WHEN MATHEMATICS TEACHERS FOCUS DISCUSSIONS ON SLOPE

– ***Swedish upper secondary teachers in a professional development initiative***

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Abstract

This thesis is the case of when mathematics teachers focus discussions on slope. The shift towards collegiality is a new setting for many teachers. Most teachers work alone, in isolation from their colleagues and collegial collaboration requires organisational structures. The aim of the study is to describe and analyse upper secondary mathematics teachers' collective practice, developed in a professional development initiative.

This study is a case study and the empirical data is generated through observations and an interview of a group of four teachers at a school who met on a weekly basis throughout a term. Their discussions focused on the mathematical concept of slope in a setting of learning study.

This study draws on Wenger's *Communities of Practice Perspective*, as a unit of analysis, and addresses the question: What are the characteristics of practice when upper secondary mathematics teachers focus discussions on slope in the setting of a learning study? The analysis accounts for characteristics of the aspects of practice, through the coherence of *mutual engagement, joint enterprise* and *shared repertoire* in the *community of practice*.

The teachers are engaged around finding small changes in their teaching that could give major effect in students learning. They negotiate what the students need to know in order to understand the relation between Δy and Δx. The characteristic of practice is a conceptual mapping of the concept of slope. It reveals students' partial understanding of related concepts due to how they were given meaning through previous teaching. The conceptual mapping of slope goes back as far as to the student's partial understanding of the meaning of subtraction. However, what emerges is in relation to the teachers' experience of avoiding students' difficulties with negative difference when teaching slope. It turns out to be a negotiation and a renegotiation of teaching slope for instrumental understanding or conceptual understanding. An overall characteristic of practice is that it develops in a present teaching culture.

Key-elements: teacher professional development, upper secondary mathematics teachers, collegiality, slope, rate of change, teaching culture, community of practice, learning study

Acknowledgements

I am an upper secondary mathematics and physics teacher myself and I have worked alone throughout most of my career. This turned out to be a starting point for applying to the school Developing Mathematics Education in 2011. I was eager for a setting in which to place a community of upper secondary mathematics teachers to reflect on teaching and learning mathematics. Conducting this research I have had the opportunity to become aware of my teaching culture, of my routines, that govern my classroom and I have come to question them. It has exerted a deep impact on me and I have several people to be grateful to.

I would like to thank my head teacher Lisbeth Friman, who approved my partial leave from school to join the DME school for two years.

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Contents

1. INTRODUCTION AND AIM

The present trend aimed at improvements in schools is through collegial collaboration. Today "communities of practice" fill the air.

> The shifts in worldview are even more fundamental than the now-historical shift from behaviourist to cognitive views of learning. (Putnam & Borko, 2000, p.4)

The shift towards collegiality is a new setting for many teachers. Teachers in secondary education primarily feel responsibility for their own classroom practice, resulting in largely autonomous and isolated work and private learning activities. Most teachers teach separate classes behind closed doors and learn about teaching by teaching (Hodkinson & Hodkinson, 2004). Working alone, the teacher may both face disengagement and boredom, on one hand, on the other freedom and privacy (Wenger, 1998). Working alone over time, every teacher develops a unique instructional repertoire, a set of personal, artful, assumptions and responses. This also means that technical communication among teachers is more difficult, since two people can teach the exact same curriculum to similar students on vastly different assumptions that are hard to explain, let alone bridge (Evans, 2012):

> The entrenched norms that prevail among teachers have always been those of autonomy and privacy, not those of "open exchange, cooperation, and growth." Trying to overcome deep-rooted norms through new collegial structures would be difficult in the best of conditions, but few schools have created these. Thus, the Professional Learning Communities begin life as "more work," rather than as "growth opportunity ". (Evans, 2012, p.2)

Many reformers who thought increased planning time, by itself, would lead to improvements in teaching have found it does not. Teachers who are told to collaborate have often found that they are not sure what they are supposed to do, or how such collaboration can help them to improve their teaching (Stiegler & Hiebert, 1999).

> "Let's just go home early and use the time at home to prepare for tomorrows lessons" was a comment from a teacher in a school district, where time for collaboration was organized for them. The teachers were complaining about the time they had to spend meeting together. (Stiegler & Hiebert, 1999, p.149)

The discussions in many staff development sessions are characterised as "style shows" that provided few opportunities for meaningful reflection and growth (Ball, 1994):

> The common view that "each teacher has to find his or her own style" is a direct result of working within a discourse of practice that maintains the individualism and isolation of teaching. This individualism not only makes it difficult to develop any sense of common standard, it also makes it difficult to disagree. Masking disagreements hides the individual struggles to practice wisely, and so removes an opportunity to learning. Politely refraining from critique and challenge, teachers have no forum for debating and improving their understanding. (Ball, 1994, p.16)

Collegiality is the least common form of relationship among adults in schools, even though it seems both obvious and compelling. There is rather a mutual supportiveness, which is about getting along well and being friendly. This provides an essential base for a faculty's sense of community and its work with students. What it does not provide however, is any meaningful attention to that work or to a culture of growth. Collegiality is de-privatising the work of teaching, and it means being able to disagree constructively about professional practice (Evans, 2012). A challenge for collegiality among teachers is that teachers are profoundly conflict avoidant; critical interactions may be against the personal and experiential nature of the teaching profession (Evans, 2012; Labaree, 2003). The focus of developing productive collaborations within projects must be for mathematics teachers to begin to engage critically with issues of practice (Males, Otten & Herbel-Eisenmann, 2010). It is more than simply sharing ideas, it means confronting traditional practice – the teacher's own and that of his or her colleagues (Lord, 1994).

In order to be successful in finding a new role for teachers they need to participate in a professional community that supports the risk taking and struggle entailed in transforming practice. When diverse groups of teachers with different competences come together, they can create rich discourse communities with deep new insights into teaching and learning. However in many schools the existing discourse community do not value critical and reflective examination of teaching practice (Putnam & Borko, 2000).

In summary; collegiality is a new setting for many teachers; collegiality requires structure that goes beyond more than simply sharing ideas, that sustains the individualism and isolation of teaching and collegiality requires de-privatizing of the work of teaching and for teachers to begin to engage critically with issues of practice.

This underpins my experience. I am an upper secondary mathematics teacher myself and I have worked alone for most of my career. I have faced freedom to develop my own teacher practice, however I have struggled too many times. The logical action would be to turn to collegiality, but I have also too many times found collegial collaboration as an obstacle. The collegial opportunities have at the most been processes of "witness" of teaching experiences. They have not resulted in any deeper reflections on learning and teaching mathematics.

The 2011 Swedish Education reform underpins collegiality in order to implement reinforced requirements of scientific basis and proven experience (National Agency for Education, 2013). Hence it requires opportunities for teachers to develop knowledge of content and methods as proven experience is to be systematically reviewed and documented.

The aim of this thesis is to describe and analyse upper secondary mathematics teachers' collective practice developed in a teacher professional development initiative.

Professional development, based on collegiality, taking place among upper secondary mathematics teachers did not give me a wide selection to choose from. A learning study turned out to give access to empirical data to examine upper secondary mathematics teachers' collective practice. The teacher professional development of learning study is a version of the Japanese lesson study and has taken place in Swedish schools since 2003.

This is a case study *about* four upper secondary mathematics teachers' collective practice in a setting of learning study; it is not a study *within* learning study. Four teachers at a school have met weekly throughout a term in a setting of learning study. Their discussions at the meetings focused on the mathematical concept of slope. This thesis is the case of when mathematics teachers focus discussions on slope.

The **research question** is formulated in the following manner:

What are the characteristics of practice when upper secondary mathematics teachers focus discussions on slope in a setting of a learning study?

How to read this

In Chapter 1 the INTRODUCTION AND AIM of the study is given and the research question is formulated.

In Chapter 2 the BACKGROUND is elaborated. I first review Swedish upper secondary mathematics teachers' practice, addressing teaching culture. Then follows a review of the mathematical concept of slope. Finally collegiality is accounted for and the professional development of learning study is described and discussed.

In Chapter 3 the THEORETICAL FRAMEWORK is presented as well as a description on how it will be used in the analysis.

In Chapter 4 the METHODOLOGY is considered and elaborated. First an account of the case study is given. The selection of the case and the generation of empirical data of the case are then discussed respectively. The approach to theory and empirical data and the analysis is then described. Finally the trustworthiness of the case study is scrutinised.

In Chapter 5, THE CASE OF WHEN MATHEMATICS TEACHERS FOCUS DISCUSSIONS ON SLOPE is presented. That is the empirical data analysed through the theoretical framework presented in the chronological order of the case.

In Chapter 6 the CONCLUSION AND DISCUSSION of the thesis is given. The research question will be answered and the theoretical contributions of the thesis will be discussed. Finally future research will be addressed.

Terms used

The teachers of the case will be referred to as *the teachers*. Hence *teachers* will be teachers in general.

Upper secondary school is year 10-12 (age 17-19).

Upper secondary mathematics teacher – a teacher educated to teach mathematics in upper secondary school in Sweden.

2. BACKGROUND

This chapter will provide the background of the thesis. Four upper secondary mathematics teachers, focusing discussions on slope constitutes the case of the thesis. Their practice in a setting of learning study will be analysed. First mathematics teaching in Swedish upper secondary school will be described. From that teaching culture will be addressed, as the concept of slope and previous research on slope will be reviewed. I will also review collegiality and the context of the study, the setting of learning study, in particular.

2.1 Swedish perspective

The Swedish upper secondary school was reformed 2011 and a new curriculum was formulated. It emphasises that teachers should cooperate with other teachers in order to achieve the education goals (National Agency for Education, 2013).

The Swedish upper secondary curriculum defines the subject of mathematics as a tool in science and different professions, but ultimately to be about discovering patterns and formulating general relationships. The curriculum states teaching in mathematics should give students the opportunity to develop their ability for mathematical working. This involves developing an understanding of mathematical concepts and methods, as well as different strategies for solving mathematical problems. The curriculum also emphasises that teaching should cover a variety of working forms and methods of working, in which investigative activities form a part (National Agency for Education, 2012). The curriculum defines seven abilities for mathematical working, to strive for in mathematics teaching:

- Use and describe the meaning of mathematical concepts and their interrelationships.
- Manage procedures and solve tasks of a standard nature with and without tools.

- Formulate, analyse and solve mathematical problems, and assess selected strategies, methods and results.
- Interpret a realistic situation and design a mathematical model, as well as use and assess the model's properties and limitations.
- Follow, apply and assess mathematical reasoning.
- Communicate mathematical thinking orally, in writing and in action.
- Relate mathematics to its importance and use in other subjects, in a professional, social and historical context (National Agency for Education, 2012, p.1).

In upper secondary school the subject of mathematics is divided in courses, into five levels (1-5), and teaching in one course should cover a specified core content. National tests are given for each course, and are compulsory to use for assessing the students' final grade (A-F).

Lundin (2008) has analysed Swedish school mathematics from a historical perspective and describes its vision as students engaging in complicated mathematical problems, carefully designed by the teacher. This vision emphasises the reasoning between students and the teacher.

Even so, research shows that Swedish school mathematics is, largely, synonymous with solving of exercises, and has remained so over time. The teacher going through a few examples on the whiteboard, while the students listen is the form that dominates this teaching. Similar exercises in the textbook will then follow, and the remaining time of the lesson is spent for individual work in the textbook. The textbooks used maintain this practice, as they contain large numbers of exercises. The exercises are solved with a specific method, and have a correct answer. This teaching reinforce that students primarily learn what seems useful in order to solve routine and non-reflected arithmetical problems. In turn it generates a mathematical discourse of numbers in calculations and the content is defined by the exercises in the textbook. This criticism against this teaching is not towards the textbook, rather to how the textbooks are used (Lundin, 2008; National Agency for Education, 2000a, 2000b).

This also coheres with the Swedish way of assessing students' mathematical knowledge; with tests with just like the exercises the student spends time with in lessons. According to Lundin (2008) the fact that students are regularly assessed with traditional tests explains the manner of teaching. Research also shows that it is the teacher-made tests that assess more computational skills, as the National tests focus on testing students conceptual understanding, and hardly require any computational skills (Boesen, 2006).

Wyndhamn, Riesbeck and Schoultz (2000) have studied the Swedish mathematical curriculums[1] and the roles of problem solving in mathematics

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[1] Lgr 62; Lgr 69; Lgr 80; Lpo 94

teaching. In the earlier curriculums an assumption seems to be that mathematics is learnt for solving problems. That the mastering of mathematical techniques will lead to competence in solving problems. From that the following curriculum includes problem solving as an explicit topic to be taught. In the later and the present curriculum rather than learning about problem solving, mathematics should be taught through problem solving.

Wenger (1998) writes that practice develops in historical, social and cultural contexts that give structure and meaning to what we do. Consequently an account for mathematics teaching addressing teaching culture will be given next. To get a wider perspective on the Swedish perspective, differences in mathematics teaching will be reviewed.

2.2 Teaching culture

In mathematics education research it seem to be a qualitative difference between on the one side a shallow, arithmetic or instrumental understanding, and on the other side a deeper, structural and relational understanding. In the first, mathematical concepts are primarily thought of as algorithms and procedures and the mastery of these is the aim, in the second the processes and mastery is regarded as a deeper relational understanding and the concept will be treated as a whole. Skemp (1976) states that relational understanding is better as it is easier to remember. There is more to learn, the connections as well as the separate rules, but the result, once learned, is more lasting. Even so, Skemp (1976) writes there can be advantages for teachers to teach for instrumental understanding only:

1. Within its own context, instrumental mathematics is usually easier to understand; sometimes much easier. Some topics, such as multiplying two negative numbers together, or dividing by a fractional number, are difficult to understand relationally. […] If what is wanted is a page of right answers, instrumental mathematics can provide this more quickly and easily.
2. So the rewards are more immediate, and more apparent. It is nice to get a page of right answers, and we must not underrate the importance of the feeling of success, which pupils get from this. […] These children need success to restore their self-confidence, and it can be argued that they can achieve this more quickly and easily in instrumental mathematics than in relational.
3. Just because less knowledge is involved, one can often get the right answer more quickly and reliably by instrumental thinking than relational. This difference, is so marked that even relational mathematicians often use instrumental thinking. [...](Skemp 1976, p.8)

Skemp (1976) also discusses the reasons for relational understanding taking too long to achieve, and that the ability to use a particular technique is all that these students are likely to need. That relational understanding of a particular topic is too difficult, but the students still need it for examination reasons. Or that a skill is needed for use in another subject (e.g. science) before it can be understood relationally. He also gives another reason teaching for instrumental understanding, i.e. when a new teacher arrives at a school where it is the practice. Assessment of whether a person understands relationally or instrumentally, is also more difficult.

Stiegler and Hiebert (1999) have written about the *Teaching gap*, referring to the gap between different teaching cultures. They have identified the high achieving Japanese teaching culture based on conceptual understanding in contrast to the U.S. teaching culture based on instrumental understanding.

> If one believes that mathematics is mostly a set of procedures and the goal is to help students become proficient executors of the procedures, then it would be understandable to believe that; mathematics is learned by mastering the material incrementally, piece by piece. Thus learning procedures occurs by practicing them many times, with later exercises becoming slightly more difficult than earlier ones. Practice should be relative error-free, with high levels of success at each point. Confusion and frustration, should be minimized, they are signs that earlier material was not mastered. (Stiegler & Hiebert, 1999, p.90)

Even though they are concerned with the U.S. teaching culture, their comparison is interesting as it sheds light on teaching mathematics for instrumental understanding. If it resembles research on Swedish teaching will be further examined.

Stiegler and Hiebert (1999) state that the role of the teacher will follow his/her assumption of the nature of learning. Teaching for instrumental understanding is then to be responsible for shaping the task into pieces that are manageable for most students. The role is to provide all the information necessary to complete the task and assigning plenty of practice. A lesson takes the form of following the teacher's directions by practicing a procedure during seat-work. Activities are more modular, with fewer connections between them, and it is not necessarily that the students are practicing the same skills in their individual work in the textbook as those the teacher presented at the beginning of the lesson.

Research shows the textbooks use of the 'explanation–example–exercises' format, dominates both the perceptions and the practices of school mathematics (Love & Pimm, 1996).

The teacher also believes his/her responsibility is to keep students engaged and attending. Moment to moment attention is fundamental, if their attention wanders they will get lost when they try to practice on their own later. Often an overhead projector is used instead of a white-board[2]. This as the teachers can turn off and on the overhead trying to keep and to control the students' attention. The teacher acts as if confusion and frustration are signs of them not succeeding at their jobs and provide quick assistance for students to

[2] An overhead projector might not be used today, as it has been replaced by modern technique.

get the back on track. Teaching in this culture is also about enhancing students' interest by increasing the pace of the activities, by praising students for their work and behaviour, by the cuteness or real-lifeness of tasks and by their own power of persuasion through enthusiasm, humour and "coolness". This teaching concerns the non-mathematical (Stiegler & Hiebert, 1999).

In contrast, mathematics teaching based on conceptual understanding, is exemplified through the Japanese teaching culture as a discourse of structured problem solving. Stiegler and Hiebert (1999) stress that there are ways other than the Japanese to teach effectively[3]. There must be opportunities for the student to learn the concepts. The importance of both facilitating students' conceptual understanding and procedural fluency is corroborated by independent research. I will still draw on the Japanese teaching culture by elaborating on the contrast, and also as this case is somehow related to the Japanese culture through the setting of learning study.

In Japan the role as a teacher is to be responsible for choosing a challenging problem to begin the lesson and, while the students are solving the problem, to monitor their solution methods in order to organise the follow up discussion when students share solutions. The teachers lead class discussions, ask questions about methods, point out features of different methods and present their own methods. The Japanese teaching culture is reinforced by that learning occurs by first letting the students struggle to solve mathematical problems, then participating in discussions of how to solve them, and then hearing about the pros and cons of different methods and the relationship between them. The teachers are not concerned about motivating the topics in a nonmathematical manner, they act as if mathematics is inherently interesting. As learning mathematics is believed to mean constructing relationships between facts, it is more important for the student to go back and think again about earlier events, and to see connections between different parts of the lesson than it is to pay attention each moment of the lesson. Further individual differences are viewed as the natural characteristics of a group, and the teachers regards them as a resource for both students and teachers. It is beneficial for the class because they produce a range of ideas and solution methods that provide material for the discussion and reflection. Japanese teachers plan lessons by using the information that they and other teachers have previously recorded about students' likely responses to particular problems and questions. If the group is sufficiently

[3] In TIMSS 1999 they included more high achieving countries (Hong Kong, Netherlands, Czeck Republic) and they state that neither of them resembled the teaching in Japan. They conclude that there is not one way to teach effectively, however a key element for all high achieving cultures is that despite different approaches all accomplished the involvement of students in active struggle with core mathematical concepts and procedures. A striking difference in high achieving countries was how teachers used problems to teach concepts. In these countries half of the problems were used to emphasise relationships and the other half were changed so that student practised procedure or recalled information they had learned before (Stiegler & Hiebert, 1999).

large, they can be quite sure they can expect the same responses from these students. Hence they can then plan the nature of the discussions that are likely to occur. Not all students will be prepared to learn the same things from each lesson, but as the different methods are discussed each students will learn something. Finally, a mathematics lesson in Japan tells a story; it has beginning a midpoint and an end, tightly connected, as a coherent story. The chalkboard is used to keep a record over the lesson, thus nothing is erased. The disposition of the chalkboard in Japan includes a slot for presenting the accumulated lesson of the day; making connections looking for coherence in the content (Stiegler & Hiebert, 1999).

Ma (1999) has found a similar contrast in the performance of Chinese and U.S. teachers; the desire to make sure the students see mathematics as a coherent whole in contrast to seeing mathematics as a set of rules, with no relation to each other, for finding correct answers to important problems.

Stiegler and Hiebert (1999) stress teaching is a cultural activity, which explains why teaching is so resistant to change. The methods teachers use, are not determined by their qualifications as much as by the culture in which they teach. They have been examining across cultures and *the teacher gap* has become visible; it is a gap between different cultures, a gap in general methods, it is not a gap in competence. They continue and say that changing culture is very difficult since cultural activities are embedded in a wider culture.

> Teaching can only change the way cultures change: gradually, steadily, over time as small changes are made. (Stiegler & Hiebert, 1999, p.13)

A teaching system is composed of elements that interact and reinforce one another. The whole is greater than the sum of the parts. Improving teaching by changing individual elements is impossible, since in a system all parts reinforce each other. If one feature is changed, the system rushes to repair the damage, perhaps by modifying the new feature so it functions the way the old one did. This point is often missed in attempts to reform teaching. It is exemplified by the story of a U.S. teacher that did change his teaching and introduced problem solving in a class, after been watching a Japanese lesson. Although he changed, the students did not and they failed to respond to the lesson, since they had not reflected on the Japanese lesson as the teacher had. The students played their traditional roles, they waited to be shown how to solve the problem. The students are part of the system, and systems of teaching are much more than what the teacher does (Stiegler & Hiebert, 1999).

Nuthall (2002) summarises a lifetime (40 years) of research with that he finally came to realise how culture shapes our understanding of both the teaching and learning process. The most significant aspect of culture is that it becomes so much a part of ourselves that we lose awareness of how it organises our lives. The more familiar it is the harder it becomes to identify how it shapes what we believe and what we do. He says:

> What we do in schools is a matter of cultural tradition rather than evidence-based practice (Nuthall, 2002, p.6)

Nuthall (2002) underpins this referring to his previous research that concludes that underlying patterns of teaching appeared to be independent of training and experience. Being an experienced teacher apparently made no difference on what their student learned. He says it also became apparent as in what teachers do is reflected in what the students then talks about and concludes that teaching has such a long history and has such a powerful hold over us and the system around teaching sustains and promotes it. In summary; patterns in teaching have been resistant to change, since schools have served as a powerful discourse communities that enculturate their participants in traditional teaching (Stiegler & Hiebert 1999; Nuthall, 2002).

Past work indicates that even experienced mathematics teachers are relatively unaware of their discourse patterns (Herbel-Eisenmann, 2000). According to Stiegler and Hiebert (1999) teachers must become aware of the cultural routines that govern classroom life and question the assumptions that underlie these routines. However, these views are often so fully integrated into teachers' worldview and the less likely they are to be questioned or even noticed.

What teaching culture means in relation to collegiality will be discussed later, first focus will be on slope.

2.3 Focusing on slope

The steepness of a line can refer to a visual perception of the graph. The word slope (gradient, incline, pitch) is used to describe the steepness of a straight line. The rate of change of a function is the ratio of change in the dependent variable to the change in the independent variable. The slope of a line is a rate of change. Thus, slope has meaning in formulae, tables, physical situations and verbal descriptions. Slope is a deep mathematical concept closely tied to the notion of function. Calculus begins with the study of derivatives and rates of change, using slopes of lines to develop these concepts. A conceptual understanding of slope is especially crucial for the study of calculus. Also physics assumes the ability of students to interpret slope, but as a functional relationship between two quantities. Linear functions often take the form: $y = mx + b$ or $ax + by = c$ when represented algebraically. Slope is calculated by finding the ratio of the "vertical change" to the "horizontal change" between (any) two points on a line (Figure 1). The slope m of the line is $m = (y_2 - y_1) /$

$(x_2 - x_1)$. Through trigonometry, m is related to its angle of incline θ by the tangent function $m = \tan \theta$ (Clapham & Nicholson, 2009).

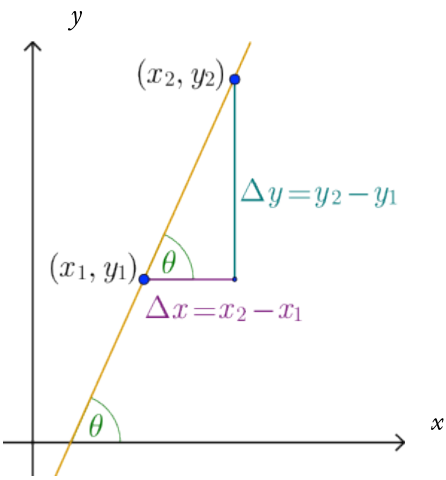

Figure 1 *Calculating slope by finding the ratio of the vertical change over horizontal change between to points on a line.*

In Sweden the students are taught the concept of function as a way of inquiring change and rate of change among other correlations. As students reach upper secondary school, slope is introduced closely to the concept of function, as a property of linear function (National Agency for Education, 2012).

A linear function is known as $y = kx+m$, thus $k = (y_2 - y_1)/(x_2 - x_1)$. The teachers focus discussions of k, as the slope of a straight line, intending the students to understand the relation between Δy and Δx in the algorithm $\Delta y / \Delta x = (y_2 - y_1)/(x_2 - x_1)$. In Swedish upper secondary school, textbooks frequently define the slope of a line as the ratio of rise in y to the run in x as you move from one point to another along the line. The model of a staircase is used to make meaning of the slope as the rate of change.

