \rho') \) represents the Green’s function. The point \( \rho \) approaches the surface from the inside and is separated by a short distance \( \delta \) from the boundary.

Let the field \( u \) in equation (1.3) represent the electric field \( E \). For the interior case, the total electric field inside the scatterer equals zero \( (u(\rho) = 0) \). A wave with the electric field in the \( z \) direction is incident on an infinite PEC (perfect electric conductor) cylinder, as shown in Fig. 1.1. The magnetic field is then transverse (TM) to the \( z \) direction \([10]\) and one obtains the EFIE. Since the electric field inside a PEC scatterer is zero and continuous at the boundary \([11]\), the Dirichlet boundary condition on the surface, \( E(\theta) = 0 \) holds, and equation (1.3) reduces to,

\[ E^i(\theta) = -\int_l G_0(\theta, \theta') \frac{\partial E(\theta')}{\partial n'} dl'. \]

(1.4)

A surface current \( J_s \) is induced in the \( z \) direction. The normal derivative of the electric field on the boundary equals,

\[ \frac{\partial E(\theta')}{\partial n'} = i k \eta J_s(\theta'). \]

(1.5)

Here \( k \) and \( \eta \) are the wavenumber and the impedance of free space, respectively. The scatterer surface should be smooth, otherwise the normal derivative \( \partial E(\theta')/\partial n' \) is not
well defined in particular at sharp edges. The final result is an integral equation for the current $J_s$,

$$E^i(\theta) = -ik\eta \int l G_0(\theta, \theta') J_s(\theta') dl'. \tag{1.6}$$

The free space Green’s function $G_0(\theta, \theta')$ in two dimensions is,

$$G_0(\theta, \theta') = \frac{i}{4} H_0^{(1)}(kR). \tag{1.7}$$

$R = |\rho - \rho'|$ is the distance between the observation and the source point and $H_0^{(1)}(kR)$ is the Hankel function of the first kind and zero order. One obtains the Electric field integral equation (EFIE) for a PEC surface,

$$E^i(\theta) = \frac{k\eta}{4} \int l H_0(kR) J_s(\theta') dl'. \tag{1.8}$$

To calculate an approximate solution for the surface current $J_s$, one divides the boundary into cells. The surface current in each cell can then be approximated as a superposition of sub-sectional basis functions. This method is called the method of moments (MoM). Spline functions can be used as basis functions. A zero order B-spline with $p = 0$ is a simple pulse function.
Higher order spline functions approximate the surface current more accurately, and will be discussed in section (1.4). The current in terms of $N$ splines is written,

$$J_s(\theta') = \sum_{j=1}^{N} c_j N_{j,p}(\theta'), \quad (1.10)$$

and one obtains,

$$E_i(\theta) = \frac{k\eta}{4} \sum_{j=1}^{N} c_j \int H_0(kR) N_{j,p}(\theta') dl'. \quad (1.11)$$

By performing a testing (collocation) at the middle of each cell on the boundary, one extracts a system of linear equations. The matrix form, in terms of the usually dense matrix $Z_{mj}$, is,

$$[E_i^t] = [Z_{mj}][c_j]. \quad (1.12)$$

This system of linear equations can be solved by means of Gaussian elimination or iterative techniques like the QMR. Typically one needs 10 basis functions per wavelength in order to get a good approximation for the surface current. This leads to a large number of unknowns and solving the system of linear equations with regular solvers will be very slow.

### 1.2 The Magnetic Field Integral Equation (MFIE)

If a wave, with the magnetic field in the $z$ direction, is incident on a PEC cylinder, the electric field is transverse (TE) to the $z$ direction. The magnetic field only is needed to calculate the surface current. The magnetic field has a vanishing normal derivative on a PEC boundary [11],

$$N_{j,0}(\theta') = \begin{cases} 
1 & \text{if } \theta' \in \text{cell } j \\
0 & \text{elsewhere}
\end{cases}. \quad (1.9)$$
\[ \frac{\partial H(\theta')}{\partial n'} = 0, \quad (1.13) \]

and the magnetic field equals the surface current \( H(\theta') = J_s(\theta') \). If the observation point \( \rho \) lies inside the scatterer, the extinction theorem [12] and equation (1.3) yields,

\[ H^i(\theta) = \frac{k}{4i} \int_l H_1(kR) \hat{R} \cdot \hat{n}' J_s(\theta') dl'. \quad (1.14) \]

\( H_1(kR) \) is the Hankel function of the first order and \( \hat{R} \) is a unit vector in the direction of \( \rho - \rho' \). This formula is called the Magnetic field integral equation (MFIE). In this case the induced surface current lies in the plane shown in Figure 1.1. A problem is that the singularity of the kernel at \( \theta = \theta' \) is stronger than for the TM case, and this complicates the quadrature. By approximating the surface current with a spline basis one obtains a linear system.