> A line leaning upwards has a positive slope. A line leaning downwards has in the opposite a negative slope. The slope of a staircase is recognised from both the height and the length of the step. The slope of a staircase can be given by a value: slope = the height of the step / the length of the step. In the same way a value can be given to represent the slope of a line. That value is called the gradient. The gradient of a line = the change in *y*-led / the change in *x*-led. Taking any point on a line with the gradient of 3; If you go one step to the right in x-led, you will have to go three steps up in y-led to reach the line again. (Szabo, Larsson, Viklund & Marklund, 2007, p.224)

Researchers have found that students at various levels have difficulty conceptualising the idea of rate of change (Thompson 1994c; Thompson & Thompson, 1994, Ubuz 2007; Wilhelm & Confrey 2003). These researchers have all noted that students are typically introduced to rate of change through the slope formula. Further, students tend to practice inputting numbers and calculating the slope of a line with little or no focus on interpreting the meaning of the result within a given context and with little consideration for units of measure. Generally, students are introduced first to slope and later to rate of change; and maybe the students are not making the connection between these two concepts.

Further Teuscher and Reys (2010) write to quantify steepness, students tended to calculate the slope of the two line segments and compare the values, disregarding the sign. In contrast, the slope and rate of change are values that quantify the relationship between the independent and the dependent variables. The sign of either the slope or the rate of change provides information about this relationship. For example, if the sign of the rate of change is negative, then as the independent variable increases, the dependent variable decreases. The ratio of the change in the output value and change in the input value of a function is called as rate of change. The meaning of steepness is different from the meaning of rate of change.

Students in the U.S. are taught the phrase "rise-over-run" as a mnemonic for the algorithm "change in y, over change in x", for calculating slope. The implication of this teaching, as the result of this instructional device, is an instrumental understanding of slope as a fraction, with the denominator as the change in x and the nominator as the change in y (Walter & Gerson, 2007). This is underpinned by Skemp (1976) however this teaching does not give understanding of the meaning of the quotient for how the line is positioned in the plane, neither does it make connections to rate of change. Previous research has found that students have a difficult time reasoning with rates as quantities rather than as the result of a computational process (Stump, 2001).

Stump (1999) writes that teachers during their years of experience with students have witnessed students' difficulties with symbolic interpretation and manipulation when teaching slope. All the teachers in her study mentioned the students' confusion between rise and run or between x and y in the formula for slope:

> Ms. B: Well, if they don't know the formula, one main thing, they put the x's over the y's. They do that all the time.
>
> Mr. C: Reversing the numerator and the denominator. In terms of the definition, change in y over change in x .... And I would say the order in, which they subtract them. It takes a little bit of time to get used to the idea that you can do it either way as long as we do it in the same order. (Stump, 1999, p.139)

Another challenge, as an important mathematical precondition for the concept of slope is proportional reasoning (Lamon, 1993). Rates involving time are the most intuitive for the student, but further abstraction is required to develop an image of rate that entails the covariation of two non-temporal quantities (Thompson, 1994).

The study by Stump (1999) plays a considerable role in this thesis as she examines secondary mathematics teachers' knowledge of teaching slope. She summarises what pre-service and in-service teachers think students must have experience of before they can truly understand slope: the Cartesian coordinate system, plotting points and graphing were among the geometric concepts and skills identified by the teachers. Variables, formulae, and solving equations were among the algebraic concepts and skills mentioned. Graphing linear equations was categorised as both geometric and algebraic. The teachers listed arithmetic concepts such as fractions, addition, and subtraction. In-service teachers additionally included the following arithmetic concepts: percentages, positive and negative numbers, and division involving zero.

Further she provides a summary of the 'analogies, illustrations, examples, or explanations, teachers think are most useful or helpful to teaching the concept of slope: teachers mentioned standard formulae and graphs, but they also mentioned real-world situations involving slope. These situations fell into two categories, referred to as physical and functional. Physical situations focused on the steepness of physical objects such as mountain roads, ski slopes, or wheelchair ramps. Functional situations emphasised the linear relationship between two varying quantities such as distance and time. The teachers were more likely to mention physical situations than functional situations.

The aim of Stump's study is to ascertain teachers' readiness to implement recommendations concerning a curriculum that focuses on concepts, connections, and functions and moves away from a focus on manipulative facility. The teachers expressed concern with students understanding of the meaning of slope:

> D: I think seeing the big picture.... I don't think it's difficult to find m. I think it's a really basic- it's basic arithmetic. But I think actually being able to go further with slope, seeing that it does mean something.
>
> F: I think a lot of times students can maybe calculate it but they don't know what it really means, what they are calculating.
>
> […]
>
> Mr.C : [I want students to know] what it means and how it would apply in a mathematical sense, to solve common problems or at least to relate to common problems.
>
> Mr.G: Just in general, what does it mean? When you say this line has a slope of 3/2 what does that mean? What does the 3 mean? What does the 2 mean?
>
> (Stump, 1999, p. 140)

In other research it has been observed that an 'experienced high school teacher's rich conceptions of functions contributed to his skilful implementation of a reform curriculum' (Lloyd & Wilson, 1998). Stump (1999) also reviews teachers' knowledge of slope and notices that many teachers did not demonstrate understanding of the notions of increase and rate of increase.

Stump (2001) found that the students demonstrated a better understanding of slope as a measure of rate of change than as a measure of steepness. The students in this investigation also exhibited a limited understanding of slope as a measure of steepness. They had trouble considering slope as a ratio. This study suggested a gap in students' understanding of slope as a measure of rate of change and implied that instruction should be focused on helping students form connections among rates involving time, rates involving other variables, and graphical representations of these relationships.

Pang (2008) illustrates an introduction to the slope of a straight line as part of Hong Kong mathematics curriculum. The study has the source of a learning study, focusing in the development the students' capabilities to understand the mathematical concept of slope. The Hong Kong study identified four common misconceptions of the students understanding of slope:

> Some students think the formula to find the slope is $(y_2 - y_1)/(x_1 - x_2)$, $(y_1 - y_2)/(x_2 - x_1)$, or $(x_2 - x_1)/(y_2 - y_1)$
>
> Some students perceive slope to be the angle of inclination with the x-axis.
>
> If two straight lines are parallel in the Cartesian coordinate system, then some students think that the longer one will have a steeper slope than the shorter one.
>
> Some students conceive that straight lines with greater values of negative slope are steeper. (Pang, 2008, p.6)

The major difficulties that the students had faced were; weak at manipulation of slope and in particular negative slope, the meaning of slope, solving mathematical problems in which the angle of inclination is larger than 90° and the relationship between $(y_2 - y_1)/(x_2 - x_1)$ and $\tan\theta$.

The Hong Kong study assesses that most students were not able to discern that a horizontal distance and vertical distance respectively are two of the critical aspects of understanding slope. Neither did the students have the preunderstanding of negative slope. Typical for the test made to assess the students' preunderstanding were that none of the questions included any formulae or numbers. In each case the students were asked to explain, rather than calculate. Each question included a figure based on two triangles, and the answer was in the comparison of the two triangles (Pang, 2008).

In this manner a background of slope is accounted for. The review is in terms of students' difficulties and misconceptions of slope. It addresses the manipulation of a formula to calculate slope and the concern for the students to understand the meaning of slope. This review will give perspective to the teachers' discussions on slope.

2.4 Collegiality

The way of working, in collegiality or in isolation, is also a part of culture (Stiegler & Hibert, 1999). In U.S., as in Sweden, the teachers work alone and collegiality is the least common form of relationship among adults in schools, even though it seems both obvious and compelling. There is rather a mutual supportiveness, which is about getting along well and being friendly. U.S. teaching is highly personal and over time, every teacher develops a unique instructional repertoire, a set of personal, artful assumptions and responses (Evans, 2012).

> Two people can teach the exact same curriculum to similar students on vastly different assumptions that are hard to explain, let alone bridge. This also means that technical communication among teachers is more difficult. (Evans, 2012, p.2)

Ball (1994) describes the many staff development sessions characterised as "style shows". She emphasises that the common view that each teacher has to "find his or her own style" is the result, but that it also maintains, the individualism and isolation of teaching. From this situation, the development of productive collaborations must focus on mathematics teachers beginning to engage critically with issues of practice (Males et. al, 2010). This is more than simply sharing ideas, it means confronting traditional practice – the teacher's own and that of his or her colleagues (Lord, 1994). In order to be successful in finding a new role for teachers they need to participate in a professional community that supports the risk taking and struggle entailed in transforming practice. When diverse groups of teachers with different competences come together, they can create rich discourse communities with deep new insights into teaching and learning. However in many schools the existing discourse community do not value critical or reflective examination of teaching practice (Putnam & Borko, 2000).

The Japanese teaching culture is an example of teachers working in collegiality.[4] The interest is in how this example can shed light on Swedish upper secondary mathematics teachers focusing discussions in collegiality.

In Japan self-reflection is a sign of competence, and colleagues' critical comments are a part of the Japanese culture (Lewis, 2002). *Kounaikenshuu* is the word used to describe the continuous process of school-based professional

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[4] It must not to be mistaken that the Japanese is the way of teaching to achieve success. Many factors differentiate Japan from the Swedish, and the teaching might not be the factor that explains the students' achievement.

development that teachers engage in once they begin their teaching careers. Japan makes no assumption that a teacher is competent once they have completed their teacher training programs. Participation in Kounaikenshuu is a considered part of teachers' work in Japan. One of the most common components of Kounaikenshuu is *jugyou kenkyuu* – lesson study[5]. The premise behind lesson study is simple; if you want to improve teaching, the most effective place to do so is in the context of a classroom lesson. If you start with the lesson, the problem of how to apply research findings in the classroom disappears. The improvements are devised within the classroom in the first place. The challenges becomes to identifying the kinds of changes that improve student learning in the classroom. So in lesson study, teachers spend hours planning a single lesson (Stiegler & Hiebert, 1999).

Yoshida (2004) says that Japanese mathematics teachers, value lesson study because they can come together to develop their pedagogical knowledge and skills, just as their own teaching can also be regarded from a realistic and grounded perspective in a lesson study:

> The conversations allows them to think about principles that could guide their everyday teaching of mathematics, and which they could then continue to experiment with and refine in their own class room. (Yoshida, 2004, p.223)

According to both Evans (2012) and Stiegler and Hiebert (1999), U.S. teachers find professional communities as "more work," and they would rather go home early to plan tomorrows lessons. And many reformers who thought increased planning time, by itself, would lead to improvements in teaching have found it does not. Yoshida (2004) writes that all work in lesson study is done after school. It is during afternoon hours[6] that most meetings are conducted, although they also often spill over into after-hours. Yoshida (2004) also gives insight into what they discuss for all those hours in lesson study. The teachers first engage in the problem from which the lesson will be launched as the Japanese mathematics lessons are based on problem solving[7]. Then the anticipated solutions, thoughts and responses that students might develop as they struggle with the problem will be explored. This is in relation to the kinds of questions that may be asked to enhance student thinking

[5] Lesson study was established in the 1960s and as the Japanese government saw the value they began to encourage schools to engage in practice. Small pockets of financial assistance were then created for lesson study. The activity has always remained voluntary, and in principle schools use this method because they choose to (Yoshida, 2004).

[6] Students in Japan finish schools between 2.40 and 3.45 p.m. Teachers are hired to work until 5 p.m and are expected to remain in the building for those hours (Yoshida, 2004).

[7] The problem solving appealed to Japanese educators because it emphasized the importance of knowledge and practice and promoted students active learning through solving problems encountered in everyday life. In the early 1990s the national recommendations were to improve children's ability to think deeply about mathematics problems. In the new textbooks that followed, the fostering of problem-solving skills was emphasized (Youshida, 2004).

during the lesson, as the type of guidance that could be given to students who show misconceptions in their thinking. The end of the lesson, the moment at which students understanding can be advanced, is carefully considered in the lesson study (Yoshida, 2004; Stiegler & Hiebert, 1999).

Yoshida (2004) reports that teachers engaging in lesson study gain in opportunities to discuss the content that they are called on to teach, and in doing so they will refine and deepen their understanding of that content. In addition, teachers can learn about how students tend to understand and approach the content. In Japan lessons are highly sharable among teachers. According to Stiegler and Hiebert (1999) the lesson is a part within a teaching culture.

Collegiality is a part (or not) of a teaching culture, as it relates to how teachers think about the nature of mathematics, how learning takes place and what a lesson is. In Japan collegiality focuses on mathematical content and the lessons are very shareable, as part of their teaching culture (Stiegler & Hiebert, 1999).

Stiegler and Hiebert (1999) were attracted to the notion of Japanese lesson study and imported it to the U.S. as they think it lays out a clear model for teacher learning that builds on collegial collaboration turning personal experience and knowledge of teaching into theory. Their aim was to change focus, from the teacher to teaching. Lewis (2009) argues that there is evidence that lesson study can be used effectively outside Japan. Her research is based on a U.S. case of mathematics teachers' learning in a lesson study setting. She reports changes in teachers' professional community:

> Motivation and capacity to improve instruction, including norms that emphasize inquiry and continuing improvement
>
> Sense of mutual accountability to provide high quality instruction
>
> Shared long-term goals for students
>
> Shared language, processes, and frameworks for analyzing instruction (Lewis, 2009, p.287)

In Hodkinson and Hodkinson's (2004) work on communities of practice and teachers' workplace learning, they conclude that a highly collaborative working culture is accompanied by a learning culture.

The case of this study is about four upper secondary mathematics teachers focusing discussions on slope in a setting of learning study. The review of collegiality, taking the lesson study into account, provides a deeper understanding of learning study, hence what emerges within the collective practice of the teachers. As the case concerns upper secondary mathematics teachers in particular, I searched in the literature for high school mathematics teachers and lesson study. Yoshida (2004) writes that most primary schools conduct lesson study, but rarely any high schools. In personal contact with Makoto Yoshida he writes there is no formal study that tells why high school mathematics teachers are not doing lesson study. However he thinks that reasons such as high school teachers being very busy with extra curricula such as coaching sports clubs might be a reason. He continues that these teachers are subject teachers so they think they are knowledgeable on their subject. And in addition the students need to pass severe and competitive entrance exams to gain entrance into colleges so instructions are more focused on acquiring knowledge and skills.

2.41 Learning study

Learning study is not very often conducted in upper secondary schools in Sweden neither. However, through a state governed financed initiative *Matematikssatsningen* 2009-2011, learning study has spread quite rapidly among primary and lower secondary school teachers. The evaluation states:

> Participating in a Learning study, the most valuable outcome of the process was the opportunity given to collegiality, together reflecting on how the content should be presented to the students in the classroom. (National Agency for Education, 2011, p. 32)

The beginning of the story, the background and the idea of learning study, is presented by Ference Marton:

> By destiny and a kind invitation I came to spend some years in South East Asia, the address was Hong Kong. There I got involved in a relative large research project, with the purpose to understand why some students in one class came to learn much more than students in another class, even though they were the same age and equally good. Their teachers seemed to treat the exact content with methods in the especially same way, being equally experienced and equally nice. We used some theoretical tools, earlier developed mainly in Gothenburg, and could conclude that the way in which the teachers treated the content, gave different opportunities to discern the content by the student. What happened with the content in the classroom made learning possible or impossible. In the following project we sought to use the gained insights of creating better learning by a more suitable treatment of the content. We took impression of collective and inquiry based forms of teacher professional development, common in that part of the world. We chose the Japanese lesson study as a model. (Maunula, 2011, p.11)

Learning study involves teachers and researchers working together to plan a research lesson. The lesson is taught by the teachers in one or several cycles, and is observed, evaluated, and modified by the team before the next cycle is taught (Marton & Lo, 2007). The difference between learning study and lesson study is that the former comes with a theory of learning. It is not prescribed, but most often a learning study draws on the theoretical assumptions of the variation theory[8], which is a theory about learning and experience described by Marton and Booth (1997), Marton and Tsui (2004), Runesson (2005) and Lo (2012). It is developed from an empirical basis, i.e. from the questions: why some people learn things better than others? and How you learn something effectively? The basic principle is that learning is always directed at something, and that learning must result in a qualitative change in the way of seeing this something (Marton & Booth, 1997). Another assumption of the variation theory is that in any learning situation the pre-understanding of the learner must always be ascertained, as a learner's misconception of a phenomenon may be a partial understanding:

> There are different degrees of partial understanding of the whole. Still answers can be totally wrong and the above does not imply that everything is a partial understanding of everything. If there exist different understandings of phenomenon we can judge the quality as which is best and worse, but it does not change the fact that there are different understandings. And that is the point of departure. (Marton, 1997, p.19)

The message from Marton (1997) is that learning must always launch from the learner's perspective, the experience of the learner. Runesson (2005) wishes to emphasise that variation theory focus on *what* to be learned. To distinguish variation theory from other theories, she says it may be regarded as a complement to the constructivist or the socio cultural perspectives on learning[9].

Previous research states that classroom instruction based on the assumptions of variation theory enhances student learning, it directs teachers to focus on content rather than method, the assumptions from the theory provide the teachers with an analytical framework for reflection of teaching and learning mathematics in the classroom, and the result is effective communication in the classroom (Pang & Marton, 2003; 2008; Olteanu & Olteanu 2011; 2012, 2013).

Desimone (2009) describes focus on content as a critical feature of professional development; content focus as an activity on subject matter content, and how students learn that content. Other critical features for intellectual and pedagogical change are; collective participation, active learning, duration and coherence. Learning study is accomplished through the

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[8] The variation theory has its roots in phenomenographic research, which accounts for how the same thing or the same situation can be seen, experienced or understood in a limited number of qualitatively different ways. It is a qualitative methodology within the interpretivist paradigm. It has the human experience as its object and the ontological assumptions are of subjectivist nature. There is only one world, experienced by humans in many different ways. The epistemological assumptions are that knowledge is a way of experiencing the world and when you gain knowledge you have gained a way of experience (Marton & Booth, 1997).

[9] However this review is not stating that the constructivist or the socio cultural perspectives on learning cannot focus on content.

active participation of teachers, as opposed to listening to a lecture. The implementation of the lesson is public and is most often video recorded in order to carry out a deeper analysis later. Runesson (2004) describes that teachers, once she or he has become used seeing him or herself teaching, also become aware of what is happening in a new way. Learning study also supports activities that are spread over a term and may include 20 hours or more of contact time, a critical feature according to Desimone (2009). Furthermore, the work should also be seen as consistent with teachers' knowledge and thoughts. Lord (1994) has identified a critical stance as necessary for transformation of practice. This review also discovers that teachers must become aware of the cultural routines that govern the classroom life, questioning the assumptions that underlie these routines (Stiegler & Hiebert, 1999).

Learning study is defined in relation to design experiments as:

> In design experiment, the theory is in the first place in the hands of the researchers themselves, as is the design. In a learning study, teachers are expected to use the theory as a tool and a resource, and to set up the design themselves. (Marton & Pang, 2006, p.196)

Stiegler and Hiebert (1999) give a metaphor regarding handling a new feature as a new tool. It is a story about recently-arrived immigrants who had mostly lived in tents or in very primitive housing. The story goes that the immigrants were not used to eating on tables, but now there was an intensive effort underway to convince them to do so. A family from Yemen then began to eat from the table. But the table was upside down with the top on the floor and the legs standing up. According to Putnam and Borko (2000) a tool can provide valuable opportunities for teachers to think in new ways, but its actual power is in relation to how teachers handle it.

The concepts of the theory used in learning study will also be defined, in order to understand what emerges in practice when teachers focus discussions on slope. Variation theory takes its starting point in the *object of learning*, which is a central concept, used to ground discussions. Seeing or experiencing an object of learning in a certain way requires the learner to be aware of certain aspects, and for the learner to be able to discern these aspects at the same time (Marton & Tsui, 2004). Since these aspects are critical to the intended way of seeing the object, they are entitled *critical aspects*. The object of learning and the critical aspect are two important concepts in the learning study.

Marton and Tsui (2004) then stress the importance of *variation* in order to be able to discern the critical aspects of an object of learning. Variation refers to variation in what is to be learned, which is variation in order to discern the aspects of the object of learning; to discern the critical aspects, their interrelation and how the aspects relate to the whole. This implies there must be different dimensions and different dimensions to consider when the content is varied. That is, *dimensions of variation*. These dimensions of variation are important as they give opportunities for effective learning. The content can be varied through; *contrast, generality, separation and fusion*. Contrast is when the content is presented in relation to what it is not. Generality is when the content is presented in different but similar ways. Separation is when the content is separated from a constant background. In fusion, all aspects of the object are varied simultaneously, hence also experienced simultaneously (Marton & Tsui, 2004; Marton & Pang, 2006). For instance, to understand a fraction, students need to be presented with a non-example of a fraction such as a whole number. To help the students to generalise the concept of ½ they need to be presented all kinds of examples that involve ½, such as half of a pizza, half an hour etc. In order to understand the relationship of the numerator to the value of a fraction, the denominator must be kept invariant as the numerator varies. The student's attention will be drawn to the numerator, which has then been separated from other critical aspects. To enable students to understand critical aspects of numerator and denominator in determining the value of the fraction, the numerator and the denominator must be varied at the same time, though systematically (Pang, 2008).

Previous research states that in many cases the object of learning is chosen to grasp mathematical content that is to extensive (National Agency for Education, 2011), but when they do not, Olteanu and Olteanu (2012) conclude that teachers improve their own knowing of meaning of all the parts of the object of learning by analysing the critical aspects of students learning. The teachers are than able to put these pieces of knowledge together into knowledge of the meaning of the object of learning by opening up dimensions of variation. Further, Holmqvist (2011) shows that teachers' theoretical insights appear to affect their ways of approaching the object of learning, such as changes in how to organize the critical features of the learning object. The results show the impact of contrasts in the pupils' learning outcome. When it comes to the concept of *variation*, it is not a general rule that the more variation in the mathematical content, the better. Critical aspects differ between different objects of learning, between different capabilities and they cannot only be reached from the nature of the subject (Runesson, 2006).

Both learning study and lesson study involves an advisor. The advisor in learning study must also be an expert on variation theory, as she or he will introduce it to the teachers. A report states that the quality of the learning study varies, but one aspect of quality is that the advisor can interpret the theory in relation to the mathematical content (National Agency for Education, 2011).

Ten years after learning study was brought to Sweden, Marton (in press) reviews its applications. He writes that learning study has been characterised as an arrangement for jointly choosing an object of learning and designing a lesson, by a group of 3-6 teachers, who are often teaching the same subject on the same level. The lesson is implemented by one of the teachers in his/her own class, and the students' understanding of the object of learning is assessed before and after the lesson. The lesson is then observed by the other teachers, and discussed afterwards. The design of the lesson is revised by the group and then taught by another teacher in his/her class, again observed by the other teachers. This second cycle is mostly repeated once more, and after the third cycle there is a final discussion. But Marton (in press) also reports on learning study which followed the model in several important respects, though not in every detail. In some cases more than one lesson has been used. In other examples there has been one cycle only and some cases one teacher might have planned the lesson on her own. He refers to these cases as "Learning study in a wider sense". According to Marton (in press), however these cases have some important things in common with learning study and he characterises "Learning study in a wider sense" as: they have a theoretical grounds, the teacher doing the teaching was involved in planning it and the students' understanding of the object of learning was explored somehow at the beginning and the end of the pedagogical encounter.

> "Learning study" thus refers to a particular model for teachers' in-service training and research. "Learning study in a wider sense" refers to studies of the relationship between learning and the conditions of learning created by teachers, on their own or together with others. (Marton, in press)

Summary

This chapter presents the background of the thesis. First an account of mathematics teaching in upper secondary school in Sweden was given. Then teaching culture was elaborated drawing heavily on the work of Stiegler and Hiebert (1999). Then slope and previous research on teaching slope was reviewed. After discussing the shift towards collegiality, learning study was addressed.

In the following chapter the theoretical framework will be presented and justified.

3. THEORETICAL FRAMEWORK

In this chapter I will present the theoretical framework that is used to analyse the characteristics of practice when upper secondary mathematics teachers focus discussions on slope in a setting of learning study. First I will address teachers in collegiality and an attempt will be made to justify Communities of practice. From that the concepts of Communities of practice that will frame the case will be described.

3.1 Addressing teachers in collegiality

There are many different kinds of learning theory. Each focuses on different aspects of learning, and thus each is useful for different purposes. Borko (2004) writes that it is a challenge to identify and measure teacher learning. To understand teacher learning, as it takes place in many different aspects of practice, we must take into account both the individual teacher and the social system in which they are participants.

This thesis addresses the teachers' practice emerging in collegiality, in a learning study.

A learning study involves teachers and researchers, and research in learning study most often analyses student and teacher learning within the phenomenographic paradigm. Variation theory is then used as an analytical tool. This thesis is not research *into* learning study, and I am not the researcher planning the lesson with the teachers. Still variation theory can be used to analyse the critical aspects addressing the teachers' understanding of teaching and learning slope. Nevertheless, the framework does not enable us to embrace the larger context of how teachers are influenced by teaching culture.

Desimone's (2009) critical features for intellectual and pedagogical change form the basis of a framework for studying the effectiveness of teacher professional development. Although teacher change in terms of teaching culture is elaborated, the focus is not on how effective learning study is.

Lord (1994) has developed a framework to examine how mathematics teachers took a more critical stance toward their own teaching practice and that of their colleagues. This thesis situates teachers at the centre of the process, and launches and takes into account critical colleagueship. However the focus is not to analyse how teachers take a critical stance towards their practice.

The framework of this case must enable; a focus on what emerges as mathematics teachers focus discussions on slope; the teachers' experience of teaching and learning slope must be viewed; how the teachers are influenced by the teaching culture, but also how they influence the collective practice. Lave and Wenger (1991) developed the social practice theory that describes how individuals in different situation shapes and are shaped by the cultural atmosphere around them. They focus on participation and describe learning as a trajectory in *communities of practice*.

3.2 Communities of practice

Within the paradigm of social practice theory, Wenger (1998) conceptualised communities of practice as a social theory of learning.

> Practice is doing in historical and social context that gives structure and meaning to what we do. […] In this sense, practice is always social practice. (Wenger, 1998, p.47)

Wenger (1998) writes that communities of practice are groups of people who share a concern or a passion for something they do and learn how to do it better as they interact regularly. A community of practice has a shared domain of interest. Membership therefore entails a commitment to the domain and a shared competence distinguishes members from other people. In pursuing their interest in their domain, members engage in joint activities and discussions and share information. They develop a shared repertoire of resources: experiences, stories, tools, ways of addressing recurring problems. Communities of practice have been around for as long as human beings have learned together. At home, at school, at work, in our hobbies, we belong to communities of practice. In some we are core members, in many we are merely peripheral. And we travel through numerous communities over the course of our lives.

The framework is not about whether the practice is right or not. It is about the active involvement and how it takes place; what is brought to the table in a community of practice (Wenger, 1998). It is an appropriate unit of analysis to frame the group of teachers as a community of practice. The teachers in the case are an active part of their community but at the same time they are influenced by a teaching culture. The teachers are accountable to the quality of the community of the practice. Their experience of teaching and learning about slope will be negotiated in the community of practice and validated as competences. The tension between competences and experience is very important for the dynamic in a community of practice. Newcomers can be carriers of innovation, and can facilitate negotiation and renegotiation of competences. When the core is too strong there is a lack of tension between competences and experience and the community of practice may become static and stand in the way of learning (Wenger, 1998). The advisor in the learning study, as an expert on variation theory, will thus play a special role in the community of practice. Variation theory is as tool for the teachers. Tools may serve to mediate in social practice, stabilise human practice, coordinate and discipline human reasoning by suggesting how to do things (Säljö, 2000)[10]. A tool may also facilitate discontinuities or continuities in a community of practice. *In this study I regard the variation theory as an artefact, an intellectual tool.*

Wenger (1998) emphasis that practice drives the process in a community of practice, still the design of the method or activity is important. The learning study, its activities, can facilitate learning but the learning process is always driven by practice; it is the practice in *the community of upper secondary mathematics teachers focusing discussions on slope* that will be analysed.

The teachers in this case have been part of the same mathematics department for some years and they might already participate in a community of practice. Previous competences have not been defined and discontinuities or continuities the learning study will make to the practice of the mathematics department will not be evaluated. No intervening changes in the teachers' professional community at their department will be identified. Teachers' reflection on their practice may imply a change in their practice. A change in a teacher culture over time is not included within this framework. It is not enabling to put words into the tension between agency and structure and change in culture over time. It is not the focus of the thesis but it might contribute a small piece to that even wider and more long-term perspective of change in teaching culture. The framework of Wenger's does not enable to measure teacher learning in a community of practice, it is rather about the process of social learning (Wenger, 1998) [11].

The framework of communities of practice has been used in previous mathematics education research examining teacher learning; with different focus, and in different ways. The framework describes how an individual shapes and is shaped by the communities of practices they participate in.

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[10] In order to understand learning in social practices, we cannot analyse these artefacts in isolation and then analyse the human thinking. We must seek to understand how people reflect in social practice with help of artefacts (Säljö, 2000).

[11] This takes me back to Borko (2004) writing that there is a challenge to identify and measure teacher learning. To understand teacher learning, as it takes place in many different aspects of practice, we must take into account both the individual teacher and the social system in which they are participants. According to Wenger (1998) learning is in what you do, but measuring learning is undefined.

Focusing on identity or community is not a change of topic but rather a shift in focuses within the same general topic (Wenger, 1998). Different uses of the framework for analysing the process of teacher learning has focused on one hand the individual, and on the other hand the community. Palmér (2013) and van Zoest (2003) are examples of studies that have used the framework to focus on the individual in communities of practice analysing the becoming of an identity. This approach to the framework is focus on community.[12]

Focusing on community, the framework has been used as both emergent and designed (Palmér, 2013). Kazemi and Franke (2004) focus teachers using their students' mathematical work as their collective inquiry. They account for the learning of teachers as a group, first designing a community of practice and then drawing on the emergent shifts in teacher's participation. Hemmi (2006) examines and structures the empirical data in her thesis by using communities of practice. Her thesis focuses a mathematics department, with a special focus on their relation to proof. Her approach to the framework is neither attempting to design, nor is she analysing if a community of practice emerges or not. She sees the mathematics institution as a community of practice; it is her unit of analysis. My approach to the theory is very much inspired by her work. *I see the mathematics teachers focusing discussions on slope as a community of practice; it is my unit of analysis*.

Graven (2004) also draws on Wenger's Communities of practice to study teacher learning. She extends the model of interrelated concepts of learning with confidence as it emerged in mathematics teachers' description and explanations of their learning in an in-service programme stimulated by curriculum change. Similarly, Admiral, Lockhorst and van der Pol (2012) developed a descriptive model of a teacher professional community, from the basis of Wenger's definition. They add to this that a community can move back and forth along three stages of development of core features; beginning, evolving and mature. They conclude the model can be used as an analytical framework when studying the design, description and effects of teachers' communities in secondary education.

This part was an attempt to justify my choice of communities of practice as a theoretical framework. It was an overall presentation of what the framework is enabling. My approach to communities of practice was also described. From this the framework will be further presented, elaborating the aspects of practice.

[12] Focusing on community rather than identity is also in consensus with the generation of empirical data. This will be elaborated on in chapter 4.3.

3.3 Aspects of practice

Lester (2005) argues that there is no data without a framework to make sense of it, just as a good framework allows us to transcend common sense. A framework provides a structure to be able to understand the case in-depth. In this thesis theory frames the empirical data of the case.[13] Wenger (1998) writes that a perspective can acts as a guide toward what to pay attention to. Communities of practice frame the empirical data of this case to the extent that it acts as a guide towards what to pay attention to. Selected concepts of that theory enable to put attention to when mathematics teachers focusing discussions on slope.

Wenger (1998) discusses basic aspects of practice and characterises these in terms of their internal dynamics. The aspect of *meaning* is the level of discourse at which practice should be understood. Meaning is defined as an experience of everyday life and is located in a process; in the *negotiation of meaning* (Wenger, 1998). Framing this case as a community of practice pays attention to the teachers' negotiation of meaning[14]. That is the negotiation of their experiences of teaching and learning about slope. Negotiation constantly changes the situations to which it gives meaning and affects the participants. It entails both interpretation and action, and this process always creates new circumstances for further negotiation and further meanings (Wenger, 1998).

The negotiation of meaning involves the interaction of two constituent processes, *participation* and *reification*. Participation is defined as active social involvement but also as personal membership (Wenger, 1998). The teachers in this case participate when they try to make meaning of the tool of the variation theory, when they discuss the concept of slope. Participation may also refer to the active involvement in planning the lesson, just as it also refers to when they, in silence, observe the lesson.

Reification is defined as a shortcut for communication, a focus, a projection of what they mean. There is a duality embedded - participation and reification cannot be considered in isolation, they come as an interacting pair. Reification always rests on participation and in turn participation always organises itself around reifications, because it involves artefacts, words and concepts that allows the negotiation of meaning to proceed (Wenger, 1998). The mathematics teachers in this case focusing discussions on slope can be assumed to reify mathematical concepts; their definitions and relations. The concepts of the tool of variation theory may also be reified in practice. Wenger (1998) states that anything that comes from the outside, as a tool or an artefact[15], must be negotiated in a community of practice. He gives a

[13] The methodological approach to theory and empirical data will be accounted for in Chapter 4.4.

[14] Negotiation does not necessarily refer to something going on between people but can as well be conceived as processes going on silently in one's head (Wenger, 1998).

[15] Lave and Wenger (1991) state that knowledge is encoded in the artefacts, and introduces the concept of transparency of artefacts. Wenger (1998) does not conceptualise it further. The concept of

metaphor of the duality; he sees reifications as anchors and participation as the waves they create. The further the distance it travels, the more meaning is lost. If the teachers are not using the shortcuts for communication, the shortcut is lost.

Another aspect is practice as *community*[16]. A community has dimensions of source of coherence through *mutual engagement*, a *joint enterprise* and a *shared repertoire*. The characteristics of practice these dimensions entail are:

Mutual engagement defines a community and being engaged gives a sense of belonging. It can give rise to differentiation as to homogeneity, as it involves competences and competences of others (Wenger, 1998). The teachers' practice does not require homogeneity. It draws on what the teachers know, and their ability to negotiate what they do not know. Wenger (1998) writes this can both be resource and a limit.

The joint enterprise is what is being negotiated and reflected upon in the community. It does not imply that everybody agrees with everything. This reflects the complexity of mutual engagement (Wenger, 1998).

> Communities of practice are not self-contained entities. They develop in larger contexts – historical, social, cultural, institutional - with specific resources and constraints. Some of these conditions are explicitly articulated. Some are implicit but are no less binding. (Wenger, 1998, p.79)

An explicit condition for the practice of this case is that it takes place in a setting of learning study. The teachers are to plan a lesson. The process of defining a joint enterprise is keeping the practice in check, just as it also pushes it forward (Wenger, 1998). A history of teaching culture may also be a binding condition, as it is so fully integrated into teachers' worldview.

A shared repertoire is the development of the joint enterprise, it is the words, tools, concepts that are produced or adopted throughout the community of practice. This is what has become a part of practice and combines the aspects of participation and reification. A resource is also to what extent it is available for further engagement (Wenger, 1998).

> The importance of our various communities of practice can thus be manifested in two ways; their ability to give rise to an experience of meaningfulness; and, conversely, to hold us hostage to that experience. (Wenger, 1998, p.85)

The aspect of *learning* is what changes our ability to engage in practice. Wenger (1998) continues that negotiation of meaning is fundamentally a

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transparency will not be included in addition to the negotiation of meaning as the duality of reification and participation in the framework.

[16] Wenger (1998) write that a community is not usually a community of practice, and practice is not necessarily a community of practice. *Community of practice* is a unit. In this thesis, just as in Wenger's book: when the term community or the term practice is written alone it is merely an abbreviation.

temporal process. Practice can be described as the teachers shared histories of learning. This is the understanding of why they engage, the struggle of defining the enterprise i.e. realising why they engage in teaching and learning about slope and what they need to negotiate and renegotiate to define that. Learning in the community is also the ability to change the resources, to renegotiate the experience of teaching about slope, to adopt tools, create and break routines.

The aspect of *boundary* means that a community of practice cannot be regarded in isolation. They must be seen as a part of the rest of the world, dependently on other practices. As we all participate in several practices, the teachers' engagement in this community of practice also entails engagement in external practices. It can be assumed that the teachers' engagement involves engagement with the students, and with the other teachers at the mathematics department in the school. Engagement can also be narrow, and the understanding in a shared repertoire is not necessarily the one that gives the members broad access to other practices. A community of practice can become an obstacle to learning by entrapping the members in its very power to sustain their identities. In these cases nothing else is taken into account, no other viewpoints that may create discontinuity (Wenger, 1998). In this case an advisor joins the community, and general discontinuities may spread, as membership is relative. Wenger (1998) defines a characteristic, as *brokering*, that is connection provided by people who can introduce elements of one practice into another. Learning study involves teachers and researcher planning a lesson. The researcher, who is also referred to as the advisor, may be seen as introducing elements from mathematics education research to the community of teachers. The job of brokering involves processes of translation, coordination and alignment between perspectives. It is complex.

These concepts form the framework when the unit of analysis is a community of practice. This is what attention will be paid to in the analysis.

Summary

In this part, a theoretical framework that will address teachers in collegiality has been presented and an attempt to justify it has been made. My approach to Communities of practice and the aspects of practice have been characterised by concepts from the theory. This is for the analysis of empirical data. In the next chapter, methods will be discussed.

4. METHODOLOGY

In this chapter how the case was captured will be discussed, and I will reflect upon and justify the methods used. My approach to the case study will be given in the first part. From this follows the selection of the case and how the empirical data was generated. The role of the researcher and ethical aspects will be taken into account. My approach to theory and empirical data will be given and then a description of how the empirical data was analysed. In the final part scrutiny of the trustworthiness of the case will be carried out.

4.1 A case study

This research has arisen from a problem encountered in my everyday work as a teacher. It has arisen in response to the shift towards collaborative work in schools which has been reviewed in the literature and elaborated on in previous chapters. The desire is to improve practice in this particular area. The aim of this study, though, is to describe and analyse upper secondary mathematics teachers' practice developed in collegiality, trying to capture its characteristics. This is a search for understanding rather than establishing explanations and looking for causes. This is a distinction between qualitative and quantitative research (Stake, 1995). The nature of this research is within the interpretative paradigm.

Catching the complexity and situatedness[17] of human behaviour, analysing and interpreting the uniqueness of real individuals and situations, are elements of case study research (Cohen et al., 2011). Stake (1995) writes that we study a case when it itself is of very special interest, when we look for details of

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[17] Situatedness can be a part of ethnomethodology i.e. searching for how people make sense of their everyday world, how statements are related to the social contexts producing them and how all accounts of social settings are mutually interdependent (Cohen, Manion, & Morrison, 2011). This case study is not defined as to applying an ethnomethodological approach. In Chapter 4.4 there is a methodological approach to theory and empirical data which enables me to properly identify my interest in characterising the practice of the mathematics teachers and in obtaining rich information and providing a deep understanding of these characteristics.

interactions with their context. He also makes a distinction between the uses of cases. Not because it is useful to categorise case studies in general, but because the methods used will be different. A case may, in itself, be of interest as we want to know, on one hand, about that particular case and on the other about a general understanding of something else. Stake (1995) denotes them as *intrinsic* and *instrumental* case studies respectively[18]. This case study is intrinsic as the aim is to understand the complexity of mathematics teachers' practice emerging in collegiality in particular. However, this case study is also instrumental, as it is whished, to develop understanding beyond the four teachers focusing discussions on slope in a learning study. Stake (1995) writes the more intrinsic a study is, the more the researchers have to restrain their special interest to discern issues that capture the uniqueness of the case. At certain points the use of the case in this study was intrinsic, at other points it was instrumental since the process of conducting case study is linear, but iterative (Yin, 2009). Capturing the case of this study has been a linear process of reviewing literature, generating empirical data, analysing and writing up the thesis. The process has also been iterative, as it required me to review and re-examine my former decisions. This will be discussed further as I underpin the approach to theory and empirical data in Chapter 4.4.

Ragin (1992) writes that researchers should ask "What is this a case of?" again and again, working through the relationships of ideas to give evidence to its answer. In this case study, what the case is a case of, has changed during the course of research. There are several key elements in this study and it has been recognised that cases may be multiple in a given piece of research (Ragin, 1992). When the results are presented in this thesis this study is *a case of when mathematics teachers focus discussions on slope.*

Bassey (1994) says that when conducting case study research the research question is the engine but he admits it is expected that research questions are modified and sometimes replaced in the course of the process. Stake (1995) writes that the best research question evolves during the case study, as they guide the work but also sharpen the meaning of previous studies and clarify the potential findings. He chooses issue questions as primary research questions, and he means that issues draw us toward observing the problems of the case and the complex background. The research question of this study was formulated early in the process. It has worked as an engine in this linear, but iterative process. Research questions have emerged as issues as the case was discerned, they have grown and been modified with evidence from empirical data and literature. The development of what this is a case of has evolved with the formulation of the research question.

According to Bryman (2001) and Stake (1995) the point of a case study cannot be to generalise to other cases or populations. The strength of a case study is rather its ability to contribute to the expansion and generalisation of

[18] In a third scenario Stake (1995) uses several cases to study, rather than just one case.

theory. This in turn can help other researchers to understand other similar cases or situations (Yin, 2009). Flyvbjerg (2006) emphasis a formal generalisation is only one of several ways to gain understanding and accumulate knowledge. A purely descriptive case study, without any attempts to formal generalisations, can have a force in itself. However, the generalisation of a case depends on the selection of the case.

Above was an outline of my approach to case study research. The implication from this will follow. From this the selection of the case will be presented

4.2 Selection of case

Yin (2009) writes that a case is not a sample. The strength of case study is that the case is only representing itself. This case study is about understanding the particular; the selection should provide rich information about mathematics teachers' practice emerging in collegiality.

However, professional development, based on collegiality, taking place among upper secondary mathematics teachers did not give me a wider selection to choose from. Two professional development initiatives based on collegiality were to take place in the surrounding area. These two possible selections were traced from a university and were both settings of learning study for upper secondary mathematics teachers. One learning study was to focus discussions on differentiation, the other on slope. Both learning studies were to start up in Aug-Sep 2012 and carry on throughout the autumn term 2012. I contacted both, and one of them; a group of four teachers and an advisor focusing discussions on slope, did not hesitate to let me join them. So a learning study turned out to give access to empirical data to examine upper secondary mathematics teachers' collective practice[19]. It was not a selection according criteria to maximise rich information, it was the selection that gave me access to fieldwork in the limited time available.

The case constitutes four upper secondary mathematics teachers and an advisor in a setting of learning study. The learning study takes place at an upper secondary school in Sweden. The school states that they work to strengthen their students' self esteem and to increase their knowledge and develop their capabilities, taking the individual into account. The school employs 100 teachers, of these 16 teach mathematics. The learning study includes 4 of these 16 mathematics teachers. These four mathematics teachers had come across a learning study at a conference for mathematics teaching and learning, and found it very interesting. They were amazed by the research results presented, how much better the students were due to teaching informed by variation theory. These teachers have initiated collegial

[19] This is case study *about* four upper secondary mathematics teachers' collective practice in a setting of learning study; it is not a study *within* learning study.

collaboration in a learning study, it did not come top down. The learning study was not forced upon them and this is something to consider when eventually generalising the case.

An external advisor[20], based at a university, is also participating in the learning study. The advisor has experience of mathematics education research and a special focus on learning study. The four teachers have been teaching mathematics in upper secondary school for 4 -12 years and they have been employed at the school for 3-12 years. At the time of the learning study, the teachers were also working on new reform (GY11) and at the same time at their school they were implementing a new platform for learning and teaching.

The advisor organised the learning study following the guidelines by Marton (2004), taking into account both the teachers and the advisors perspective on what was manageable. This project was taking place in addition to the advisors ordinary work at a university, and it was not founded on the advisors behalf in any way. The teachers agreed that the advisor should not participate in each meeting, but in those that were extra important.

The teachers and advisor met on 7 occasions and in between the teachers were set up for work. Each meeting has a purpose and the work in between is also defined. The advisor introduced the concepts of variation theory throughout and during the meetings with the teachers. In addition the teachers were given Pang´s (2008) article *Using the Learning Study Grounded on the Variation Theory to Improve Student's Mathematical Understanding*[21]. This article is based on a learning study in Hong-Kong aiming at improving students understanding of slope.

The learning study took place during a period of 15 weeks[22] and the meetings with the advisor took place almost every other week. They got help from the head of the school to organise it practically. Their timetables were therefore set to make room for space for the work with the learning study. Since they now had this slot of time together in the timetable, they decided to use the time every week since they were also set up to work in between the meetings. They met every week, each time for about 2 hours. The time slots were scheduled in the late afternoon. Participating in the learning study for such an extensive time, the teachers were assured a week off from school.

The first meeting took place at the university, but the other 14 meetings were, of the convenience of the teachers, held at their school. Those meetings took place in various spaces at the school, such as classroom, group rooms, areas for preparation etc.

In Figure 2 the idea and the elements of a learning study according to Marton (2004) is given. These elements have been illustrated in boxes to place focus

[20] I am not the external advisor in the learning study.
[21] The study is accounted for in chapter 2.3.
[22] 15 weeks is almost a full term in Sweden.

on different steps of the teachers' professional development. The arrows are representing the iterative process.

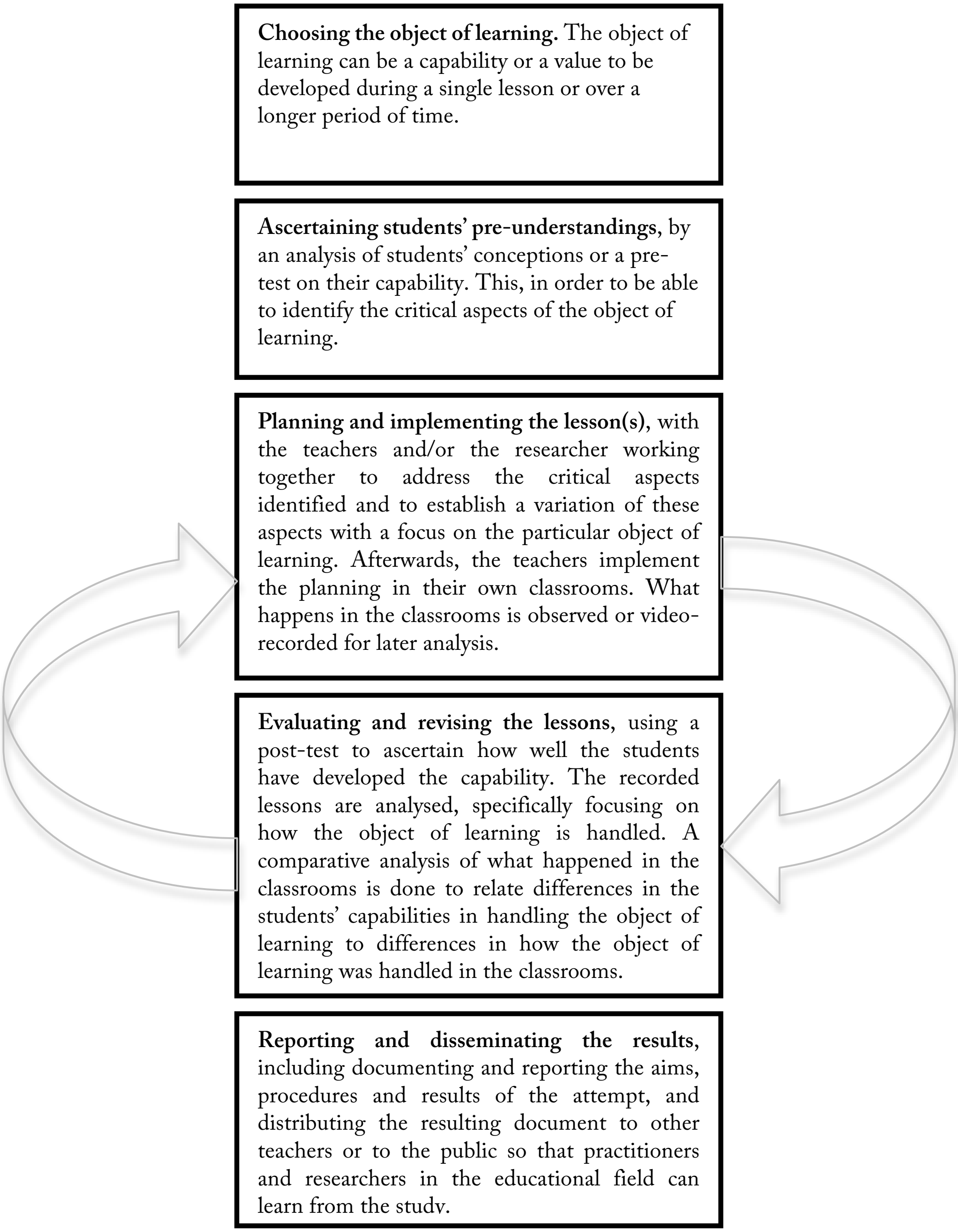

Figure 2 *The elements of a learning study (Marton, 2004).*

The iterative process included three cycles, hence three groups of students. The teachers, in agreement with the head of the school, decided which groups of students should be included in the learning study. They decided to choose from the Technology educations program students in three parallel classes. These students were in their first year in upper secondary school, taking Mathematics course 1c at the time of the learning study.

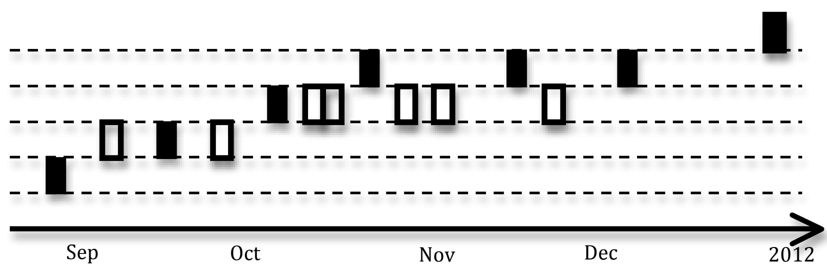

Figure 3 *The meetings that took place throughout Sep – Dec, in 2012*

Figure 3 illustrates meetings that took place throughout Sep – Dec in 2012. Each rectangle represents a two-hour long meeting. In order to capture what activities that took place in the different meetings, the five different steps of the learning study (given in Figure 2) are indicated using different levels. Seven meetings included both the advisor and the teachers and this is shown by black rectangles. In between these meetings the teachers are set up to work and these meetings are indicated with a black frame only.