### 1.3 The Combined Field Integral Equation (CFIE)

Surface integral equations like the EFIE and the MFIE have internal resonance problems. At, or near, resonance frequencies, the system of linear equations will be ill-conditioned. A way to avoid this problem is to use a formulation known as the combined field integral equation. The CFIE is a linear combination of the EFIE and the MFIE [13]. The linear combination operator is given by,

\[ \mathcal{P} = \zeta + \frac{\partial}{\partial n}, \quad (1.15) \]

where \( \zeta \) is a complex constant. A suitable value for \( \zeta \) is given by the wave number of the incident field \( |\zeta| \approx k \). This type of integral equation has a unique solution and is stable for high frequencies. The CFIE for the TM and TE cases are given by:

\[ \mathcal{P} E^i(\theta) = \frac{k\eta}{4} \int_l \mathcal{P} H_0(kR) J_s(\theta') dl'. \quad (1.16) \]
and,

$$\mathcal{P} H^i(\theta) = \frac{k}{4i} \int \mathcal{P} H_1(kR) \hat{R} \cdot \hat{n}' J_\sigma(\theta') dl'. \quad (1.17)$$

### 1.4 Uniform and nonuniform B-splines

There are many types of basis functions that can be used in the method of moments [14], [5]. Higher order B-splines are smooth polynomials and have good approximation properties. The recursive form for B-splines of order $p$ can be written as [15]:

$$N_{j,p}(\theta) = \frac{\theta - \theta_j}{\theta_{j+p} - \theta_j} N_{j,p-1}(\theta) + \frac{\theta_{j+p+1} - \theta}{\theta_{j+p+1} - \theta_{j+1}} N_{j+1,p-1}(\theta). \quad (1.18)$$

If $p = 0$, one has a pulse function with support $[\theta_j, \theta_{j+1}]$; $\theta$ could also be complex. Zero order splines are easy to apply in the MoM for a closed surface, since there is no overlap between the basis functions at the end points. To obtain accurate results, we have used higher order B-splines for the surface current and the B-splines then connect near the end points of $[0, 2\pi]$.

Dividing the boundary into equi-distant $\theta$ intervals leads to uniform B-splines. For certain problems, one could use nonuniform B-splines. Nonuniform B-splines are obtained by modifying the knot vector that defines the intervals for the splines. An increased resolution may be needed for problematic regions, such as the deep shadow (standing wave pattern) or at edges on the scatterer.

### 1.5 Integration along a complex contour with a decaying kernel

According to Cauchy’s integral theorem, the integration in Equations (1.11) and (1.15) does not depend on the path of integration, provided that the integrand is analytic. Doing the integration along a real path leads to a dense matrix. One could find a suitable path in the complex plane and get essentially the same results, but with a sparse matrix [4]. It is not so easy to find a suitable path when the observation point $\rho$
lies in the shadow of the scatterer (paper I), since the integrand does not decay rapidly. The consequence is that the contour has to cross a number of basis functions in order to reach small values. There are some iterative methods for the steepest descent path [16], but they converge slowly. An approximate path is defined by means of line segments, as discussed in paper I.

1.6 Numerical quadrature

Quadrature refers to any method that approximates an integral. A quadrature with $n$ points includes nodes $\theta_j$ and weights $w_j$,

$$\int_{a}^{b} f(\theta) d\theta \approx \frac{b - a}{2} \sum_{j=1}^{n} w_j f\left(\frac{b - a}{2} \theta_j + \frac{b + a}{2}\right).$$ \hspace{1cm} (1.19)

There are many types of quadrature depending on the selection of nodes and weights: Gauss-Legendre, Gauss-Chebyshev and Gauss-Hermite. Gauss-Legendre quadrature is used in this thesis for an integrand that is both oscillatory and near singular along the path of integration. The nodes are given by the nulls of the Legendre polynomial $P_n$ and the weights are given by,

$$w_j = -\frac{2}{(1 - \theta_j^2)[P'_n(\theta_j)]^2}. \hspace{1cm} (1.20)$$

In order to evaluate highly oscillatory integrals accurately and efficiently, more advanced quadratures are of interest. Recent papers have presented methods that deal with this problem [17],[18]. The quadrature has not been the main focus in this thesis.

1.7 Direct and iterative solvers for linear systems

A system of linear equations can be solved with methods of two types:

- Direct methods (Gaussian elimination).
- Iterative methods (GMRES, QMR).
Direct methods find the solution after a fixed number of operations. Direct methods are stable and accurate and could be used for banded systems.

If the matrix is well conditioned, an approximate solution can be obtained with fast iterative methods, but preconditioners are normally needed. Iterative methods are suitable for sparse matrices. In paper I and II, Gaussian elimination is used to solve the linear system of equations in order to obtain high accuracy.

1.8 Summary of paper I

The paper explores the use of a priori information about the phase, as a way to reduce the number of basis functions, and the use of contour deformation as a means to obtain sparsity. A new ansatz that introduces scaling with amplitude variation in the shadow is shown to improve accuracy, especially for the TE case.

1.9 Summary of paper II

A hybrid approach to the scattering problem is presented. The lit region is treated as in paper I, while an analytic solution is used in the shadow. A problem that is solved fully, is the matching between the numerical solution and the analytic solution for the current in the shadow.
References