The first step of a learning study is to choose an object of learning, which in this case, was identified by the teachers to be the concept of slope. At the first meeting they narrowed down the concept of slope to the rate of change. The second step is to ascertain students' preunderstandings thus the purpose of the second meeting was to construct a pre-test. Before the second meeting with the advisor, the teachers met to choose questions, preparing for the construction of the pre-test. The students (58), from three different groups took the pre-test on the same day and the teachers would analyse the results before the third meeting with the advisor. At the third meeting the purpose was to identify critical aspects based on the analysis of the pre-test. From this followed the planning of a lesson with the aim of opening up dimensions of variation to make the discernment of critical aspects possible for the pupils. One of the teachers taught the planned lesson in one of the groups. The lesson was video-recorded and the other teachers were advised to observe the implementation of the lesson too. The fourth step was to evaluate and revise the lesson. The students took a post-test and the teachers analysed the results in relation to the result of the pre-test. The purpose of the fourth meeting with the advisor was to evaluate the lesson in relation to the analysis of the test results and to what happened in the classroom. The lesson was reviewed by the group through its video-recording. The iterative process followed and there was a second and a third and final cycle in the learning study. Different teachers implemented the second lesson and the third lesson, with other groups of students. In the last step, the teachers were to document the learning study in a report for external communication. That process is not focus of this thesis however. At the final and seventh meeting with the expert, the teachers reflected on the processes that had taken place during the learning study. Within this wrap-up the purpose was also to try to identify critical aspects in other areas of mathematics.

The first lesson was implemented in a class that had no relationship to any of the teachers in the learning study. This choice was for organisational reasons. It was clear that implementing a lesson demands some organisation and they needed to engage teachers outside the learning study to cover for them and to make space for the lesson. None of the classes were about to learn about the relation between Δy and Δx at the time when the learning study was carried out. The content is in their syllabus but not to be covered yet. So when the teacher took the lesson in a class, the students were learning something else and the lesson was not in a sequence with previous and following lesson.

Stake (1995) says the uniqueness and the context of the case must be considered. The case is framed as a community of practice, which enables us to describe and analyse its context even further. This is presented in Chapter 5. In the finally discussion of the case, looking for theoretical contributions in chapter 6.2, the selection of the case will be reviewed. Even if it was not a case selected on criteria I can discuss what type of case it turned out to be. Next the generation of empirical data will be described.

4.3 Generation of empirical data

Bassey (1994) says the research question also defines what should be done, as case study research has no specific method unique to it. The research question has developed and been modified throughout the conduct, but not to such an extent that it changed the method used. The choice of method in this case study is based on it being qualitative research as well as on the nature of the selection of case. Through the setting of the learning study there was access to 14 two-hour meetings. Empirical data was therefore generated through observation of these meetings. As a complement an interview was also implemented.

Previous research regarding learning study has mostly aimed at developing practice and the advisor and the researcher is then the same person. In this case study, I was a strict observer (Bryman, 2001) meaning that I did not interact[23] with the respondents. I have informed them about the case of upper secondary mathematics teachers participating in collegiality, but without any further details that could impact their participation. The research question was not stated, nor any expectations.

The participants of the study have all agreed to their contribution and are aware of their ability to call off the study. It is also confirmed with the participants, in line with the basic principles of conducting research with regard of ethical aspects, that any information about them will be stored and processed with confidentiality. And that it will only be used for the purpose of this research (HSFR, 2002). In a democratic society we can expect the freedom to investigate and ask questions, and with that follows the responsibility of respecting truth and the individual integrity. The researcher is expected not to intentionally mislead others. This should permeate the work of the researcher reporting the findings, in the analysis and even before that, in the generation of empirical data (Bassey, 1994). Pring (2004) writes that observations do not occur except on the understanding of what I as a researcher bring to the observation. And Bryman (2001) states that this will filter how I interpret the observations, as well as what I observe. From a philosophical perspective, this is fundamental for the generation of data of this case study, hence:

> The data never come in the shape of pure drops from an original virgin source; they are always merged with a theory at the very moment of their genesis. (Alvesson & Sköldberg, 2000, p.17)

This is a subjective research paradigm, but not as a consequence, it is the intention of qualitative research (Stake, 1995). Flyvberg (2006) summaries several previous in-depth case studies reporting on that the case material have shown their assumptions to be wrong. This also brings me back to Ragin (1992) pointing out that a cases may be multiple in a given piece of research since ideas and evidence may be linked in many different ways. Flyvberg (2006) writes that the most advanced understanding is achieved when the researchers put themselves in the context of the study. Also, Wenger (1998) writes that the joint enterprise can never fully be determined by an outside mandate. I am a non-participant when observing the meetings in this study. Hence I will not fully capture the case framed in a community of practice as the participants understand it[24].

As a method, direct observations are faithful to the real-life and holistic nature of a case (Cohen et al, 2011). Field notes were taken during these observations, and transcribed as soon as possible after the observation. The

[23] Even so, in the strict observation periods the respondents will be aware of me, which surely will exert some impact on them.
[24] This is also an indication of that the study cannot be defined as an ethnomethodology.

field notes did not follow a structure or include any categories. At this time the case was used as an intrinsic case, to explore the case for its own sake (Stake, 1995). I was writing down my immediate reflections, trying to make sense of the case as it unfolded in practice (Flyvberg, 2006).

The meetings were also video-recorded. A small web-camera was used which was placed on the table around which the meeting took place. Direct observations were made on 13 occasions. On one occasion the teachers video-recorded themselves and shared them with me. The video-recordings have not been important for the purpose of hearning the exact words, it was the meaning that was important. It gave access to the source of empirical data again, and again. Hence I will not capitalise on making sense of the case:

> It is like unexpectedly running across someone we have not seen for years. At first we don't recognize them, then with surprising suddenness the face fits into patterns that we do recognize. We wonder why we didn't recognize them in the first place. (Stake, 1995, p.72)

The interview took place once the learning study was conducted, a month later. I met the four teachers at their school and the interview was held in the group. I wanted to make sure I had not misunderstood or misheard what I had seen and heard. Hence interview questions were used to confirm the empirical data (Bryman, 2001). I wanted the interview to provide a complement, to find out what I had not understood or not heard through my observations. The interview was consequently semi-structured, i.e. a set of questions had been prepared but there was also space for further questions. The answers as a confirmation or as a complement guided me towards the next question. Confirming empirical data does not imply that interview questions were formulated as statements for the respondents to confirm or reject. The questions were open enough to provide both complement and confirmation to empirical data. The themes focused on in the interview were the teachers' expectations and experience of collegiality. The interview was also aimed at complementing and confirming issues of the setting of learning study. The interview guide is given in Figure 4.

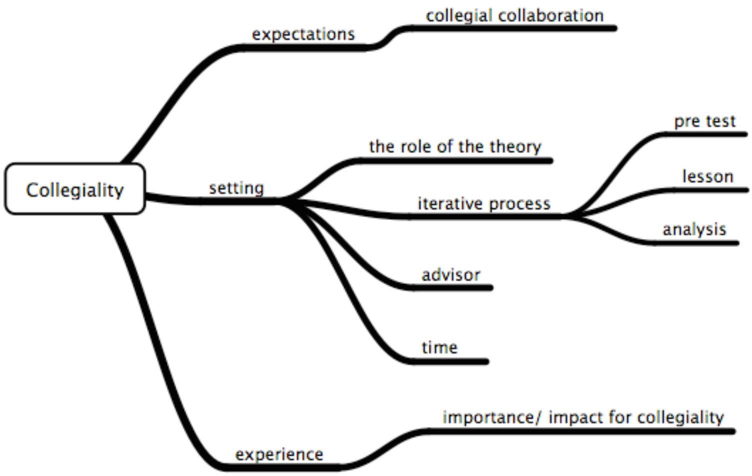

Figure 4 *The interview guide*

In this part the generation of empirical data has been elaborated, taking the role of the researcher into account. In the next part the approach to theory and empirical data will be discussed.

4.4 Approach to theory and empirical data

Earlier it was discussed that the strength of case study is about its ability to contribute to the expansion and generalisation of theory. Further Bryman (2001) says that the generalisation regards the quality of the theoretical thinking. For me theory began by wondering. How and why, I wondered, are the issues that were discerned in the case linked to each other and to the context? This was of interest in order to characterise the practice of the upper secondary mathematics teachers, to gain rich information and provide a deep understanding of these characteristics. This case study has an abductive approach, rather than a deductive or an inductive approach[25]. Eriksson and Lindström (1997) say abduction is a way to discover meaningful underlying patterns. It makes possible to connect surface and deep structures. There are

[25] In a deductive approach the researcher deduces hypotheses from a theoretical perspective that will be empirically tested. The empirical analysis will either confirm or reject the theory and might lead to a redeveloped theory. Roughly described, an inductive approach implies the direction opposite of the deductive approach. With an inductive approach the theory is the result of the research. Qualitative research more often implies inductive research, as a case study does not imply inductive nor deductive research explicitly (Bryman, 2001).

also links between abduction and pragmatism, which is about making our ideas clear and to open up for new ways of thinking[26].

The abduction has a starting point in interpreted knowledge. The interpretations of patterns are thus made in terms of theory-laden empiricism. The interpretation is made in a wide sense, including literature, conceptual analyses and historical sources. The perspectives determine which of the deep structures that are tangible. From this approach new knowledge is established (Eriksson & Lindtröm, 1997). In this case study the starting point was in the interpreted knowledge with the researcher and the empirical data of the case. The theory-laden empirical data has been elaborated through different literature and theories and examined from perspectives such as teaching culture. By focusing on different aspects, the interpretations have helped to define the unit of analysis. The research questions, and the "what is this a case of" has evolved and developed in this process. Gellert, Becerra, and Chapman (2013) write that research questions must relate to the paradigmatic questions of the theory. Hence I will also say that the research question has been modified alongside the unit of analysis in the abductive process[27].

Alvesson and Sköldberg (2000) define abduction as entailing a commutation between data and theory in a scientific and systematic way to look for answers to research questions of interest. They continue that the researcher is minimising the risk of interpreting what they think they are seeing in light of their own unreflected preunderstandings or to reinvent the same theory but in new words and concepts. Eriksson and Lindström (1997) write that surface structures may become the prison of induction, and the deductive approach may bring models of explanation that may work counter to deeper understanding. The researcher might become convinced, without reflecting.

Abduction is a significant form of reasoning because it generates new possible explanations. The given interpretation is followed and strengthened by a series of new observations. These new observations are not produced by a kind of mix of deduction and induction, but are really new and specific elements according to Alvesson and Sköldberg (2000). They illustrate the essence of the abductive approach with the case of the black swan. Saying an abduction would, just as an induction, observe a swan with a certain colour at first but then also show how the underlying pattern of bird´s genetic structure might lead to a certain colouring. By that an abductive approach is explaining a certain case. In this case study I observed teachers discussions regarding their experience of avoiding student difficulties when teaching about slope. With the abductive approach I tried to find the underlying patterns for this

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[26] Pierce reintroduced abductive thinking in modern time, he is also known for pragmatism. Pierce's collected papers (1931-1935) were translated and published in Sweden in 1990.

[27] I have learned, (since I am indeed a learner in this process) that slightly changing the wording can affect the unit of analysis vastly, since terms come with deep-rooted connotations. Hence, this has also been a process of defining the language of description of the case.

observation. This is when the use of the case was instrumental at certain points, when an attempt was made to establish the meaning to the issues that arose.

Alvesson and Sköldberg (2000) also bring the abduction and hermeneutics together. Saying that abduction is the hermeneutic spiral: interpretations of things, which we already have some understanding of. Eriksson and Lindström (1997) distinguish abduction from hermeneutics; hermeneutic implies reading between the lines, whereas abduction implies reading beyond the lines.

Using the abductive approach methods for analyse of empirical data will follow.

4.5 Analysis of empirical data

Pring (2004) and Lester (2005) writes that a theoretical framework will have an important role to play to structure and frame the observations to what is to be observed. The unit of analysis of this study is a community of practice, drawing on Wenger's (1998) work. It frames what to pay attention to in the observations. I have described how a few concepts of the theory of Communities of practice enable the capture of the case. The aspects of practice are *meaning*, *community*, *learning* and *boundary*, which are characterised through the concepts of *participation*, *reification*, *mutual engagement*, *joint enterprise*, *shared repertoire* and *brokering*. This theoretical framework pays attention to the context, which is important for a case study trying to understand the particular.

In this case study the analysis has taken place throughout and after the observations.

> There is no particular moment when data analysis begins. Analysis is a matter of giving meaning to first impressions as well as to final compilations. Analysis essentially means taking something apart […] We need to take new impressions apart, giving meaning to the parts. Not the beginning, middle and the end, not those parts but the parts that are important to us. (Stake, 1995, p.71)

There is no single or correct way to analyse qualitative data. It may involve organising and accounting for empirical data, making sense of it in terms of the participants definitions of the situation, noting patterns, themes, categories and regularities. The researcher must follow a principle of fitness for purpose (Cohen et al, 2011). The aim is to describe and interpret. The case is used to primarily to understand this particular case, but also to create understanding beyond the four teachers focusing discussions on slope. The issues discerned in the empirical data have been examined to reveal underlying patterns and deep structures. Analysis methods have also evolved in the course of the project. The framing of Communities of practice pays attention to properties as unique events in the context of the case. It was most suitable to present this in chronologically order, with issues raised (Cohen et al, 2011). Bassey (1994) suggests that, in analysis, the researcher should seek to précis qualitative data into meaningful statements. In this chronologically analysis an attempt has been made to capture the case through short, impressionistic scenes that focus on one moment or give a particular insight into meaning, community, learning and boundary (Stake, 1995). These scenes have been captured in relation to mathematical content that gives deeper understanding to the characteristics of practice. It turned out that these scenes were captured from every meting except the last (the 14th). The last meting was analysed, however that empirical data did not add anything to that was captured before. The background of the case, presented in Chapter 2, is important when connecting the issues raised in the community of practice with underlying patterns.

Access to the unit of analysis was given through observations and an interview. Earlier it has been described that I would not risk capitalising on making sense of the case, hence everything was also video-recorded. The video-recordings (34 hours) have been imported into NVivo software. The videos have been examined, but only those scenes that provided an understanding of the case were transcribed. In the making of the chronological analysis the software was only used for transcribing.[28] Transcribing video recordings in Swedish into English was also a matter of interpreting another language. Much thought was necessary, especially regarding mathematical concepts. Due to this situation translations regarding the mathematical content will be supplied. To the Swedish reader:

- *k*-value refers to the coefficient *k* in the equation of the straight line $y = kx+m$
- *slope* is translated from the Swedish word *lutning*
- *difference* is translated from the Swedish word *differens* or *skillnad*
- *distance* is translated from the Swedish word *avstånd*
- *change* is translated from the Swedish word *förändring*

In the analysis and writing up the thesis, I tried to regard the respect for person and truth (Bassey, 1994). The teachers occur in the scenes as *Teacher 1, Teacher 2, Teacher 3*, and *Teacher 4.* They are labelled as to in which order they appear in the first scene. They are not presented with gender or age, with respect for person. Choosing not to bring a particular scene with ethical respect for person is not with respect for truth. It is a balance. Choosing what

[28] The NVivo software was very appealing as it uses nodes (codes) in the analysis, to be able to dissect empirical data. However, when creating nodes that paid attention to the framework it soon became clear, with this massive load of empirical data and as the concepts are so tightly interrelated in the framework, the nodes did not help to see beyond the transcripts to make some kind of meaning of the case.

scenes to include or not was in relation to the capture of the uniqueness and context of the case.

This was a description of the analysis of empirical data, which points direction towards the presentation of the case. Before that and to close this chapter, I will evaluate the methods used by discussing the trustworthiness of this case study.

4.6 Trustworthiness

Bassey (1994) says that trustworthiness becomes significant for case study, rather than reliability and validity. He draws on the work of Lincoln and Guba (1985), who give the criteria of *credibility, transferability, dependability* and *confirmability*[29] to ensure the trustworthiness of qualitative research. They argue that ensuring credibility is the most important factor establishing trustworthiness. Credibility deals with how well the findings correspond with reality. How this case study can ensure its credibility will be elaborated. The main source of empirical data is a natural setting[30] and the selection of the case has previously been described. The research methods used, observations and an interview, are established methods in case study research. Characteristic of case study research is detailed empirical data from a wide source (Cohen et al, 2011). The triangulation, the width, lies in different methods. Different methods can put attention to different issues, thus the interview was held to complement the observations. The aim of the interview was also to confirm the observations and in some sense to ensure the participants' validation. However, the source of empirical data of this case study is also wide as concerns time and occasions. The triangulation is hence within the method.

The credibility of the researchers is important, as they are the major instrument of data generation and analysis. Hence I have placed some focus on myself in terms of preunderstandings. My personal views and feelings have appeared in the choice of research area. This was given in the introduction, in terms of my personal experience of collegiality in school. The case was chosen from an upper secondary mathematics teacher's practice. I also realise the limits imposed by that:

> When research is driven by extreme interpretivism research depends on the insiders' perspective. Since they know the behaviours and ideas that have meaning to people like themselves who regularly participate in the practice, they are unlikely to recognize the patterns of group life of which their actions are a part. (Eisenhart & Borko, 1991, p. 147)

[29] These criteria are in preference to criteria employed by a positivist; *internal validity, external validity/generalizability, reliability and objectivity* (Lincoln & Guba, 1985).

[30] By natural setting, I do not imply collegiality in general is a natural setting for these teachers.

I was an insider, a part of a traditional Swedish mathematics teaching as the participants in the case. The case study was driven from practice, from my teaching culture, but it was not analysed from that perspective. Conducting this case study I have had the opportunity to become aware of my teaching culture, to acknowledge teaching as a culture.[31] I have become aware of my routines that govern my classroom and have come to question them. Conducting this research I have considered the structural features and causes of social practice, and as I analysed the case these features are not unreflected common sense of an insiders' life. When I return to my practice I now bring an outsiders' perspectives. That is another story though.

The process has been described as linear, but iterative to stress re-examination and review of interpretations in accordance to previous research findings. The abductive approach also emphasises reflection and the responsibility of the researcher. Case study places focus on context and this may also provide credibility as it gives the reader insight to interpretations.

The transferability of the case study is a criterion that concerns to what extent the findings can be applied to other situations (Lincoln & Guba, 1985). This is elaborated as generalising case study in Chapter 4.1 and will be discussed further in Chapter 6.2.

In order to establish trustworthiness, the criteria of dependability regard how detailed the report is. So a researcher may repeat the study but not necessarily obtain the same results. This criteria is tied to the criteria of credibility (Lincon & Guba, 1985). Dependability places focus on how in-depth the methodological descriptions are. The generation and analysis of empirical data has been previously described - what was planned and what was realised.

Finally, the confirmability of the case study concerns that the findings is not merely reflecting researcher biases (Linoln & Guba, 1985). Objectivity is not the aim of the interpretative paradigm, this is subjective research (Stake, 1995). It has been discussed that cases may be multiple in a given piece of research since ideas and evidence may be linked in many different ways. The confirmability of this case lies in how well I allow the reader is allowed to follow the conduct to learn how the empirical data lead to the findings. The credibility ensured above and the dependability of the case study in the first place also show that this single case presented is not captured by my bias. Trustworthiness is important for the contribution of the case.

[31] In order to conduct this research, I have been on a temporary leave (2 years) from my work as an upper secondary mathematics teacher.

Summary

In the first part of this chapter the approach to the case study was described. The implications of this followed and were then presented in terms of the case selected and how empirical data was generated. The role of the researcher and ethical aspects have been taken into account. After the abductive approach to theory and empirical data was discussed a description of how the empirical data was analysed was given. In the final part the trustworthiness of the case was scrutinised. Consequently how the case was captured, and also my choices were justified and reflected upon. In the following chapter the, case of when mathematics teachers focus discussions on slope will be presented.

5. WHEN MATHEMATICS TEACHERS FOCUS DISCUSSIONS ON SLOPE

The case of this thesis is the case of when mathematics teachers focus discussions on slope. In this chapter the case will be present in chronological order. In order to understand the issues that arise, they are presented in relation to the context. The structure of the presentation hence follows the setting of the learning study in terms of its activities, given in Figure 2 and 3 respectively in Chapter 4.2. Scenes that are important and hence capture the case are given as the community chooses the object of learning, ascertains the students' preunderstanding, plans the lesson[32], and evaluates and revises the lesson. Finally scenes form the iterative process of planning a second and a third lesson is presented. Some issues are also given in relation to the teachers' reflections from the interview.

5.1[33] Choosing the object of learning

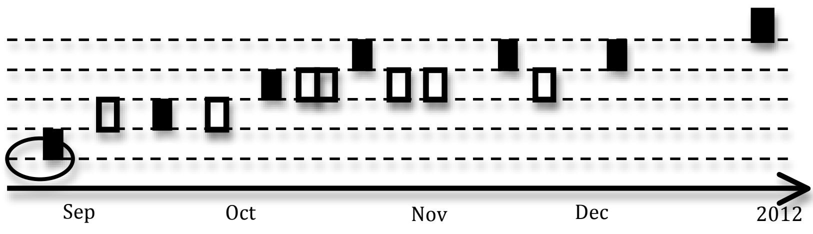

Figure 5 *The black ellipse points out the source of empirical data that will bee analysed and presented in this section. In addition, empirical data from the interview has also been analysed.*

[32] The implementations of the lessons are not analysed and presented.

[33] The level of title indicates what activity that is carried out in the learning study. Titles of lower level indicate issues that emerged in practice and capture the case.

Finding the focus of the discussions

The first activity is to choose the object of learning. The teachers have brought the idea of the equation of a straight line, $y = kx + m$, to the table. It is up to the teachers to define what they want the students to understand, what they will focus on:

Excerpt 1

Advisor	*What do you want the students to develop, what do we want them to understand?*
Teacher 1	*I come to think about procedure, to determine the slope, the coefficient k. It can be done in different ways with different procedures.* *[…]*
Advisor	*So the students should understand what the k-value means? What do we want them to understand?*
Teacher 2	*Do you mean what they [the students] need to know to understand k?*
Advisor	*No, what do we want them to know, is it the mathematical expression of Δy over Δx or is it that it graphically means an increase, decrease or constant we want them to understand.*
Teacher 2	*I would say that one aspect supports the other aspect.* *[…]*
Teacher 3	*It is a problem, singling out just one issue. You cannot introduce the value of m without talking about the slope of the line.*
Advisor	*I agree again, but this is not about what should and what should not be included in the lesson, it is about what the focus of the lesson is. Will the lesson focus on whether the straight line is going up or down due to a positive or negative value or is it focus on that is a rate of change; the relation between Δy and Δx and the value of k.*

They choose the *relationship between Δy and Δx* to be their focus, i.e. the rate of change.

This was a negotiation of meaning of the object of learning. It facilitates, through the advisor, the teachers with the question: *What do we want the students to understand?* This is their joint enterprise. As they reflect on that, it becomes visible that the process of defining the object of learning is also a negotiation of meaning of the concept of slope; as meaning of what slope actually describes (the line goes up or down) and the process for determining slope. The teachers are negotiating the concept of object of learning in terms of what it is and what it is not in relation to their experience. Their experience is that you cannot choose just one of the aspects to create meaning of the concept of slope. They negotiate that the connection between what slope actually describes and the process for determining slope is necessary, but this content is too wide to be treated in a lesson. This is their shared repertoire, which is illustrated in Figure 6. The advisor is guiding them and it becomes clear that defining the object of learning is finding the focus of the discussions. They will focus the process for determining slope.

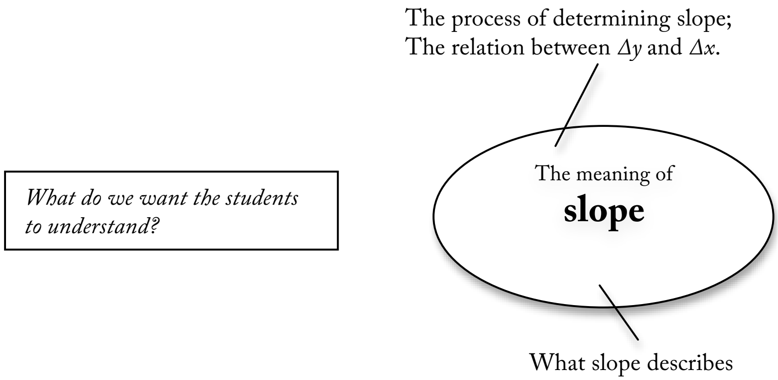

Figure 6 *The joint enterprise and the shared repertoire*

The joint enterprise and the shared repertoire characterises practice in coherence with the mutual engagement (Wenger, 1998). After the learning study was finished I returned to the teachers. Asking them about their initial expectations of the learning study, they said:

Excerpt 2

Teacher 1	*An opportunity to learn, to professionally develop […]*
Teacher 2	*Capability to explain in the classroom […]*
Teacher 3	*It is really a matter of observing each others' lessons, but there is a lot of trust involved in that. I had an idea we would observe lessons and then sit down and discuss the lesson. And it is the discussion afterwards that I find so interesting, that I want to get hold of. A learning study is not exactly that, but still I think it is in line with that. This also works, since you discuss the lesson before its implementation and by doing that I have elaborated on and shared my believes and teaching experience. In these discussions my teaching and what I do can be challenged. […]I felt I wanted to develop professionally, and wanted someone to examine it with me. I have video recorded my lessons before, but it is extremely boring to watch on your own. It is totally different to have someone to discuss it with.*

The reason why they participated in a learning study and their expectations are an important context of the case. The teachers share the engagement in an interest in learning more, for professional development. They can also imagine the power of learning with and from each other. In institutions where knowledge is regarded as an individual strength and is not seen as a give and take, collegial collaboration as a teacher professional development initiative may form an obstacle. Working alone, the teacher may both face disengagement and boredom, on the one hand, on the other freedom and privacy. There may be a strategy not to participate in collegial collaboration, due to the costs of freedom. There is a balancing act between boredom and freedom, and it is clear that collegial collaboration takes time (Wenger, 1998).

These teachers have insisted on collegial collaboration in a learning study, as a necessity and an opportunity to develop their teaching, and it was not imposed top down.

5.2 Ascertaining the student's preunderstanding

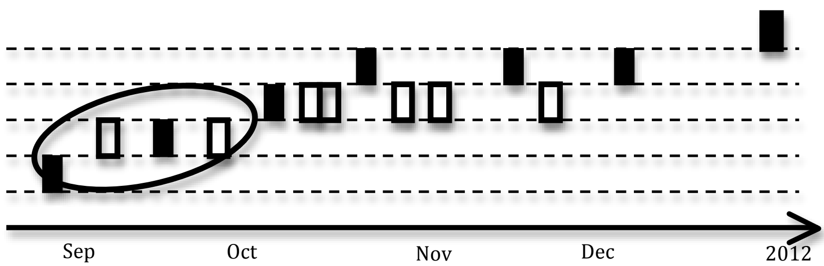

Figure 7 *The black ellipse points out the source of empirical data that will be analysed and presented in this section. In addition, empirical data from the interview will also be analysed.*

Starting from student's difficulties

Next they are to ascertain the students' preunderstanding about the relationship between Δy and Δx. They are to identify what they think are critical aspects. The teachers confess that they have not grasped the theoretical concepts of the theory yet:

Excerpt 3

Advisor	*What is a critical aspect?*
Teacher 2	*What is it that the student has to know in order to understand the –whatever it is.*
Advisor	*It is not the mistakes that the students make that is the critical aspect; it is that the student must know not to make that mistake and how to reach the next level. So for example, if we notice that the student does not understand the rules of*

priority, it is not the rule of priory that is a critical aspect. It is what we need to focus on in order for the student to understand the rules of priority that is a critical aspect.
[…]

Advisor — *It [critical aspects] is often, and it must not be misinterpreted, as the mistakes students make and their problems. Difficulties are not the same as critical aspect.*

From that they try to identify critical aspects of the relationship between Δy and Δx.

Excerpt 4

Teacher 1 — *They need to understand the meaning of Δ (delta).*

Teacher 2 — *The relationships in a triangle, when two points are given.*

Advisor — *What do the students need to understand, in order to understand that?*

Teacher 2 — *We must ascertain if they understand the meaning of a coordinate, to be able to read it.*

Teacher 1 — *It is a difficulty, handling a coordinate in quartiles other than the first one, but also the fundamental that a coordinate is written as (x, y) and not in the opposite way (y, x).*

The analysis shows that the community identifies that the students need to know the meaning of Δ, in order to understand the relationship between Δy and Δx. From that they negotiate the preunderstanding of the meaning of a coordinate. In this tracking or conceptual mapping of the relationship between Δy and Δx, the negotiation often starts from students' difficulties. The advisor emphasis that the concept of critical aspects not should be treated as a difficulty, even so the meaning of a critical aspect in relation to a difficulty does not become clear. A student difficulty negotiated is concerned with when coordinates are located in quartiles other than the first, when the students need to handle negative values of x and y.

Students' preunderstanding of fractions

What more, says the advisor, asking for potential critical aspects:

Excerpt 5

Teacher 1 — *A fraction.*
[…]

Teacher 1 — *Whenever a student sees a fraction he or she wants to turn it into a decimal.*

Teacher 2 — *We do not want the students to turn the fraction into a decimal in this context.*

Teacher 1	*If the fraction is 2/4 or 2/3 makes a difference and in an example it should rather be 2/3 so that the students will not immediately turn it into a decimal.*
Advisor	*If we state a difference 2 and a difference 3 and then ask them to state the meaning of 2/3.*
Teacher 3	*This is difficult, my experience says that once the k-value is introduced the next step is to include k-values between 0 and 1 and that is difficult for the students.*
Teacher 2	*We can apply 2/3 to another context, to cost per kg.*

They discuss that the meaning of a fraction is important, but they go back to the object of learning to focus their discussions again. The advisor alerts them to the fact that they should be careful so the object of learning does not changes into the meaning of fractions. Even so the teachers cannot let go of the students' preunderstanding of the fractions:

Excerpt 6

Advisor	*Let me make a point of the students preunderstanding of fractions. Thinking about lower schools, how the students have been taught the fractions, as two bars out of three, as shadowing shapes. So they have this kind of understanding, but when they come here 2 means difference of x-coordinate and 3 means difference of y-coordinate and the fraction is then to be understood different from the chocolate bars!*
Teacher 2	*Pizzas and chocolate bars!*
Advisor	*How can a fraction be represented in two dimensions?*
Teacher 1	*[Laughing] We need to get away from that, all the pizza slices and pies*
	[…]
Advisor	*Coming back to the students' preunderstanding of fractions, as chocolate bars, that might be a critical aspect when connecting a fraction to the k-value.*

The analysis shows that as they try to identify potential critical aspects they also reflect on other concepts[34] and the relationships to other concepts, such as fractions. They negotiate the concept of fraction to be a potential critical aspect and they become engaged in how students handle fractions; it identifies that what numbers Δy and Δx should represent make a difference to how the student might interpret and understand it. Students' difficulties are again negotiated; the students' difficulties in interpreting a fraction, or even an integer, as slope.

[34] Concepts refers to mathematical concepts, if nothing else is stated.

They also engage around what meaning the concept of fractions is given to the student in previous teaching. How they as teachers, through textbooks, provide the students with the meaning of fractions as shaded parts of pizzas and chocolate bars. They realise they need to get a way from this. This is an example of a discontinuity in the community, as they value and negotiate their teaching experience from a new perspective. The advisor is a broker, as she brings a new perspective to their practice.

The community is engaged in discussions about students' partial understanding of fractions. They realise that they are about to shift focus towards fractions; the focus is object of learning. In this situation the artefact is coordinating and keeping focus in the community.

The conceptual mapping also includes fractions. This mapping gives the community an awareness of students preunderstanding, from what and how related concepts were taught earlier. It trains focus on teaching.

Assessing "mathematical thinking"

From this they are to construct questions for a pre-test[35] to assess the students' preunderstanding. The teachers now have an outline of what to include in the pre-test. The emphasis is on the importance of the questions are included. The advisor says:

> ***Excerpt 7***
> *Advisor* *As it matters to the test, when we are choosing questions we must know what can we learn from the result of that sort of question. […] So it is not important whether the question gets two marks or not, this is rather what could be called a qualitative assessment of a test.*

They summarise the questions to include in a pre-test:

> ***Excerpt 8***
> *Teacher 1* *We have three questions and three levels; can the students read a coordinate, in the first question, can they calculate a distance if both coordinates are positive, in the second level, and the third level if the coordinates are both positive and negative.*

The teachers then meet, without the advisor participating, and they return to the discussion of interpreting *2/3*, as a slope and they try to formulate a question. They are considering asking the student to mark out a point *B* if point *A* is given as well as distances $\Delta y = 2$ and $\Delta x = 3$:

[35] The pre-test is taken by the three groups of students. See the pre-test in the appendix.

Excerpt 9

Teacher 3	*The next question would then be to find the y-coordinate of the next point on the line five steps along the x-axis.*
Teacher 1	*Ahhh, that's mean!*
Teacher 3	*Alright, make it six steps instead.[…] Let's make a staircase model with different steps.*
Teacher 1	*Exactly, then they will have to know how to [pointing at the staircase in the booklet] We should also ask: How did you reach that answer?*
Teacher 3	*How do we [teachers] become wiser of if a student knows how to solve this question?*
Teacher 2	*That they can follow the levels in the questions. [Laughing]*
Teacher 3	*That they know how to count a grid ….hm I really want to get hold of their mathematical thinking, I want to have control of their preunderstandings, of their mathematical thinking. This becomes so mechanical, can they find the points and so on. I am not saying this is wrong, but what can we learn from it as we proceed?* […] *I find it important to include the question: How did you reach the answer? And can we in any way help them [the student], so we will not get the answer "I calculated it ". Can we pave the way for them to show mathematical symbols or mathematical thinking?*

They agree that it is of interest to follow up with such a question as: How were you thinking? or How did you reach the answer?

They discuss the next step of the question.

Excerpt 10

Teacher 2	*Then we should take a big step [to find a new point on the straight line], you would say? Or?*
Teacher 1	*I was thinking, that they are forced to calculate it. It must be outside the coordinate system anyhow.*
	[…]
Teacher 3	*If they counted the grid there (pointing at the first step of the question), then they cannot do it this time.*
	[…]
Teacher 3	*Can we then add the question: "How is y changing, as x is increasing by one unit?"*
Teacher 1	*Well done. "How is y changing, as x is increasing by one?"*
Teacher 3	*That is a standard question I always ask, because then you have found the k-value.*
Teacher 1	*If they answer 2/3 then they have already solved it. This is the best question we have constructed.*

The analysis shows how the teachers negotiate the pre-test to help them ascertain the mathematical thinking of the students. They reflect on " *What is this question telling us?*" and "*How do we (as teachers) become wiser of if the student knows how to solve this question*?" They are engaged around these questions as they prepare questions for the pre-test.

As they are not sure they will capture the mathematical thinking in an ordinary calculation, they also ask the students to write how they reached the answer. They negotiate how to ascertain the students preunderstanding and they are not sure of how to pave the way so that the student does not answer I calculated it. They want to ascertain what the student understands, beyond "the mechanical". One idea that is brought to the table is to construct questions where it is not possible to "count in a grid", where the answer rather lies out of sight of the student. So the students "is forced to calculate" it, rather than just count.

An interpretation is that this practice develops in a teaching culture not used to qualitative assessment. They negotiate the fact that they want to find the "mathematical thinking".

Another terminology

When they are about to wrap up after two hours and they go through the planning to prepare for the next meeting they conclude:

Excerpt 11

Teacher 2	*[…] Identifying critical aspects, we should have had that in mind already, while constructing the pretest. Right?*
Teacher 3	*I think that is what we will see from the analysis of the pretest. […]*
Teacher 2	*But we have ideas of what we think are critical aspects. That might be how it is, we have not articulated it though [in terms of the theoretical concepts]. We have used another terminology.*
Teacher 3	*I am not sure how aware I am about it [the theory], we have been talking about different things, we have formulated questions to identify the parts, but is that the critical aspects?*
Teacher 2	*Using this we will try to decide what we think are critical aspects and the test will show if it is or not.*

It seems that the community has reified the theory even if they have not "used the terminology" as they say. The question that they keep asking about the students' preunderstanding, is a product of how they have negotiated meaning into the theory.

Returning to the teachers I asked them for any suggestions to the set up of learning study:

Excerpt 12

Teacher 2	*I would have appreciated a package before, to get into it.*
Teacher 1	*A course in variation theory!*

I asked them about the role of variation theory for them as participants in the learning study:

Excerpt 13

Teacher 2	*To elaborate on the content of the lesson, and to put that into words. To be able to talk to each other about it.*
Teacher 4	*It is a great advantage to have a language, from a theory based in research. To be able to explain how you teach the content in different ways by using contrast and such things.*
Teacher 3	*I feel like cheating all the time and we tried to describe without really knowing the concepts.*
Teacher 4	*At the end [of the process] though we had talked more about contrast etc.*
Teacher 2	*Sometimes I realised – this is a contrast! [laughing].*
Teacher 1	*This feels like a first step, but it should rather be like – here we need to create a contrast.*
Teacher 3	*I feel that I have not grasped variation theory. That was wrong from the beginning, I should have known more.*
Teacher 1	*Yes you know too little about the theory.*
Teacher 4	*But we are still talking in those terms, we always tried to make sense of it and we understood more every time we met our advisor. The advisor explained; here is a separation and so on.*
Teacher 2	*We were aware of that, I did not have the time to learn the basics, which I wanted to. I wish, I had studied more parallel [with the learning study]. […]*
Teacher 4	*I think we got much better though.*

I have defined the variation theory as an artefact, an intellectual tool. Tools may serve for mediating in social practice, stabilising human practice, co-ordinating and disciplining human reasoning by suggesting how to do things (Säljö, 2000). A tool may also facilitate discontinuities or continuities in a community of practice. The analysis shows that the teachers find the role of the theory to elaborate on the content of the lesson, and to put that into words, as a language. However the analysis also shows it was initially problematic to reify the artefact in the community.

The construction of the Cartesian coordinate system

The next step in the process of constructing the pre-test is to categorise expected answers to each question in order to structure the qualitative analysis. The advisor joins the group again and they go through the questions the teachers have prepared:

Excerpt 14

Advisor	*What type of answers can we expect from the first question, which is to read a coordinate [coordinate on the y-axis].*
Teacher 2	*To do it correctly, but also to write in reverse order, as (y,x)*
Advisor	*But this, that the coordinate is on the y-axis may also be problematic, it is another aspect.*
Teacher 1	*If they understand that the x-coordinate is zero.*
Advisor	*Yes, be aware of that if a student gives the y-coordinate only in the answer.*

The expected answers to the next question, a point in the second quartile, is categorised as; the correct answer, the reverse order *(y,x)*, and no negative values stated:

Excerpt 15

Advisor	*What can we learn from this?*
Teacher 3	*That they have partial understanding of negative numbers.*
Teacher 2	*That they do not have the fundamental knowledge on the construction of the coordinate system.*

The community develops categories for each question constructed in the pre-test, from what their experience says about expected answers - the incorrect answers. The process of going through the questions and categorising the expected answers is also a negotiation of what they really aim at testing. As the advisor said: What can we learn? By identifying expected answers they also renegotiate the question. Earlier they brought to the shared repertoire that the meaning of a coordinate was critical for students understanding of the relationship between Δy and Δx. In this analysis they bring that construction of a Cartesian coordinate system must also be an aspect.

That what we actually teach

In one question of the pre-test a coordinate is noted as *B = (3, 4)*.

Excerpt 16

Advisor	*I have a question. It is written B = (3, 4) Why is it equals?*
Teacher 1	*I changed that. I had written nothing before, but then I looked in the math book and they write equals, so therefore I chose equals. I wanted the students to recognise it.*

Advisor	*I think it is a good discussion. I have seen that in textbooks, but what do you think. Is it right? What experience do the students have of the equivalent sign? We have given the equals sign the meaning of that the left side is of the same value as the right hand side. Writing the coordinate like this, we will give it another meaning. We must always be aware of the students' preunderstanding, moving away from that another way of writing can in itself create a critical aspect for the student.*
Teacher 4	*Ahh, I have come across this regarding the use of the signs of implication and equivalence. Some books are not consistent in the use of the two signs. I can tell that the students sometimes are confused by it.*

The teachers are engaged, they seem enlightened by the discussion. They discuss how misunderstandings can be eliminated. How a coordinate is written and how the coordinate axes are graded. Depending on how they will grade the axis, they will bring different aspects out, thus they can learn different things from the students understanding. The advisor emphasises that then this is one more aspect that will vary in the question, another aspects to consider.

The analysis shows that the renegotiation of the questions also concerns aspects that were previously hidden from the community. Aspects that it was not intended to assess in the pre-test. A negotiation of how to eliminate students' misunderstandings emerged in the community. It brings an awareness of what we actually teach, what meaning we give into concepts when we use them.

When I returned to the group I asked them about what they knew about variation theory one of the teachers said:

Excerpt 17

Teacher 3	*I had read about it [variation theory] on the web, it was from a lower secondary school. But even so I was drawn to what they said regarding their lessons; the major effect the small changes in their teaching had implied. I am confident that it is all about finding these small differences in the teaching. And if the nuances are so small I will need help to find them, I might not discover them on my own. […]*
Teacher 1	*It can be content you have taught in the same way throughout the years. […]*
Teacher 2	*Sometimes you are not aware of what you say.*

The collaborative work of reflection on the nuances in mathematics teaching was their launch point for participation in a learning study. They called for collegial collaboration, they needed each other to identify and reflect on this nuance in the teaching, on what is said or not said in the classroom. Their mutual engagement was about finding the small changes in their teaching that could give major effect in students learning. The issue of the notation of a coordinate was an example of a small change to make. This negotiation was in line with their mutual engagement and perhaps why they invested so much energy into it.

Distance and change

As they categorise the aspects of the question in the pre-test they realise that two questions were similar: *Determine the distance in x-led between the two points* and *Calculate the change in x-led between the two points* respectively. From this the reflection goes:

Excerpt 18

Advisor	*Distance and change.*
Teacher 1	*The change becomes negative in the example a distance is always positive.*
Advisor	*It may be interesting to open up that dimension of variation.*
Teacher 3	*Generalisation?*
Advisor	*Yes generalisation […]. It will be interesting to see if the students can discern that generalisation in the analysis of the test.*

The community negotiates the questions are similar in that both questions assess the line segment between two points in a coordinate system, but as a distance and as a change respectively. The artefact of the theory is used as a language as they negotiate two aspects of the same content; distance and change. This scene is included since a lot of the negotiation in the community will concern this matter. It is interesting how they identified this aspect as they where renegotiating the questions in terms of “What is this telling us?”.

Summary

All four teachers participate in the meetings, and it is possible to analyse their membership via their active involvement. Teacher 4 is not actively involved in the community, but the negotiation of meaning is also what goes on silently in our heads. The negotiation of meaning does not only exist between people (Wenger, 1998)[36].

As they are to ascertain the students’ preunderstanding regarding the relationship between Δy and Δx, the community is mutually engaged around

[36] This analysis can only capture what goes on in between people though.

finding the small nuances in the teaching that makes big difference to the students' learning. The joint enterprise is a negotiation of what the students need to know in order to understand the relationship between Δy and Δx; this is given in Figure 8, framed in a rectangle. Figure 8 also illustrates the shared repertoire, which is a projection of their negotiations. It is a conceptual mapping starting from the relationship between Δy and Δx framed as an ellipse. The mapping extends to content that the teachers distinguish as related concepts, content that is not in focus of their discussions. Hence negative numbers and fractions are framed as ellipses with dotted edges.

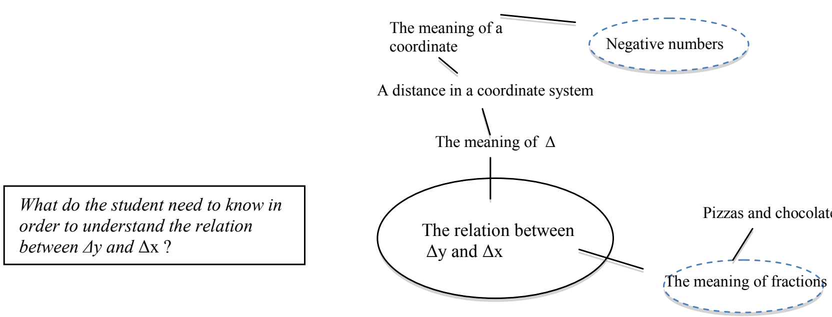

Figure 8 *The joint enterprise and the shared repertoire, as they ascertain the student's preunderstanding regarding the relation between Δy and Δx.*

5.3 Planning the lesson

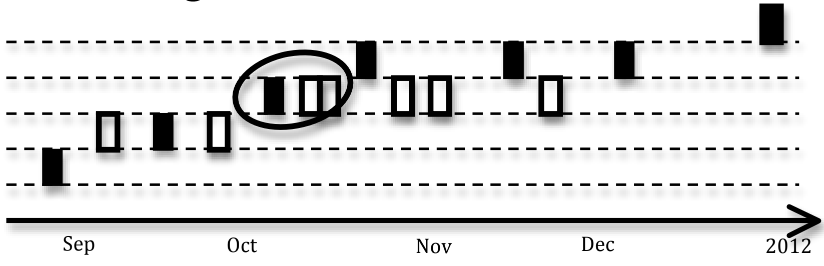

Figure 9 *The black ellipse indicates the source of empirical data that is analysed and presented in this part. It begins at the third meeting with the advisor and continues for two more meetings. In addition, empirical data from the interview is also analysed.*

Always treat Δx as positive

According to the planning of the learning study they will identify critical aspects from the analysis of the pre-test and discuss dimensions of variation to open up in the first lesson. The teachers have made a frequency table of the expected answers. They can immediately identify that some students write the coordinate as *(y, x)*, rather than *(x, y):*

Excerpt 19

Advisor	*What is the critical aspect [as the student write the coordinate in reverse order]?*
Teacher 1	*They need to understand the construction of the coordinate system.*

Some students get a negative answer when they are asked to determine the distance in *x*-led between two points in the second quadrant:

Excerpt 20

Advisor	*What do they need to understand to determine the distance between two points?*
Teacher 2	*The number line.*
Advisor	*The meaning of horizontal and vertical axis and how they are related to each other.*
Teacher 2	*The construction of the coordinate system. Again*
Advisor	*The meaning of direction.*
Teacher 1	*I think it is a matter of that they did not consider if it is an absolute value or a difference. And they draw the conclusion that if the distance is on the negative part then the distance is negative.*
Advisor	*Is the critical aspect to see the difference between a number and its absolute value?*
Teacher 2	*Do they have to know what an absolute value is to calculate Δx?*
Teacher 3	*I think ….. we are heading towards the rate of change, so we should rather focus on change then absolute value.*
Advisor	*Still it is a critical aspect, so I'll write it down.*

They discuss what the students need to understand to see Δx and Δy as changes.

Excerpt 21

Teacher 3	*Why do we need to consider Δx as a negative difference, why don't we always treat is a positive? If we have time on the x*

	– axis, then we always take the later value minus the first value.. and then it is only Δy that changes it sign.
Teacher 2	*Δy, what did you say?*
Teacher 3	*Δx is always positive, Δy can change sign. This is what you do later on talking about increase and decrease, and this is how you do in physics.*
Advisor	*We will need to consider how change and distance are related.*

The community has spent a lot of time identifying potential critical aspects and preparing the pre-test, so it was a fairly rapid process to identify real critical aspects from the analysis of the students' answers. The construction of the Cartesian coordinate system is identified in order to understand the meaning of a coordinate, but also to understand and determine a distance between two points. From that they negotiate the meaning of direction in order to understand the difference between two points and its absolute value. This came up and was identified first after the analysis of the pre-test, the community did not explicitly identify it as a potential critical aspect earlier. The aspect is renegotiated in the community, as a member suggests they should "always treat Δx as positive". The teacher argues that it is because of how you will do later as you apply it in physics; as increase and decrease with respect to time. The competence from previous teaching experience, and from taking a wider perspective of the relationship between Δy and Δx is valued in the community. The relationship between distance and change is renegotiated – it is still a critical aspect.

The conceptual mapping tracks the meaning of the concepts in students' preunderstanding. They negotiate the partial understanding of the student. The advisor provides the community with meaning of direction as important. It is negotiated that, in order to understand distance it must be discerned in relation to change and vice versa. However practice develops in a teaching culture, to always treat *Δx* as positive.

In relation to related concepts

In the last question in the pre-test it was stated that a slope can be written as a fraction. The students were then asked to draw a straight line going through a point A with the slope *5/7* in a diagram. The teachers see that many students have interpreted the fraction as the coordinate *(5, 7)*. Many students have then drawn a line, not passing through point A but through the origin. It takes them back to their earlier discussions about the students' preunderstanding of fractions as chocolate bars and pies. They also reflect:

Excerpt 22

Teacher 1	*So they interpret the fraction as a coordinate and they do not understand that they need two points to create a line.*

Teacher 3 *Taking into account that the students have preunderstanding of the concept of proportionality, it is not so strange that they draw the line through the origin.*

Advisor *I recall an article… where students interpreted a fraction as a decimal, they thought that the fraction sign could be swapped for the comma sign.*

Teacher 2 *That seems very odd. Do you think that is what happened here?*

Teacher 1 *It is interesting; if a fraction can be interpreted as a decimal then a fraction can be interpreted as a coordinate. It is also interesting what you say about proportionality.*

Teacher 3 *That is one of the things one have to make sure with proportionality at the beginning…that they don't write the x-coordinate in the nominator.*

Teacher 1 *Yes, that is true.*

The analysis shows how the community negotiated the students' answers in relation to their preunderstanding of other related mathematical concepts. They renegotiated the meaning of fractions and how students related it to decimals or coordinates. The community also reflects on the concept of proportionality, as an underlying concept that creates difficulties for students handling the straight line. They conclude that it might be that the students have interpreted the fraction as a coordinate, since this is something they have experience of students struggling with; the coordinate in relation to proportionality. This brings the interrelations between concepts of fractions, proportionality and straight lines into the shared repertoire. The conceptual mapping is now a mapping relation to related concepts, such as proportionality.

Using a tool

They decide to start to plan the lesson (45-60 min).

Excerpt 23

Advisor *Let us create dimensions of variation to discern the structure of the coordinate system.*

Teacher 3 *I have experienced it many times, that you have said that y is equal to zero on the x- axis, they work along, and then some time later you ask them Where is y equal to zero? and then they point at the origin.*

Teacher 1 *Yes, that is also my experience.*

[…]

Teacher 3 *If we just throw in random points in the coordinate system then it is fusion, which we do not want.*

Teacher 2 *We can walk along the axis.*

Teacher 4	*I would say it is some sort of separation. We will let the x - coordinate vary as we keep the y-coordinate constant.*

They discuss drawing line segments of the same value along a horizontal line to let the student discern that the distance is positive everywhere in the coordinate system.

Excerpt 24

Advisor	*What variations do we have?*
Teacher 3	*What is the contrast here?*
Advisor	*Between the positive and the negative [pointing at the distance on the positive side and the negative side of the x-axis]. And the separation is translating the distance on the line, that is kept constant. The same will be done in y-led.*

The analysis shows as they plan the lesson in the community the concept of dimension of variation is reified. The artefact is used as a tool, as an instruction of: *How can we let the student discern the critical aspects?* This is the joint enterprise, and the advisor says, ”Let us create dimensions of variation”. And they do, they are creating a separation in the first example. In the other example the theoretical concepts of contrast and separation is used to reflect on the lesson plan. At this stage of the process the concept of dimension of variation is both an action in a lesson plan and a reflection of a lesson plan. It is a negotiation of the mathematical content and a suggestion of how to teach the content.

When I returned to the group for an interview I asked them to reflect on the variation theory. The teachers say:

Excerpt 25

Teacher 1	*Looking in the maths book I ask myself, what is this question about? […]*
Teacher 2	*And we ask each other, is this generalisation or is it separation. […]*
Teacher 1	*The learning study has already made a great impact on me in my daily work. It has started a process of continuous and spontaneous reflection. Today a conversation about the meaning of parenthesis emerged in the classroom.*

In the previous analysis I wrote that the artefact facilitated a language for the community. The above underpins this and gives also response to a reflective artefact that facilitates discontinuities in the teachers' daily work.

Avoiding minus signs

Next in the lesson plan they will let the student discern the relation between distance and change:

Excerpt 26

Advisor	*We want the students to discern what a distance is and what a positive and negative change is.*
Teacher 3	*But then…..I don't know. As I do, I always let Δx be positive, should we talk about negative and positive change on the x-axis then. We will talk about increasing and decreasing functions later. That is not being consistent.*
Advisor	*I have experienced that students do not have the meaning of positive and negative change. I always say to them to follow the direction of the axis. If you go with the axis then it is a positive direction and if you go in the other direction then it is negative.*
Teacher 2	*That is a way of going through the structure of the coordinate system!*
Advisor	*This is instead of up and down, to the left and to the right. I emphasise going in positive or negative direction along an axis.*
Teacher 1	*This is the structure. Teaching vectors this is fundamental and the students follow this.*
Teacher 3	*What comes with this is that Δx can become negative and then we have the negative sign in the denominator to handle. If we instead always treat Δx as positive then if Δy is negative the k-value is also negative. We are not heading towards vectors, we are heading towards differentiation. It becomes logical for me.*
Teacher 2	*It is still the structure of the coordinate system.*
Teacher 1	*Our aim is teach the k-value, hence the sign of k becomes important. Thus the direction becomes important*
Teacher 3	*For me it is not.....Which point is point one and which is point two?*
Advisor	*[…] Why should they have to think of which point is the first or the second? It does not matter. What matter is in what direction.*
Teacher 3	*We create a natural structure for the student, we say we call them point one and two. It says in their formula-booklet.*
Advisor	*Yes, but this could be point one and this point two, it doesn't matter which point is point one.*
Teacher 2	*I think, taking the analysis of the pre-test into account, we should not decide which point is the first and the second. It is rather the structure [of the coordinate system].*

Teacher 3	*You think? I do not! I think the students will drown in minus signs and they need to consider going left or right.*
Teacher 2	*Why should we be afraid of minus signs?*
Teacher 3	*Because it becomes wrong. Minus signs are shit. [Laughing]*

There have been examples of negotiation and renegotiation of meaning of mathematical concepts that I interpret as discontinuities in the community. This scene rather captures a lack of discontinuity; it shows a static core in the community[37]. The challenge in the community is to allow discontinuity, to keep the tension between competence and experience (Wenger, 1999). The competence is formulating questions from your experience, but from a new perspective.

Also some members were core members, perspectives of teaching and learning are validated in the community, and the perspectives of the core members are more often considered. The core is very static regarding the negotiation of Δx; students' difficulties with negative numbers cause problems in the classroom, thus it is better to avoid it in the teaching. The idea is to always let the leftmost pint be point number 1 and hence the right most point number two. Then the students can use the algorithm in the formula booklet, without any risk of ending up with a negative denominator. It saves students from "drowning in minus signs". This is a new situation in the community, as they do not agree when they are negotiating how to teach the content in the lesson. A shared repertoire does not imply shared as in a common view on what is negotiated (Wenger, 1998).

Be alert, says the advisor

The advisor points out that they have to make up their mind regarding this:

Excerpt 27

Teacher 2	*It depends if we only focus on this lesson, on Δx and Δy, or if we are building on something else.*
Teacher 1	*I think, when we talk about the k-value of a straight line, to find the k-value we try to make meaning by: If you go one step forward, will you go up or down to reach the line? So we are using the positive direction [of Δx]*
Advisor	*Be alert, the students have to understand the difference between distance and change, that a change can be both positive and negative. What is happening with these mathematical concepts when we say that Δx is always positive?*

[37] Reminding the reader of that this case study is not about if the teachers are wright or wrong. It is about their active involvement and how it takes place and the underlying structures and patterns.

The advisor makes an example to emphasis her earlier statement. She marks out two points *(3, -5)* and *(6, -8)* in a coordinate system, searching for Δy:

Excerpt 28

Advisor	*How will the student handle (-8) - (-5)?*
Teacher 1	*No we are dealing with difficult stuff. We have both the big and the small minus [ironic]* *[Laughing]*
Advisor	*How would you explain to them [the students] (-8) - (-5)? By the help of the number line*
Teacher 3	*No, I would rather not. [Laughing]*
Teacher 1	*We put ourselves on (-8) and then we will subtract from that (-5).*
Teacher 4	*The minus sign, it is the difference between them. What is the difference between the numbers (-8) and (-5) on the number line?*
Advisor	*What is the difference? (-3). Right? [...]*
Teacher 4	*I would rather go with the meaning of subtraction as the difference?*
Advisor	*Yes, but how do you get the negative difference then? We need the direction even to explain this.* *[Silence]*

The advisor has an important role in facilitating new questions to negotiate and renegotiate and she puts a lot of effort into renegotiating the meaning of the structure of the Cartesian coordinate system. At this point she is negotiating, from her experience of teaching and learning, rather than pure theoretical assumptions. The importance of direction is emphasised. The teachers' experience is subsumed and the advisor points out that this learning study belongs to them as the community renegotiates the idea of always treating Δx as positive, when she says: What will you do? The advisor renegotiates it as far as she says; "What is happening to the mathematical concepts when you say that Δx is always positive?"

They decide to avoid an aspect, but they are not explicitly stating its value. One teacher says that it depends on if they focus on the lesson or if they are building on something else. They realise the shift in value and the mutual engagement is not longer clear. They value the engagement of finding the nuances in this particular lesson, in relation to a wider engagement.

The experience of avoiding negative signs is shared in the community as students' have difficulties with handling negative numbers. The analysis captures a scene when the community has renegotiated the fact to avoid negative signs to be impossible. Even if they avoid it saying that Δx is always positive, they still have to handle it when calculating Δy, given by the example *(-8) - (-5)*. In this negotiation there is a lot of trust in the community, as the teachers admit that they would rather not like to explain the subtraction of a negative number. They reflect on the meaning of the subtraction as a difference, but conclude it is not enough to explain a negative difference. So the point that the advisor makes is that if they do not give the aspect of direction, the student will give meaning to the subtraction, the difference, as a distance. That is how the community has negotiated how to treat Δx. Point is taken.

The conceptual mapping reveals not only students' difficulties but also what they find difficult to teach – subtraction of negative numbers. What they negotiate, their joint enterprise is not taking place in isolation from the world.

The advisor says to me:

> ***Excerpt 29***
> *It is sometimes a challenge to supervise a group of teachers. They have so many years of experience. I have to be careful being the advisor, the teachers must not make the mistake of thinking that I am disregarding their teaching.*

After, when asking the teachers to reflect on the role of the advisor, they say:

Excerpt 30

Teacher 2	*She brought up things that we had not considered, such as showing the coordinate system.*

I asked if it was ideas about the mathematical content that the advisor brought up? The teachers say:

Excerpt 31

Teacher 3	*Yes, that too, or a pedagogical perspective. The advisor brought the concepts from the variation theory.*
Teacher 2	*The problem you have in the classroom, you can now explain in a different way. And how you can change that in your teaching. You can define it. I have missed that throughout [my career].*
Teacher 1	*When the advisor was with us, she pulled it in a direction. She listened to us and tried to put it into thoughts and words. It felt as if the advisor was running it.*

The teachers also say that without the advisor it would have been impossible for them to organise everything. They needed her to run it for them.

Interpreting variation

They now have an outline of a lesson plan, but the teachers will meet two more times to make a more detailed plan. Teacher 1 has volunteered to implement the first lesson:

Excerpt 32

Teacher 2	*What do you think? How do you think you will begin [the lesson]?*
Teacher 1	*I have four points drawn [on the y-axis], without any coordinates given. And then I think we will reason about the coordinates. I think that we will talk about that the x-coordinate is zero on the y-axis.* *[…]*
Teacher 3	*The student will pay attention to the things that vary and if we let the y- coordinate vary then it is the y - coordinate that will be paid attention to.* *[…]*
Teacher 1	*But maybe we then can compensate that [the variation] and stress the fact that x is zero on the y/axis.*

The community negotiates meaning into the concept of variation as an opposite of keeping constant, and that variation is always paid attention to by the learner. They negotiate how to let the student discern that x is always zero on the y-axis, and they mark out four points on the y-axis. This is to separate these points from other points, which is a variation through separation. The product of that separation is that they will have to emphasise that x is zero on the y-axis, verbally, since the variation is embedded in the y-value, as it takes different values, as it varies. They want the focus to be on the x-value, but attention will rather be drawn to the y-value. When the advisor is not participating, the community reifies the concept of variation differently to what they have done before.

How to teach

The planning continues:

Excerpt 33

Teacher 1	*So now I have drawn four points here. […] Then they will be named A, B, C and D.*
Teacher 3	*Will you name them A, B and C?*
Teacher 1	*They can be named anything. Or?*
Teacher 3	*I was thinking that you have the points and that you fill out the coordinates, the coordinates should not be given. […] Or will you display all the three coordinates at the same time?*

Teacher 1	*Four [points]. But it might be a smart idea to present one point at a time.[…] I did not plan to write the coordinates out, but of course you can do that as well.*
Teacher 2	*It becomes clear if you write them out.*
Teacher 3	*Why did you not want to write them out?*
Teacher 1	*I can write them out!*
Teacher 3	*I think there is a value in introducing one point at a time.[…] I was just thinking how to get some more activity among the student, not to give a lecture.*
Teacher 2	*That is important, we want the students to be active.*
Teacher 1	*[…] but I can indeed print and make copies of these [slides in a presentation] for the student to get some more activities. They can get copies without the coordinates, to mark out by themselves. In pairs!*

They want the student to take an active role, so they decided to make worksheets for the students.

What also becomes visible, as the community meet without the advisor, is that there is more focus on how the learning takes place in the class room, the negotiation is more concerned with techniques for how to present the content. The community is concerned with how the points should be labelled, if they should be introduced one by one, if they are to write them up on the board. The analysis also shows that activity of the students is important according to their experience. The lesson is planned with opportunities for the students to be active and to interact in the classroom and they negotiate the activity that will emerge as the students fill out a worksheet in pairs. The artefact is not suggesting how learning takes place it rather suggests how to teach the content. Hence they do not have a tool to coordinate practice that facilitates discontinuity regarding how learning takes place.

To return to the interview with the teachers. From the overall conversation regarding the setting and the experience of participating in a learning study the teachers say:

Excerpt 34

Teacher 4	*Variation theory is interesting; I had always thought of variation with a focus on method in some way, this was kind of – wow.*
Teacher 1	*I agree. It is the content that I can vary and it has effect – Wow!*
Teacher 2	*I agree – it is not just varying method. I feel, I am not that worried anymore about varying method. It is good, since I do not have that many methods, you reorganise them in groups to discuss and do things.*
Teacher 3	*Exactly. I also think it is hard to vary mathematics teaching.*

This negotiation underpins that the artefact did facilitate discontinuity in practice. The artefact made them aware of their own teaching culture and the teachers also seem relieved that it is not the method that must be varied in the lesson. However, planning a lesson is complex and they are concerned with how learning takes place and there is a lack of coordination in practice regarding this.

Four teachers – four perspectives!

One teacher stands next to the whiteboard and draws a line in a coordinate system parallel to the x - axis. She starts to mark out points on the line. This example is discussed in their previous meeting with the advisor and it is about letting the student discern the aspect of distance and from that its relationship to change:

Excerpt 35

Teacher 3	*So now it is two points?*
Teacher 1	*Shouldn't we have that? [rubs the board]*
Teacher 3	*I was thinking that we first find the coordinates and then find the distance between them.*
Teacher 4	*Or should we start to find the points from given coordinates and then draw the line through them. Then we will find the distance and we will introduce Δx.*
Teacher 2	*Will we put one of the points on the y-axis, to follow up what we did before? [points on the y-axis]*

The teacher rubs the whiteboard again.

The analysis shows four teachers, giving four perspectives. It is typical for this case that as the community plans the lesson they start to negotiate teaching, it gets harder to coordinate practice to move forward. Everything is argued about.

To give a dimension to the above their previous experience of collegiality, will be included. I asked them about their experience of working together:

Excerpt 36

Teacher 2	*We have never experienced anything like this [learning study] together. We are very traditional, those who teach the same courses in parallel classes might construct tests, mark tests and assess students grades together. Sometimes we discuss what we are going to do, assign a problem to the students together.*

Asking to what extent they have experience of planning lessons together, they say:

Excerpt 37

Teacher 1	*Yes, sometimes, as an outline of a lesson. It is more often the activity we plan, rather than the lesson. We plan the courses together, in term of its schedule; let's cover this chapter by then, let the students take a test then and so on. In addition we also talk about what we have done today as in how far [in the text book] we have come. We have done this for a long time, more or less. When it suits us. Then we plan the experiments, but that is in physics.*

The teachers have no previous experience of planning lessons together; they are rather collaboratively engaged in more organisational matters in the faculty.

A renegotiation

Excerpt 38

Teacher 2	*Will we put up a point on the other side [the negative side on the x-axis], to find the same distance?*
Teacher 3	*No, not at the same time. If we will do, what I have proposed earlier, we will have to introduce that now, that these two points have an interrelated order. We can start to index the points now. We say that index is always noted from the left. Maybe, we should not say index? We say that we name the points one and two.*
Teacher 2	*Is it important to put out one and two just now? Is that your aim, that the students should be able to handle the formula booklet by the end of the lesson?*

As they discuss, the teacher at the whiteboard has written the coordinates with index on the board, (x_1, y_1) and (x_2, y_2). She also labels the points *1* and *2*.

Excerpt 39

Teacher 3	*That is how I mean [pointing at 1 and 2 written next to the points]. If we will talk about* Δx*, then we have to tell how we calculate* Δx*. You take* x_2 *minus* x_1*.*
Teacher 2	*What do we want the attention to be drawn to, is it the difference between two x-coordinates or is it as you say that it is* x_2 *minus* x_1*. I feel it is… What do we want them to pay attention to?*
Teacher 4	*Maybe it is too much to talk about take* x_2 *and* x_1*?*
[…]	
Teacher 3	*Or maybe we can assume that they can calculate that* Δx *is four. And then we stress that they have to take the last point minus the first point. Otherwise it may become minus four.*

Teacher 4	*Can we not ask them how they reached the answer, that distance?*
Teacher 3	*Shall we? Then we are talking about the absolute value, but why should we talk about distance when we want change later on. As soon as we talk about the distance then it is the absolute value, then it is positive wherever it is in the coordinate system. If we talk about distance then we have to make it clear what it is.*
Teacher 2	*That is what we are doing! [Pointing at the example on the whiteboard] So, how will we do that? Will we not use the word distance in the lesson? What will we talk about then, what will we ask the students: the line between the two points? [Laughing] [...]*
Teacher 3	*We can say Δx.*
Teacher 4	*Delta is difference.*
Teacher 1	*Distance is always positive!*
Teacher 3	*We must not end up with that the student think that distance is the same as change.*
Teacher 2	*I have understood that we are going to introduce distance and from that continue to introduce change. To stress that they can be different.*

Planning the lesson in detail turns out to be a renegotiation of the critical aspects of the relationship between distance and change. One teacher wants to avoid this aspect in the lesson. Even though the community has earlier given the outline by creating a contrast to discern a distance. They negotiate how to denote and give meaning to Δx, without using the term distance. However this renegotiation is also a negotiation of the concepts of delta, difference, change and distance.

The analysis also shows that the members are engaged around different things. Some engages around what they have learned and negotiated earlier in the community and other engage around what they know from their experience. One teacher is more engaged in imposing his experience on the community and therefore they have to spend time renegotiating this matter again and again. I have earlier given examples of the power of the artefact to coordinate and to facilitate discontinuities in the community, but as they are about to implement the lesson the power of a core member is stronger. Practice has become static regarding this.

Summary

As the community starts to plan the lesson, the mutual engagement starts to fade, as they are not longer engaged around the same matters. One teacher takes a more active role in terms of renegotiating what was already agreed on in practice. The joint enterprise is concerned with how they will let the students discern the critical aspect in the lesson. The projection of that turns out to be avoiding the critical aspect rather than allowing the students discern it. However the conceptual mapping continues and it has shed light the relationship to related concepts and what the teachers find difficult to teach. The shared repertoire is illustrated in Figure 11. It is more typical that the joint enterprise evolves in an existing teaching culture, of avoiding students' difficulties. This is illustrated in Figure 11, framed in a rectangle. The arrow is illustrating a negotiation as a reaction to students' difficulties handling negative numbers.

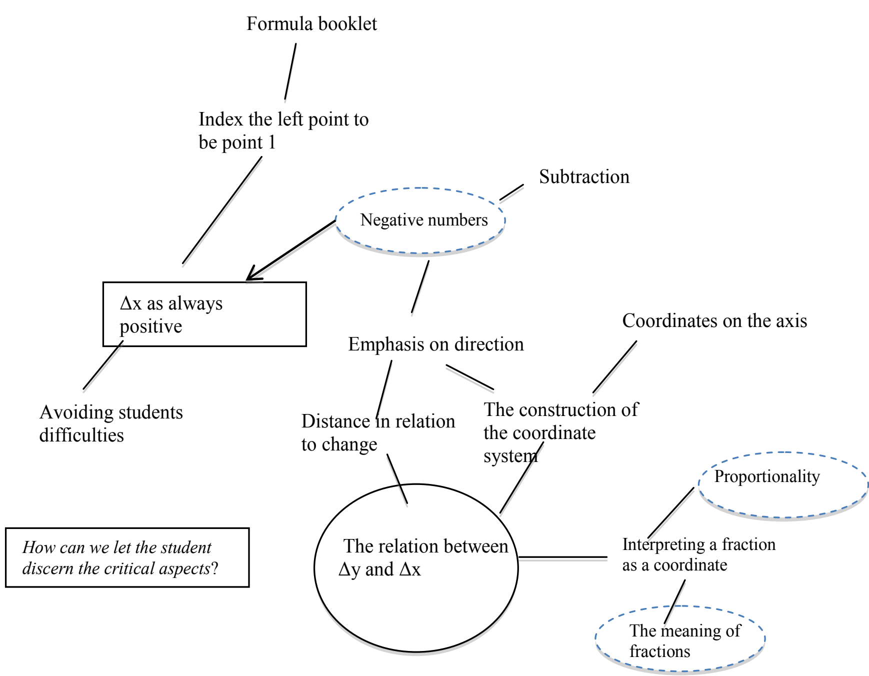

Figure 11 *The joint enterprise and the shared repertoire as the community starts to plan the lesson.*

5.4 Evaluating and revising the lessons

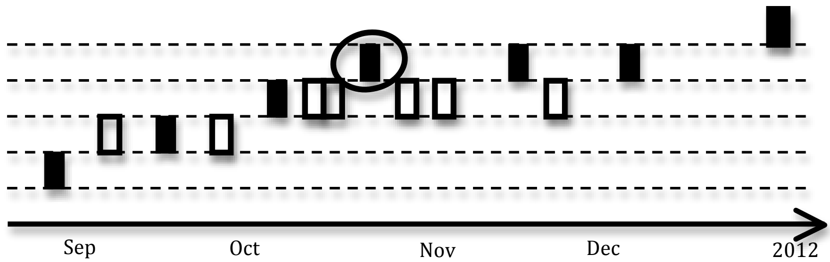

Figure 12 *The black ellipse is indicating the source of empirical data that will be analysed and presented in this section. It consists of the beginning and most of the fourth meeting with the advisor. Empirical data from the interview is also reported in this section.*

Different from a ”normal” lesson

In the fourth step of the process they are evaluating the first lesson[38]. The students from the first lesson have taken the post-test and the teachers have done a qualitative analysis, just as they did with the pre-test. The expert and the teachers then meet. The advisor asks if the planned lesson is different from what they normally do. The teachers say:

Excerpt 40

Teacher 3	*I can say I would not have put so much focus on the coordinate system as we do, maybe I would have randomly thrown some points on the board.*
Teacher 4	*This is different because we have focused on the ratio, other wise I would have drawn some lines and had started to walk along them in a coordinate system.*

They start to go through the student's results in order to compare the pre-test by the post-test. They are analysing the results in the post-test in relation to the pre-test, and they realise that most of the students have understanding of coordinates, and that most of the student had that knowledge even as they did the pre-test. From that experience, they reflect that they should not have spent so much time on coordinates in the lesson.

So, the post-test was initially used to negotiate how big a part they would take in the lesson. In the analysis of the post-test they also reflect that they have misinterpreted the analysis of the pre-test. The teachers say that the first lesson was different from what they normally would do, they do not spend as

[38] It is the negotiations concerned with the evaluation of the lesson that takes place in the community that is of interest, hence I will not analyse and evaluate the lesson.

much time on coordinates. Their analysis is hence along with their experience of a normal lesson.

I recalled that at one point they said that a lesson like this does not exist in reality, and afterwards I asked the teachers to explain that further. They said:

Excerpt 41

Teacher 3	*How the lesson turned out, with all the details. You do not do that normally. […]*
Teacher 2	*In a normal lesson I am more flexible, this time I hade to stay to a manuscript all the time. I was not used to that. I mean, I do have a plan, but sometimes something else emerges in the classroom.*

The lesson was different as the teacher was ruled to a tight manuscript.

The meaning gets lost

In the next step of the evaluation process they watch the video-recorded lesson together:

Excerpt 42

Advisor	*Let us see what dimensions of variation we have created in the lesson!*
Teacher	*Can we review that again [dimensions of variation]*

They revise the concepts of variation once more, before they analyse the video-recorded lesson.

Wenger (1998) writes that reification always rests on participation and in turn participation always organises itself around reifications. As the community is not using the concepts of the theory the meaning gets lost. The concepts of object of learning and critical aspect have been given meaning to and reified in the community. Not saying the theoretical concepts has been given the correct meaning, following the assumptions. The concepts of dimension of variation are not used as frequently as the others, hence the meaning also gets lost.

Avoiding the term distance

After watching the video-recorded lesson they reflect that there are too many aspects that vary at the same time when they introduce Δx, they decide not to bring the aspect of index into the revised lesson.

The advisor reviews the fact that the teachers have created both a contrast and a generalisation in the same example. The teachers are listening carefully.

Excerpt 43

Advisor	*Why did you create the generalisation of always subtracting the left from the right point?*
Teacher 3	*We said that we wanted to get into that quite soon, (x_1, y_1) and (x_2, y_2) since it is in their formula booklet.*
Advisor	*Yes, but when you are saying distance and…*
Teacher 1	*No, I did not say distance because we decided to only use change, we would not use the word distance in the lesson. I said change.*
Advisor	*Ok, you said change. But the point is that the students need to understand what change is before you generalise.*

The analysis shows that the concepts of dimension of variation are used as a language of how the content was presented in the lesson. They negotiate that there are too many aspects that vary in the first example, and from that they decide not to index the numbers too soon. The advisor renegotiates the idea of always keeping Δx positive, in terms of their language: "Why did you create a generalisation of always subtracting the left from the right point?"

This also captures that they had decided not to mention the concept of distance in the lesson, as a way to avoid the critical aspect of relationship between distance and change. The advisor brings it back to them, saying that the student still needs to understand the meaning of change, and that meaning could be emphasised in contrast to the concept of distance.

Disposition of the whiteboard

Another point the advisor makes is how the teachers are using a presentation. Each slide projected on a screen, is replaced by a new slide. Hence each coordinate system is replaced by a new coordinate system. This means the students miss out on the sequence, the variation of separation. Figure 13 shows how the advisor emphasises a separation by drawing three coordinate systems next to each other on the whiteboard.

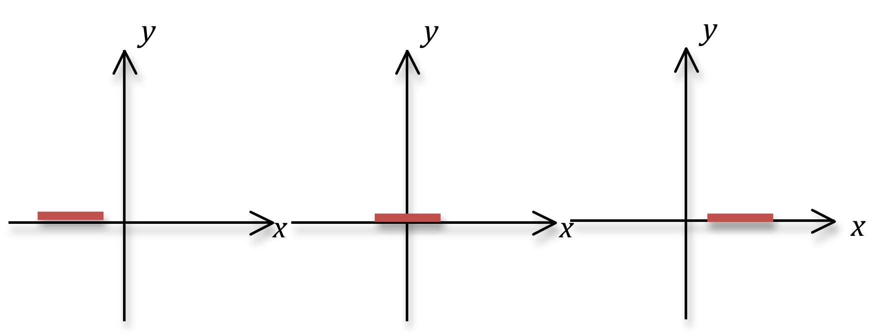

Figure 13 *A separation to discern a distance in a coordinate system*

This shows that considering the disposition of the whiteboard is also brought to the shared repertoire, but also the limits of displaying a presentation on a screen providing a moment, to moment attention.

Issues of implementation

In the evaluation of the lesson the advisor also says she misses out the interaction with the students, the discussions, in the classroom.

> ***Excerpt 44***
>
> *Teacher 4* *You said, when I met you after you had taught the lesson that it turned out different. But watching the video I think it was just as we planned it.*
>
> *Teacher 1* *Well, I kept to the thread. But then, I think I did not know the students, they were not my students, and that is why I wanted to have this contact with the students. I did not go around in the room and help them individually. I think it would have been different teaching the lesson in my own class. It will be different for you [taking the lesson in their own classes].*

They reflected that taking the lesson in a class when the teachers did not know the students meant the interaction they had planned for did not emerge.

As I returned to the teachers I asked why it was necessary to implement the lesson?

> ***Excerpt 45***
>
> *Teacher 3* *To be able to modify the lesson, then it is necessary to teach it. You have to try it in reality, it was from that experience we started to reconsider. It does not matter that much the lesson did not turn out to be a hit.*
>
> *Teacher 4* *It was just then [when teaching the lesson] I shaped up. Teaching the lesson made you more alert. […]*
>
> *Teacher 2* *It does not feel that the primary goal of this learning study is to plan a perfect lesson. What is important to me is that I have got something from this. When I go and teach, in my lessons later that are not in a learning study, then I take this with me. Then it is not ruled by a manuscript.*
>
> *Teacher 4* *That is also my experience, that it was everything around that gave me that good felling when processing the lesson. The lesson was very tight and I felt by the end, as I was teaching that the students were quite exhausted. Normally I would have cut it then, or done something different. It rarely*

happens that you have such a controlled lesson for 60 min. The last 20 min you often let them work on their own.

As the teachers reflect about the role of the lesson they reflect that the lesson is not the primary goal of the learning study, it is the teachers' professional development that is their mutual engagement. Still the lesson has a value, for them to imagine and to engage around.

Summary

When the community evaluated and revised the first lesson it becomes clear that they had negotiated to avoid, hence not to let the student discern distance in relation to change in the lesson. From that the evaluation of the lesson was once more a renegotiation of this issue. The analysis shows that the teachers find value in the unit of analysis in the lesson and to implement it in a class. However their mutual engagement is the teacher professional development, and everything that has been negotiated when focusing discussions on slope. Taking this into account of what is captured in the analysis; it becomes more complicated to coordinate practice and to find a mutual engagement as they start to plan the lesson.

5.5 The iterative process

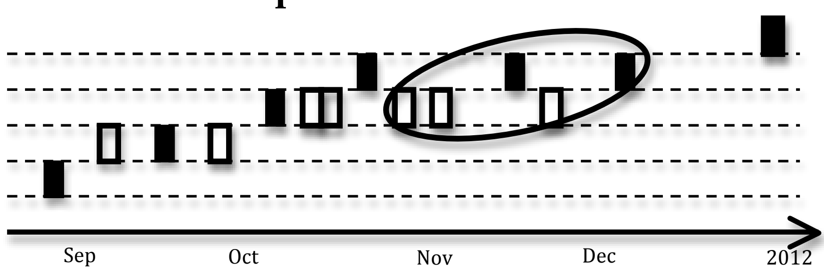

Figure 14 *The black ellips indicates the source of empirical data that is analysed and presented in this section. It is the iterative process that is in focus.*

Spending the time on what?

The teachers meet to plan the second lesson. They initially decide to project the presentation onto the whiteboard instead of using the screen. This choice means they can take an active role and fill out the points in the coordinate system, on the whiteboard. They also plan to present several coordinate systems on the board at the same time. Teacher 2 will implement the second lesson, in her own class. They start to plan:

Excerpt 46

Teacher 1 *Then we will need to label the points. 1 and 2 or A and B?*
Teacher 2 *We do as we have said before, 1 and 2, since he is so stubborn regarding that.*
Teacher 3 *I am not stubborn!*
Teacher1 *Oh yes you are really stubborn, especially with your 1 and 2 [writing 1 and 2 on the air and circle them]. You have been going on about that every time!*
Teacher 2 *Yes, a lot. [laughing] We have spent too much energy on that all ready, so now it will be 1 and 2.*
Teacher 1 *Then I suggest A and B….*
Teacher 2 *No, no [laughing]*
Teacher 1 *I think A and B is better.*
Teacher 2 *[laughing] Let us stick to this now.*
Teacher 1 *I can tell you why I think A and B is better, because when we have four points in the following example shall we then call the points 1, 2, 3 and 4 and will we then index them* x_1, x_2, x_3, x_4*? Then it does not work with your formula.*
[…]
Teacher 2 *What do you think?*
Teacher 4 *I think we should move on.*

This shows that the teachers are aware that they are engaging around a minor matter such as the labelling of the points. They negotiate that they have hade enough time spent on this discussion already, so even if they now have more arguments for labelling them *A* and *B* they will stick to *1* and *2*. From that it seems their mutual engagement has turned to the value of "getting this done" as the joint enterprise is how to label points on the whiteboard.

Which order do the concepts follow?

A teacher reflects on the introduction of the lesson, and in which order they introduce the mathematical concepts of *k*-value, slope, rate of change and straight line:

Excerpt 47

Teacher 4 *Is that the right order? We start to talk about the k-value and then we reach that a k-value is a slope.*
Teacher 3 *No, I think we will begin with rate of change, and the k-value and slope as we have introduced a straight line.*

They discuss how they will introduce slope of a straight line from rate of change:

Excerpt 48

Teacher 2	*How does the sign of the rate of change affect the slope of the line? We can ask the students that.*
Teacher 3	*And what will they answer then?*
Teacher 4	*Upward, downwards to the right.*
Teacher 1	*Rising and falling.*
Teacher 3	*They will not say increasing or decreasing at least.*
Teacher 1	*How will the student make meaning of the slope of a line?*
Teacher 3	*I think they will get the relation to the sign*
Teacher 1	*So do I.*
Teacher 4	*If we should have a contrast, a line with no slope.*

As they are revising the first lesson into the second the order in which the concepts will be introduced is also a matter of negotiation. What comes first; the rate of change, the *k*-value or the slope. It is initially a negotiation of the relationships of the concepts. In the second cycle, planning the second lesson they negotiate introducing rate of change and from that relate it to a slope and *k*-value of a straight line. One teacher finds meaning in, and suggests, a contrast to make meaning of the slope of a straight line.

The mapping concerns the relationship of aspects, the order in which they follow.

What is the meaning of finding the rate of change between two points?

The teacher, who brought up the issue of order of introduction, reformulates his question next time they meet to evaluate the second lesson with the advisor:

Excerpt 49

Teacher 4	*There is one thing that I was thinking about now when watching the video. Why are we introducing the rate of change before we introduce the straight line? […]*
Teacher 2	*That is why I had the idea initially to introduce a straight line as an introduction to the lesson.*
Advisor	*Maybe we should have a coordinate system with lines in it. And we say [as an introduction to the students]: How will we explain that some lines are falling and some are rising? [...]*
Teacher 3	*I thought that we should not bring too much, fusion, in the beginning. That is why we took it step by step.*
Teacher 2	*Yes, I agree, but now it struck me; What is the meaning of finding the rate of change between two points? The students also asked that in the lesson. Why did we not draw the line?*

The teacher who teachers the second lesson feels that it moved too slowly at certain points; hence her idea is to get to Δy much quicker next time. This is the same evaluation as of the first lesson, that the lesson was slow. The evaluation is about cutting and condensing the lesson. The post-test is negotiated in relation to the pre-test and the community analyses the students' results as having improved.

Evaluating the second lesson the community negotiates the introduction of the lesson. They negotiate to begin with a straight line and then reach slope rather than the opposite. In addition to what is described above, it turns out to be a negotiation of the concept of fusion. The community will no longer have a general rule not to create fusions in the introduction. From this they also return to the definition of the object of learning when they decided to focus on the relationship between Δy and Δx, rather than increasing or decreasing slope. It is brought to the shared repertoire that an object of learning exists in a mathematical context and the concepts of the theory are in relation to that.

Engagement around students

They also discuss that as a teacher you must not take the voice of one or two student as representing a whole class. A few students in the class from the second lesson told the teacher that the content was too simple. They noticed that there were a lot of students quietly working hard with the coordinate system and its structure in both lessons. A teacher says:

Excerpt 50

Teacher 3	*We must not forget. This lesson plan that we so carefully have processed, is because we want to reach more students than we normally do in the classroom.*

As they evaluate the second lesson the mutual engagement is concerned with value for the students. But not necessarily value for all students, they negotiate a value for students quietly working hard in a lesson. They are engaged to reach more students than they normally do.

Limits of always letting the left point be the first point

They decide to reduce the third lesson in the same manner as in the previous review. This is as the qualitative analysis of the post-test of the second group still showed a good understanding of the construction of a coordinate system. The following is from a meeting planning the third lesson.

Excerpt 51

Teacher 1	*Maybe we can reduce the number of coordinate systems with Δx, there are four now. Two of them could show Δy instead.*
Teacher 3	*There is a problem then if we want to find Δy for two vertical points. We cannot apply what I have done before*

with always taking the right point minus the left point. Which point is point 1 and which point is point 2? There is no such thing as the left and the right point. [...]We need to consider how we will handle this.

Still the community is negotiating the definition of point *1* and point *2*. The advisor has stressed it and the community has negotiated the critical aspect of distance in relation to change, over and over. Earlier in the process the teachers' way of handling this was to avoid the aspect rather than let the student discern it. In the first lesson they decided to avoid the concept of distance, in the second they let the student discern this. Still in the second lesson, they reinforce the way of defining point *1* as the left point and point *2* as the point to the right. By that they are avoiding negative difference in the lesson. Planning lesson three, they are concerned about this again, since now they cannot use this definition when they want to find the difference between two vertical points. It does not hold true.

The meaning of subtraction

Teacher 4 will take the third lesson in his own class. The teachers meet for a final meeting before the third lesson is to be implemented.

Excerpt 52

Teacher 1	*Yesterday I attended a seminar concerning a thesis on subtraction of negative numbers. It was explaining a bit. This is introduced in year six, then in year eight and that's it. On the number line 5 - (-2) is interpreted as the distance between the values and it is obvious that it must be plus [that –(-) = +]. But if you turn it around, taking the smaller value first (-2) - 5, then it is not interpreted as that distance anymore.*
Teacher 4	*No, and that is strange*
Teacher 1	*It certainly is. It can be confusing when reasoning about distance and we have elaborated on this.*
Teacher 2	*Our advisor talked about changing direction.*
Teacher 3	*Yes. It is really difficult.*
Teacher 1	*Yes that is why I feel we should have introduced direction here.*
Teacher 4	*This is what we have been discussing, all the time.*

A teacher brings an aspect to the community of how students interpret the meaning of subtraction. They renegotiate that it is this that they have been talking about throughout the entire process – the meaning of subtraction, in particular with negative numbers. It brings them back to the importance of direction in a coordinate system. The mapping is now extended to the meaning of subtraction of negative numbers. They have traced the students' partial understanding of the meaning of subtraction.

Some understanding, that is easy to grasp

Excerpt 53

Teacher 3 *But if you consider an increase, then it is positive, if it is a decrease then it is negative. […] Is the latter value greater than or smaller than the first value. If it is an increase then it must be positive, if it is a decrease then it must be negative. Right? That is why we should define the left point to be point 1. Right? […] Our textbooks make the mistake, it says it does not matter in which order you define point 1 and point 2. You just have to be consistent [when calculating Δx and Δy]. It makes another twist for the students.*

Teacher 2 *Are you thinking about applications of mathematics too much now. If you will go beyond the applications? Only pure mathematics?*

Teacher 3 *Well this is some sort of understanding, it is easy to grasp.[...] The other way, by the vectors, going left in the coordinate system is negative direction. You will lose the understanding.*

Teacher 1 *I agree with you.*

[Silence]

Teacher 2 *There is no point [laughing] we will not change anything today.*

[…]

They write on a slide in the presentation for the third and final lesson; increase means positive change, decrease means negative change.

Excerpt 54

Teacher 1 *What would happen if we did this cycle eight times, eight lessons [laughing]. Would we still be sitting here going back and forth regarding distance and change.*

[...]

Teacher 3 *I think I would have tried the other approach with vectors. Now we are stuck in many ways here. I am not comfortable at all with teaching it in that way [direction], and it might open up new things for me. Maybe I will then conclude it is more rational. I do not know.*

One teacher still insists that it is easier for a student to grasp the object of learning in ”his way”. The argument is for helping the student to understand, but also the argument they have to react to that the students do not have the pre-understanding. This may be a negotiation developed in a teaching culture.

By the end it is also clear that the teacher cannot imagine himself teaching subtraction of negative numbers making meaning of it in terms of direction, hence neither will he engage around it.

In the process of renegotiating this matter they also negotiate that they are stuck in this manner of presenting the lesson.

The contribution of the meaning of distance in a coordinate system

In the evaluation of the third lesson, they realise they have cut the lesson too far. As they analyse the results of the post-test they note that something must have happened as many students have stated a distance to be negative in the answers.

Excerpt 55

Teacher 4	*We have condensed the lesson quite well. Earlier [in lesson two] we had three examples of a distance along the x-axis, one on the positive side [of the x-axis], one on the negative and one covering both. Now we did not.*
Teacher 2	*In my lesson [lesson 2] no students made that mistake on the post-test.*

So far the community has used the analysis of the post-test to confirm their experience and feeling that the lesson is too slow, thus they have in the iterative process condensed some parts and cut others. The last part of the analysis shows they realise they have cut it too far as they analyse the results from the post-test. They negotiate the fact that in the third lesson they have cut the separation created in the previous lesson. It was no longer possible for the students to discern that distance is always positive. The concept of separation as an opportunity for student learning is reified in practice.

The meaning of *3/2*

The advisor takes note of what has happened in the third lesson regarding the fraction *3/2*, and brings them back to their earlier discussions:

Excerpt 56

Advisor	*In this example with the fraction, is it "three halves" or is it "three divided by two"?*
Teacher 4	*Well, I stress [in the lesson] that we will keep it as a fraction, hence it should be three halves. Right?*
Advisor	*Yes three halves, since when you say three divided by two it implies an operation, the student will divide three by two.*

Teacher 4 is more actively involved throughout the iterative process, and also said that implementing the final lesson contributed to this. As they evaluate the third lesson the community recognises a nuance in the teaching. They negotiate the difference in saying three divided by two, rather than three halves to give meaning to a fraction. Earlier in the result an example of when the teachers negotiate that as soon as a student sees a fraction he or she wants to turn it into a decimal, that is, to carry out the operation, to divide. By using this example the advisor brings this negotiation back to the community. This is another example of what is brought to the shared repertoire by the advisor; what do we say and what meaning do we give to the concept when we say it.

Summary

The iterative process of planning and evaluating the second and the third lessons, was a renegotiation of emphasis on direction to let the student discern distance in relation to change. These aspects are illustrated and framed in rectangles in Figure 15, the arrows are showing the process of renegotiation as a circular moment. The limit of keeping Δx positive by always treating the left point to be the first point is negotiated in practice. The mapping was extended to the meaning of subtraction, but also in which order the straight line, k-value and rate of change are related. It is also negotiated that the meaning we give to the concept follows from what we say.

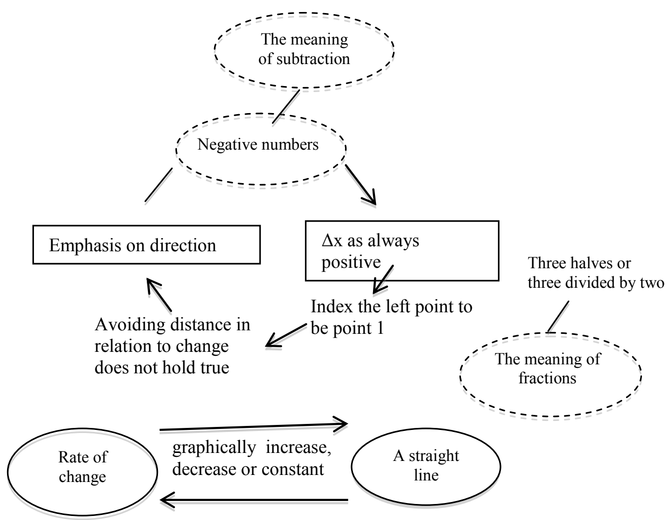

Figure 15 *The shared repertoire as the community engage in the iterative process.*

This was the case of when mathematics teachers focused discussions on slope. The analysis will underpin the answer to the research questions that will follow in the following chapter.

6. CONCLUSION AND DISCUSSION

In this chapter the research question; *What are the characteristics of practice when upper secondary mathematics teaches focus discussions on slope in a setting of learning study?* will be answered. This is the conclusion of what was captured of the case. Then the theoretical contributions of the case, addressing the theoretical framework and the selection of case will be discussed. Finally the case will be linked to future research.

6.1 What are the characteristics of practice?

Flyvberg (2006) says in case study the point is to open up rather than close the case. However I will answer the research question, trying to conclude what characteristics of practice that turned out to be central for the case. As an overall characteristic I conclude; *When mathematics teachers focus discussions on slope, practice is developed in the present teaching culture,* illustrated in Figure 16.

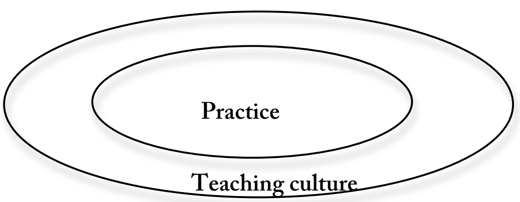

Figure 16 *Practice developed in a present teaching culture.*

Wenger (1998) emphasis that communities of practices are not self-contained entities. They develop in a larger context - historical and cultural. This case captures the present teaching culture, which is also interrelated to other characteristics of practice. Hence, as I next will discuss the characteristics of practice, I will also discuss its interrelation to the present teaching culture.

The community is mutually engaged around finding the small nuances in the teaching that exerts a major effect on the student learning. The joint enterprise is a negotiation of what they want the students to know, but also what the students need to know in order to understand the relation between Δy and Δx. This could be regared as an example of an infrastructure for sharing professional knowledge (Stiegler & Hiebert, 1999). The shared repertoire is a projection of their negotiations. The conceptual mapping of the relationship between Δy and Δx is the projection of their infrastructure for sharing professional knowledge. This is illustrated in Figure 17, which is a summary of Figure 6, Figure 8, Figure 11 and Figure 15.

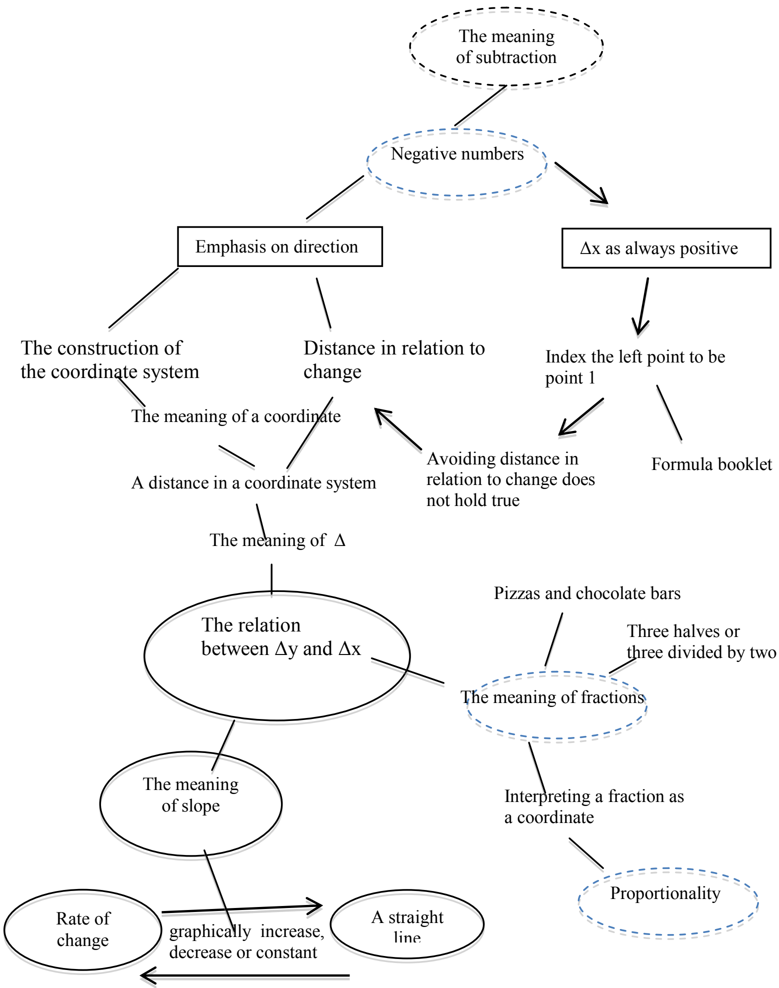

Figure 17 *The conceptual mapping of slope.*

The community identifies that the students need to know the meaning of Δ, in order to understand the relationship between Δy and Δx. From that they negotiate the preunderstanding of the meaning of a distance and the meaning of a coordinate to be necessary. Conceptual mapping also includes fractions. The mapping gives the community an awareness of students preunderstanding of fractions, from what and how it was taught earlier. It places focus on their previous teaching.

The negotiations start from students' difficulties with the content and the concept of critical aspect was negotiated as a difficulty. This is reviewed to be typical when handling a new feature in an old system, the new feature is modified so it functions the old way (Stiegler & Hiebert, 1999). A review of the literature regarding teaching and learning slope shows that it also concerns students' difficulties and misconceptions.

Further the construction of the Cartesian coordinate system is identified in order to understand the meaning of a coordinate. The advisor provides the community with meaning of direction as important to understand the construction of the Cartesian coordinate system. It is negotiated, in order to understand distance it must be discerned in relation change and vice versa. These aspects are in practice considered to be critical aspects in order to understand the relationship between Δy and Δx, thus they are written in bold in Figure 17.

Practice develops in a present teaching culture, of always treating Δx as positive. Hence the aspect of emphasis on direction is negotiated and renegotiated in practice.

As the community starts to plan the lesson, mutual engagement starts to fade as they are not longer engaged around the same matters. One teacher takes a more active role in terms of renegotiating what was already agreed on in practice. The joint enterprise is concerned with how they will let the students discern the construction of the Cartesian coordinate system and distance in relation to changes in the lesson. The projection of that turns out to be a negotiation of avoiding an aspect rather than let the students discern it. However the conceptual mapping continues and it sheds light to the relationship with proportionality. It becomes more typical that the joint enterprise evolves in an existing teaching culture, that of avoiding students' difficulties.

The mapping provided practice with opportunities for both conceptual understanding and hence teaching for relational understanding, but they did not seize the latter opportunity when they started to plan the lesson. The analysis shows the experience of trying to avoid minus signs in their teaching, since they know that it is a student difficulty. From that, as a reaction to that negative numbers are a student difficulty, the idea and experience is to always treat Δx as being positive. This could be done if you always let the left point be the first point, since the formula to manipulate is $\Delta x = x_2 - x_1$.

The analysis also shows that it is not possible to avoid minus signs and the advisor demonstrates this. It comes down to that fact that the teachers also find it difficult to teach subtraction of negative numbers, and in particular to explain the meaning of a negative difference. Typical of this case is what is brought to the shared repertoire. The community has started by negotiating the students' difficulties and ended up in a negotiation of their own difficulties[39].

The teachers find value in the lesson, and to implement it in a class. The analysis also shows that it becomes more complicated to coordinate practice, to find a mutual engagement, as they start to plan the lesson. The mutual engagement is the teacher professional development, and everything that has been negotiated when focusing discussions on slope, rather than to produce a perfect lesson.

The iterative process of planning and evaluating the second and the third lesson, was a renegotiation of emphasis on direction to let the student discern distance in relation to change. The limit of keeping Δx positive by always treating the left point to be the first point is negotiated in practice. The mapping was extended to the meaning of subtraction, but also in which order the straight line, *k*-value and rate of change are related. It is also negotiated that the meaning we give to the concept follows from what we say.

I will from the above conclude: *The coherence of characteristics*[40] *of practice is the conceptual mapping of the concept of slope*.

The teachers negotiate the concept of slope as the rate of change and what it describes. Initially they negotiate to introduce rate of change and from that reach if it describes an increase or decrease for a straight line. In the iterative process they change the order and decide to introduce slope as the steepness of a line and from that find the rate of change as a way to quantify the steepness. This is according to previous research, regarding how most students are introduced to slope. Previous research also gives that students tend to only consider the two line segments and compare the values, disregarding the sign, when quantifying steepness (Teuscher & Reys, 2010). This case captures a negotiation of the contrast of this. The negotiation of letting the students discern the change rather than a line segment, and how they relate to each other. But within a present teaching culture it is also negotiated to quantify steepness by comparing the line segments and "if it is an increase it is positive and if it is a decrease it is negative".

The literature gives that U.S. students are taught a phrase for the algorithm, as a device for instrumental understanding of rate of change

[39] The matter that students have difficulties learning minus signs and that teachers find it difficult to teach the same, is not something typical for just this case study. I am not trying to identify a correlation between the teachers' and the students' difficulties neither. The typical is that these matters are brought to the shared repertoire in the community.

[40] The coherence of the characteristics of *mutual engagement, joint enterprise, shared repertoire* from the core of *negotiation of meaning*.

(Walter & Gerson, 2007). In this case they teach a device of always letting the left point be the first point. Previous research gives that it takes a bit of time for the students to get used to that it does not matter in which order the points are labelled (Stump, 1999). Giving the students a devise of always letting the left point be the first point in the algorithm may be a way of avoiding confusion. This as students get confused of that it does not matter which point is the first or the second. Thompson (1994) have found that rates involving time is the most intuitive for the student, hence it might be easier for the student to give meaning to the device of always letting the left point be the first point as it can be related to time. As long as the students not fully understand the meaning of subtraction and the relation between distance and change they might find it confusing to give meaning to that it does not matter in which order they are subtracted.

Previous research have also found, just as this case captures, the Cartesian coordinate system, subtraction, negative numbers to be important aspects of slope and rate of change. It is reviewed a concern of how the students make meaning of rate of change, from manipulating a formula to seeing that it means something (Stump, 1999). The aspect of understanding the relation between distance and change, brought to the shared repertoire in this community is an aspect that gives meaning to slope. To be able to understand what change is, it is negotiated in the community that the student must understand the meaning of negative numbers, just as the meaning of subtraction. This challenge the teaching of subtraction as always subtracting the smaller number from the bigger, which is also in consensus with always letting the left point be the first point.

The background of the thesis gives a concern of the students understanding of slope as a fraction. The teachers in Stump's study (1999) discuss an identical example as this case captures; When you say the line has a slope of 2/3, what does 2 and 3 mean respectively? Negotiating the aspect of fraction, the teachers also become aware of what partial understanding students have from previous teaching of fractions. As the students are introduced to slope they are previously given meaning to fractions as parts of a whole, represented as slices of pizzas in the textbooks. Further Lamon (1995) writes a precondition for the concept of slope is proportional reasoning, and even regarding this, the teachers in this case become aware of the students misconceptions of slope from their partial understanding of proportionality. They negotiate how students do not understand that they need two points on a line to find the rate of change, is due to that finding the proportionality the students have only been concerned with one point.

When they focus discussions on slope it is the nature of the subject that becomes visible, and another opportunity to be aware of their own teaching culture regarding the nature of the subject. Initially the community projects emphasis onto concepts and connections, and by this they govern teaching and learning mathematics for relational understanding, rather than an instrumental understanding. Where the first is defined as knowing both how and why to carry out a mathematical operation, as the last is problematic to be called understanding since it lacks the why. The understanding lies in the rule, to be able to handle a formula, to get the right answer, without knowing why (Skemp, 1976).

The importance for teachers' conceptual understanding of slope to teach for concepts and connections, away from focus of manipulative facility (Lloyd & Wilson, 1998) has been reviewed and the community in this case is given the opportunity to do both. The discussions were more than simply sharing ideas, they were confronting traditional practice (Lord, 1994).

As the community is engaged with constructing questions for the pre-test, they negotiate how to ascertain the mathematical thinking of the student. They have negotiated that the calculations in the questions do not tell what the student really understands, a teacher says; "it is so mechanical". I interpret this as the community is trying to assess relational understanding, rather than the instrumental understanding mentioned above (Skemp, 1976). This may be put in relation to Pang's (2008) Hong Kong learning study also addressing slope. In that learning study none of the questions in the pre-test were concerned with calculations. The questions in the pre-test of this case study all include calculations, which the students also are asked to comment on. Also in the interview they comment that they have never constructed questions for qualitative analysis before. This also coheres with research stating that Swedish students are regularly assessed with teacher made traditional test that assesses more computational skills (Lundin, 2008; Boesen, 2006). All the parts in a teaching culture interrelate and maintain each other, and changing one isolated part will not make any improving effect on practice (Stiegler & Hiebert, 1999). The National tests in Sweden puts focus on conceptual understanding, and hardly require any computational skills (Boesen, 2006) and it might be an example of an isolated part that is not bigger than the whole of a teaching culture.

The teachers state that the value of participating and to invest energy in collegiality in a learning study is to find the small nuances in their teaching that exerts major effect to student learning. This impacts how and what they will negotiate meaning into as concerns mutual engagement. The analysis shows that their mutual engagement changes throughout the process. By the end it is clear that the members engage around different things. This gives an understanding of Stiegler and Hieberts (1999) statement that the success of a lesson study finally depends on the teachers, as they are the core. This case adds it also depends on the present teaching culture, as this is where the core is situated. The mutual engagement can give rise to differentiation and to homogeneity. This, as it involves competences and competences of others. The teachers' practice draws on what the teachers know, and the ability to negotiate what they do not know (Wenger, 1998). To some extent it can be interpreted as a limit of mutual engagement when not being aware of their routines that governs the classroom teaching. The progression of the shared repertoire of the community was a negotiation and renegotiation of instrumental and relational understanding of the relationship between Δy and Δx respectively. This indeed can make the teachers aware of their own teaching culture.

In the background reasons for teaching for instrumental understanding are reviewed. Skemp (1976) says to be able to handle a formula only is to get the right answer. The student would approve he or she understands, since they will always get the right answer. Always letting the left point be the first point, is to be able to handle a formula. And the students might only care for the rule, to get the right answer. Within its own context, instrumental mathematics is usually easier to understand (Skemp, 1976). As the case captures; that this way of teaching gives some understanding that is easy to grasp. Instrumental understanding involves less knowledge, hence it is easier to get the right answer more quickly than by relational understanding. The difference is so immediate that even relational understanding often uses instrumental thinking (Skemp, 1976). The Swedish curriculum also defines both abilities (out of 7) to mathematical working;

- Use and describe the meaning of mathematical concepts and their interrelations.
- Manage procedures and solve tasks of standard nature with and without tools. (National Agency for Education, 2012, p.1)

This can underpin the use of a device to always letting the left point be the first point, but only once the relation between distance and change and the meaning of change is taught to the student.

Earlier Marton's point that their aim of their learning study was to create better learning by suitable treatment of content was stated. Variation theory has been framed to be an intellectual artefact and the case captures that it facilitates continuities and coordinates practice, but also discontinuities in practice. A language for technical communication can serve to bridge teachers' practice in collegiality (Evans, 2012). This case also captures that the artefact was used as a language in the community. The artefact of variation theory facilitated questions to engage around, it helps them to negotiate and reflect on the mathematical content. The assumption of *what* to teach focused on mathematical content was brought to the shared repertoire. They focused discussions on slope in a nonmathematical manner. Their discussions focused on mathematical content and did not pay attention to cuteness or real-lifeness of tasks (Stiegler & Hiebert, 1999). The teachers said they felt relieved that variation does not imply variation in method, which was the only thing they had varied before. However, the Swedish curriculum also emphasis teaching should cover a variety of working forms and methods of working (National

Agency for Education, 2013), hence it gives understanding to the teachers' experience.

The 2011 Swedish Education reform reinforce requirements of scientific basis and proven experience (National Agency for Education, 2013). Nuthall (2002) however, say that what we do in schools is not evidence-based practice it is rather a matter of cultural tradition. This case captures the importance of the advisors participation, primarily as a broker bringing new perspectives of teaching and learning slope to practice. It is also captured that, when the advisor was not there, they tended to negotiate and renegotiate things so it functioned in the old way.

This part was an attempt to answer the research question. Flyvbjerg (2006) says a purely descriptive case study can have a force in itself and a formal generalisation is only one of several ways to gain understanding and accumulate knowledge. However, from the conclusion, theoretical contributions of the case will be elaborated.

6.2 Theoretical contributions of the case

The transferability of the case study is the criteria for establishing trustworthiness. Transferability concerns to what extent the findings can be applied to other situations (Lincon & Guba, 1985). I will take the framework and the selection of the case into account addressing the theoretical contributions of the case.

The framing sharpens the meaning of previous studies and illuminates the differential utility of prospective findings (Stake, 1995). I was inspired by Hemmi's (2006) use of the framework as unit of analysis. Concepts of the theory framed this case and a community of practice was my unit of analysis. Palmér (2013) has reviewed that Communities of practice has been used as both emergent and designed. I locked for what emerged in practice, however I think a theoretical contribution from this case could also be to use the frame when designing practices. In both cases taking into account that practice develops in the present teaching culture.

The use of the framework in previous research has focused on the one hand on the individual and on the other hand the community. That is not a change of topic rather a shift in focus in the same general topic (Wenger 1998). I have used the framework focusing on the community, which also coheres with the aim and research question and not at least with the generation of empirical data through observations. However, the analysis showed that the teachers are not equally actively involved through their participation. What was analysed was the negotiations between members, not the negotiations that went on silently in their heads. It was neither any empirical data generated from negotiations outside the meetings included. To deeply understand what happened when they focused discussions on slope I could have observed the teachers' participation in collegial settings, in their previous community of practice. Optimising the analysis would have been to use the whole framework focusing both the community and the individual. Though that would also have required other types of methods for generating empirical data, and that would have been another case study.

The interaction between people have gone through a shift; learning takes place in other spaces than face to face, even so the domain of the community of practice still works[41]. It is not too narrow and not too broad for this case. Communities of practice frame the empirical data of this case to the extent that it acts as a guide towards what to pay attention to. Selected concepts of that theory enabled to put attention on when mathematics teachers focusing discussions on slope. The framework puts into words how and why the teachers interact, still it allows focus on the mathematical content in this case study. It is a complex framework that opens up rather than closes the case (Stake, 2005) and that gives possibilities for the reader to experience multiple cases from the same unit of analysis (Ragin, 1992). Focusing the negotiation and renegotiation of teaching and learning about slope the characteristics of practice became visible. The overall characteristic of that practice developed in the present teaching culture could be captured. The role of the teachers as members as well as the role of the advisor to be a broker were captured and gave understanding of that was emerged. As the shared repertoire is the projection of the joint enterprise and coheres with the mutual engagement, it also gives understanding to what can be generalised from this case. From this I will take the selection of case into account.

The selection of the case aimed to provide rich information about practice emerging in collegiality, which in turn can help other researchers to understand other similar cases or situations. I have described that the selection of the case was not a case picked by criteria, it was the only case that gave me access to empirical date in the time available. Flyvbjerg (2006) continues that *extreme, critical* and *paradigmatic* cases can be more appropriate in regards of giving rich information[42]. If the purpose is to maximize utility of a single case, *extreme* cases that are especially problematic or good, may be used. Several types of extreme cases may even be used. *Critical* cases may capture information that permits logical deduction of the type that if it is not valid for

[41] I met Wenger in May 2012 participating in his 5-day workshop focusing Communities of practice and social learning. He and his colleague Trayner had then taken the social theory of learning beyond Communities of practices, into Social learning spaces. The new shift focuses on landscapes of practice rather than communities of practices as a consequence of how the world looks like today. Learning as a trajectory is then forming an identity across a landscape, not through a community of practice. A community of practice is a social learning space, but all social learning spaces are not communities of practices. The developed framework put emphasis on the identity in a complex world.

[42] One common misunderstanding of case study is that one cannot generalise from a single case and generalisability of case study can be increased by the selection of representative and random cases Flyvbjerg (2006). Looking for theoretical contributions of this case I am rather trying to generalise to theory. From this in turn can help other researchers to understand other cases.

this case, then it applies for no other cases (or all cases). *Paradigmatic* cases can capture or develop a domain that the case concerns. It might not be possible, or even necessary, to identify the type of case that is captured in this thesis[43]. However once the case now is captured I will discuss it in relation to different types of cases, in order to scrutinise on the generalisations that can be made.

When mathematics teachers focus discussions slope practice is characterised by a conceptual mapping of slope, developed in a present teaching culture. I think this case manages to get a point across of what may emerge in collegiality. The case may in that sense be regarded as extreme as it captures deep underlying structures and not only what emerges and how frequently they do. The generalisation from this case is the knowledge that, in order to understand mathematics teachers in collegiality in-depth, the present teaching culture must be acknowledged. The case might be critical as what emerged and the teachers' engagement must be understood in relation to the fact that they insisted to participate in collegiality. The learning study was not imposed top down, it was not forced upon them. The engagement of the teachers in this case cannot be generalised to any random group of upper secondary mathematics teachers in collegiality. This case is not what most likely (or less likely) would happen when mathematics teachers focus discussions on slope. The setting of the learning study, variation theory and the advisor play a major role facilitating an infrastructure for sharing professional knowledge. Variation theory facilitates questions in the community, which follow from the concepts of the theory and these questions define the joint enterprise in the community. This is what they will engage around, and it is coordinating practice. The concept of object of learning facilitated the joint enterprise of: "*What do we want the students to understand*", the concept of critical aspect facilitated: *"What do the students need to know in order to understand the object of learning?"* and the concept of dimension of variation facilitated: *"How can we let the students discern the critical aspects?"* The product of the joint enterprise, which is the shared repertoire, the conceptual mapping of the concept of slope was a projection of that. Theoretical contributions from this however, may be when implementing this infrastructure for sharing professional knowledge regarding mathematical concepts; a conceptual mapping and an opportunity for teaching for relational understanding may emerge. What is rather representative is the nature of the mathematical subject, the relationship between Δy and Δx and how it relates to other concepts, i.e. that it tracks students' partial understanding of the meaning of subtraction.

The role of the advisor is critical as she brings new perspective on the mathematical content to practice. A general contribution to theory from this

[43] As I have discussed the case as intrinsic and instrumental, the focus was on how the case was used to establish understanding for the characteristics of practice. The selection of case here refers to its relationship to other cases.

case stresses the importance and the impact of the advisor. Especially when the community is not fully aware of the routines that govern their teaching.

Paradigmatic cases highlight more general characteristics of the societies in question. No standard sets the standard of a paradigmatic case as it sets the standard (Flyvbjerg, 2006). I think this case captures the practice and its underlying patterns and addresses future research on what might be central for teacher learning in collegiality.

> Teaching can only change the way cultures change: gradually, steadily, over time as small changes are made. (Stigler and Hiebert 2004, p. 13)

6.3 Future research

I have problematised that collegiality is the least common form of relationship among adults in schools, even though it seems both obvious and compelling. There is rather a mutual supportiveness which is about getting along well and being friendly (Evans, 2012). Ball (1994) stresses the common view that ”each teacher has to find his or her own style” is a direct result of working within a discourse of practice that maintains the individualism and isolation of teaching. This case focuses on discussions on mathematical content, which are elaborated on as a critical feature for the effectiveness of teacher professional development initiatives (Desimonde, 2009). The case captures how the mutual engagement in the community changes as they begin to plan the lesson. This is when the core becomes static and does not allow any discontinuities from the teaching experience. In a learning study the teachers are to design the lesson using tool of variation theory. This case might address future research considering a limit of Marton's and Pangs (2006) idea of letting the teachers set up the design for themselves. The case shows that the teachers do question their perspectives regarding teaching and learning slope, it provided opportunities for the teacher to think in new ways. But even so they decide to design the lesson according to their previous experience.

This case also captures the fact that the teachers value the lesson and negotiates its importance, but their mutual engagement is in regard to their teacher professional development. They said they were engaged around everything that was learned as they planned the lesson, it was not an engagement to produce a perfect lesson in itself. Stiegler and Hiebert (1999) write that the unit of the lesson has a validity for the teachers, as it does not lack of generalisation to real life experience. A single lesson retains the key complexities that must be taken into account when improving classroom learning. I have reviewed that the idea that a lesson is a part of the teaching culture and coincides with teacher beliefs on the nature of mathematics and how learning takes place. I have also reviewed that the Japanese mathematics lesson tells a story, it is tightly connected with a beginning, a midpoint and an end. They are different from the Swedish and US mathematics lesson, which are described to be more modular with fever connections. Yoshida (2004) writes that a lesson or a sequence of lessons are highly sharable among teachers in Japan. The lesson is not a unit in Sweden or in the U.S. in the same meaning as it is in Japan. I think it gives insight into why the teachers in the community says the lesson is not usable, to the extent it is not a fixed unit they will teach again. In future research the idea of a lesson seems important to reflect on otherwise it is challenged as the unit of analysis. This might also address Marton's (in press) definition of "Learning study in a wider sense".

In the introduction I have problematised that; collegiality is a new setting for many teachers; collegiality requires structure that goes beyond simply sharing ideas, that sustains the individualism and isolation of teaching and collegiality requires de-privatising of the work of teachers to start to engage critically with issues of practice. Learning from this case the future might address if *the sharable lesson as unit of analysis is critical to maintain collegiality as a part of a teaching culture.*

Summary in Swedish

Kollegialt lärande för ökad måluppfyllelse genomsyrar diskussionen och är numera trenden och något som eftersträvas i skolan. Men övergången till kollegialt lärande är något nytt för många lärare, som inte heller alltid uppfattar vinsterna. De flesta gymnasielärare i matematik undervisar separata klasser bakom stängda dörrar och lära sig att undervisa genom att undervisa. Det här resulterar i hög grad i ett självständigt, men även ett isolerat arbete. Dessa lärare vill ofta hellre gå hem för att planera morgondagens lektioner än att delta i kollegiala forum. Många gånger kan forumen karaktäriseras som ”style shows”, där lärare visar upp och ger exempel från sin undervisning. Att utgå från individuella lärares unika stil gör det inte bara svårare att utveckla någon gemensam standard, det gör det även svårare för kollegor att inte hålla med och att vara kritiska. Det här snarare upprätthåller den isolerade undervisningen än att ge möjlighet till kollegialt lärande. Kollegialt lärande kräver struktur som går utöver mer än att bara dela idéer och det kräver ett kritiskt förhållningssätt mot den undervisning som bedrivs. Syftet med denna avhandling är att beskriva och analysera den gemensamma praktik som utvecklas när gymnasielärare i matematik deltar i ett strukturerat kollegialt forum.

En learning study, en svensk version av en japansk fortbildning som bygger på kollegialt lärande, gav tillgång till empirisk data för studien. Fyra lärare på en gymnasieskola har träffats varje vecka under en termin och deras diskussioner vid mötena fokuserade på det matematiska begreppet lutning, med målet att planera en gemensam lektion. Denna avhandling handlar om fallet när matematiklärare fokuserar diskussionerna på begreppet lutning. Fallstudien underbyggs av Wenger’s *Communities of Practice*, som analysenhet, för att svara på forskningsfrågan: Vad kännetecknar den gemensamma praktiken när gymnasielärare i matematik fokuserar diskussionerna på lutning i en learning study? Analysen inriktas på samstämmighet av *mutual engagement, joint enterprise* och *shared repertoire* för att beskriva vad som uppstår och som kännetecknar praktiken.

Det empiriska materialet har genererats via observationer av de 14 möten som denna learning study gav tillgång till och för att bekräfta men även för att komplettera observationerna generades även empiri genom en intervju. Mötena och intervjun har filmats och sedan transkriberats. Fallstudien syftar till att förstå komplexiteten i den praktik som växer fram i kollegialitet i just det här fallet, men den avser även att utveckla förståelse bortom fallet. Denna fallstudie har en abduktiv metodologisk ansats för att både karaktärisera den gemensamma praktiken och för att upptäcka meningsfulla underliggande mönster. De scener ur empirisk data som fångar fallet på det här sättet har presenterats i kronologisk ordning.

Lärarna i fallet är engagerade kring att hitta de små nyanser i undervisningen som gör stor inverkan på studenternas lärande. De förhandlar

vad eleverna behöver veta för att förstå sambandet mellan Δy och Δx i algoritmen för en linjes lutning. Praktiken kännetecknas av en konceptuell kartläggning av begreppet lutning, vilket även avslöjar elevernas partiella förståelse på grund av hur de skapades mening kring relaterade begrepp i tidigare undervisning. Den konceptuella kartläggningen av lutning behandlar även bråk och proportionalitet, och går tillbaka så långt som till elevernas partiella förståelse av innebörden av subtraktion. Analysen visar även att de förhandlingar som uppstår är i relation till en kultur av att undvika elevernas svårigheter i undervisningen. Fallet fångar förhandlingar och omförhandlingar av undervisning av lutning för instrumentell förståelse eller begreppsförståelse. Ett övergripande kännetecken för praktiken är att den utvecklas i den nuvarande undervisningskulturen, vilket även är ett betydande teoretiskt bidrag av fallstudien. Allt hänger ihop i en undervisningskultur och förklarar och ger förståelse åt praktiken. För att kunna utveckla undervisningen så är det en utgångspunkt i att vara medveten i hur de olika delarna hänger ihop och inte minst hur de upprätthåller varandra. Det här är något andra studier kan lära av att oavsett om det handlar om att designa en communtiy of practice eller att analysera vad som uppstår i en community of practice.

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Appendix

Förtest

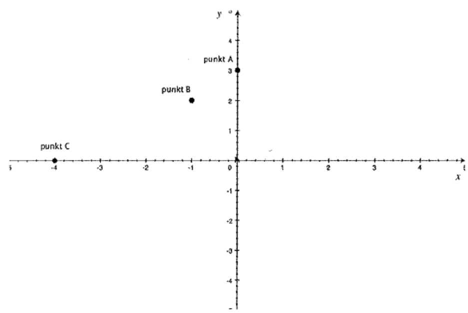
1. I koordinatsystemet är tre punkter markerade.

a: bestäm koordinaterna för punkterna

A:

B:

C

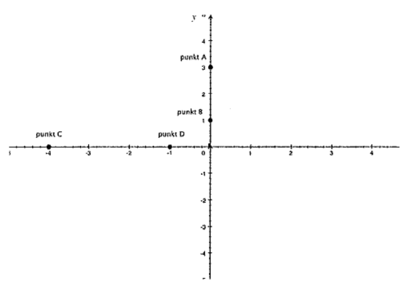


b: rita in punkterna

D: (0 ; 4)
E: (-1 ; 0)
F: (-4 ; -1)

2.
I koordinatsystemet är fyra punkter markerade.

a: bestäm avståndet i y-led mellan punkterna A och B

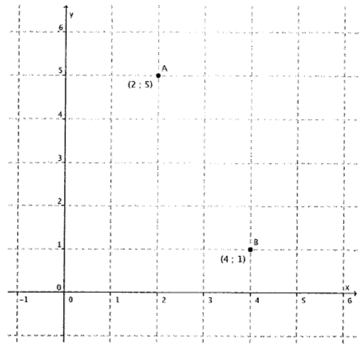


b: bestäm avståndet i x-led mellan punkterna C och D

c: hur bestämde du avstånden mellan punkterna i de föregående uppgifterna?

3. Avläs ur figuren:

a: Hur beräknar man ändringen i x-led när man "går från **A** till **B**"?

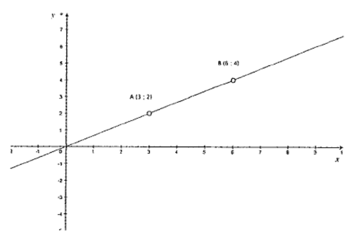


b: Beräkna ändringen i x-led.

c: Hur beräknar man ändringen i y-led när man "går från **A** till **B**"?

d: Beräkna ändringen i y-led.

4.
Punkterna A: (3 ; 2) och B: (6 ; 4) är markerade i koordinatsystemet.



a: Punkten C ligger sex steg i x-led (åt höger) från punkten A.

Ange C:s koordinater.

b: Hur kom du fram till C:s koordinater?

c: Punkten D ligger 24 steg i x-led (åt höger) från punkten A.

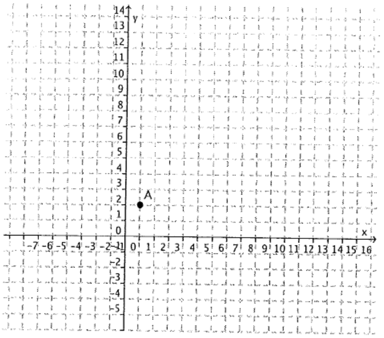
Ange D:s koordinater.

d: Visa med beräkningar hur du bestämde D:s koordinater.

e: Beskriv hur y-värdet förändras då x-värdet ökar med ett steg.

5. En lutning kan skrivas som ett bråk t.ex. $\frac{5}{7}$

a: Rita en rät linje som går genom punkten **A** och har lutningen $\frac{5}{7}$



b: Hur tolkar du siffran 5 i bråket?

c: Hur tolkar du siffran 7 i bråket?